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Acyclic and Star coloring of Bistar Graph Families

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Abstract- In this paper, we discuss about the acyclic chromatic number and star chromatic number of (i) middle graph (ii) central graph and (iii) total graph of Bistar graph $B_{m,n}$, denoted by $M(B_{m,n})$, $C(B_{m,n})$ and $T(B_{m,n})$ respectively. In fact, we discuss the relationship between these two chromatic numbers of the above graphs.

Index Terms- central graph, middle graph, total graph, acyclic coloring and star coloring

I. Introduction

Throughout this paper, by a graph, we mean a finite, undirected, simple graph. Let G be a graph with vertex set V(G) and edge set E(G). The middle graph (Michalak,1981) of G, denoted by M(G), is defined as follows. The vertex set of M(G) is V(G)UE(G).Any two vertices x,y in M(G) are adjacent in M(G) if one of the following case holds.

(i) x,y are in E(G) and x,y are adjacent in G.
(ii) x is in V(G), y is in E(G) and x,y are incident in G.

For a given graph G –(V,E) we do an operation on G by subdividing each edge exactly once and joining all the non-adjacent vertices of G. The graph obtained by this process is called central graph(Vernold et al., 2009 a,b) of G and is denoted by C(G).

Let G be a graph such that G =(V,E).The total graph (Michalak 1981,Harary 1969) of G,denoted by T(G), is defined as follows. The vertex set of T(G) is V(G)UE(G).Any two vertices x,y in the vertex set of T(G) are adjacent in T(G) if one of the following cases holds.

(i) x,y are in V(G) and x is adjacent to y in G.
(ii) x,y are in E(G) and x,y are adjacent in G.
(iii) x is in V(G), y is in E(G) and x,y are incident in G.

The Bistar graph $B_{m,n}$ is the graph obtained from K_2 by joining m pendent edges to one end and n pendent edges to the other end of K_2. An acyclic coloring [ ] of a graph G is the proper coloring such that the subgraph induced by 2 colors α and β is a forest. The minimum number of colors necessary to acyclically color G is called the acyclic chromatic number and is denoted by a(G).

The notion of star chromatic number was introduced by Grunbaum(1973).A star coloring of a graph G is a proper vertex coloring in which every path on 4 vertices uses atleast 3 distinct colors. Equivalently, in star coloring, the induced subgraph formed by the vertices of any 2 colors has connected components that are star graphs. The star chromatic number, $X_s(G)$ of G, is the least number of colors needed to star color G.

Let V_1,V_2 be the vertices of K_2 and $U_1,U_2,...,U_m$ and U_{m+1}, $U_{m+2},..,U_{m+n}$ be the pendent vertices joined to V_1 and V_2 respectively. Let V_{12} be the newly added vertex on the edge joining V_1 and V_2. Let $W_{1i}(1\leq i \leq m)$ be the newly introduced vertices on the edge joining V_1 and $U_i(1\leq i \leq m)$ and W_{2j} $(m+1\leq j\leq m+n)$ be the newly introduced vertices on the edge joining V_2 and $U_j(m+1\leq j\leq m+n)$.

II. Acyclic and Star coloring of $M[B_{M,N}]$

A. *Acyclic coloring of $M[B_{m,n}]$*

i. Theorem

The acyclic chromatic number of $M[B_{m,n}]$ is given by

$a[M [B_{m,n}] = \max(m,n) +2$, when $m\neq n$
$= n + 2$, when $m = n$.

Proof;

By the definition of middle graph,$\langle V_1,V_{12},W_{1i}$; i = 1 to m$\rangle$ form a clique of order m+2 in $M[B_{m,n}]$.Similarly, $\langle V_2,V_{12},W_{2j}$; j= m+1 to m+n$\rangle$ form a clique of order n+2 in $M[B_{m,n}]$.Now, consider the coloring C of $M[B_{m,n}]$ as follows.

Assign C_1 to V_{12} and C_2 to V_1 and V_2.Assign C_2 to U_i, i= 1 to m+n. Assign C_{i+2} to W_{1i}(i = 1 to m) and C_{k+2} to $W_{2,m+k}$ (k=1 to n). Now, we can show that the coloring C is acyclic by discussing the following cases.

Case (i)

Since V_{12} is the only vertex with color C_1, $M[B_{m,n}]$ has no bicolored $(C_1\text{-}C_j)[2\leq j\leq\max(m,n)+2]$ cycle.

Case (ii)

Consider C_2 and C_j where $3\leq j\leq \max(m,n)+2$. The induced subgraph of these color classes contains disjoint bicolored path, $U_{j-2}W_{1,j-2}V_1$ or $V_2W_{2,m+j-2}U_{m+j-2}$ or both. As V_1 and V_2 as well as U_{j-2} and U_{m+j-2} are non-adjacent , $M[B_{m,n}]$ has no bicolored $(C_2\text{-}C_j)$ cycle.

Case (iii)

Consider C_i and C_j ,$3 \leq i, j \leq \max(m,n)+2$. The induced subgraph of these color classes contains one or both of the edges $\{W_{1,i-2}W_{1,j-2}$, $W_{2,m+i-2}W_{2,m+j-2}\}$ and hence $M[B_{m,n}]$ has no bicolored $(C_i\text{-}C_j)$cycle.

Thus, $M[B_{m,n}]$ contains no bicolored cycle in the coloring C. Hence, the coloring C is acyclic.

Therefore, $a(M[B_{m,n}]) = \max(m,n)+2$, $m\neq n$
$= n+2$, $m= n$.

B. *Star coloring of M[Bm,n]*

ii. Theorem

The star chromatic number of middle graph of Bistar graph $B_{m,n}$ is given by

$X_S(M[B_{m,n}]) = \max(m,n) + 2$, $m\neq n$
$= n + 2$, $m = n$.

Proof;

Consider the same coloring C of $M[B_{m,n}]$. We can show that it is also a star coloring by discussing the following cases.

Case (i)

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Consider C_1 and C_2. The induced subgraph of these color classes contains the bicolored path (of length 2) $V_1V_{12}V_2$ and isolated vertices $\{U_1,U_2,\ldots,U_{m+n}\}$.Thus, $M[B_{m,n}]$ has no bicolored (C_1-C_2) path of length 3 in C.

Case (ii)

Consider C_1 and C_j($3 \leq j \leq \max(m,n) + 2$). The induced subgraph of these color classes contains the stargraphs,$W_{1,j-2}$ $V_{12}W_{2,m+j-2}$ or $V_{12}W_{2,m+j-2}$. Thus, $M[B_{m,n}]$ has no bicolored (C_2-C_j) path of length 3.

Case (iii)

Consider C_2 and C_j ($3 \leq j \leq \max(m,n) + 2$). The induced subgraph of these color classes has star graphs $U_{j-2}W_{1,j-2}V_1$ and $U_{m+j-2}W_{2,m+j-2}V_2$. Thus, $M[B_{m,n}]$ has no bicolored (C_2-C_j) path of length 3.

Case (iv)

Consider C_i and C_j($3 \leq i, j \leq \max(m,n) + 2$). The induced subgraph of these color classes contains the edge $W_{1,i-2}W_{1,j-2}$ or $W_{2,i-2}W_{2,j-2}$ or both. So, $M[B_{m,n}]$ has no bicolored (C_i-C_j) path of length 3.

Thus, the coloring C is a star coloring. Therefore,

$$X_S(M[B_{m,n}]) = \max(m,n) + 2, \text{ when } m \neq n$$
$$= n + 2 , \text{ when } m = n.$$

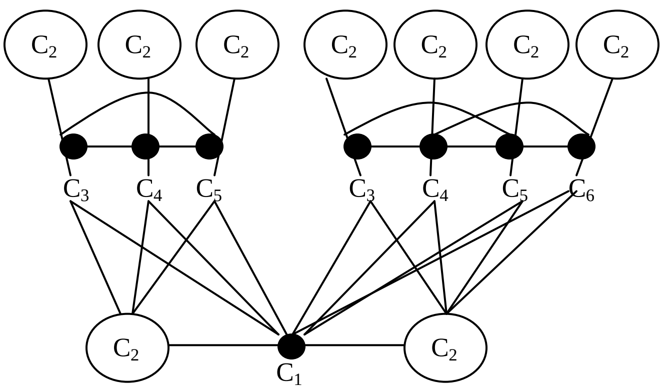

Fig.1: a($M[B_{3,4}]$) = 6 = X_S($M[B_{3,4}]$)

III. ACYCLIC AND STAR COLORING OF C[$B_{M,N}$]

A. *Acyclic coloring of C[$B_{m,n}$]*

i. *Theorem*

For any Bistar graph $B_{m,n}$, the acyclic chromatic number of its central graph is given by

$$a (C[B_{m,n}]) = m + n , \ m,n \geq 3.$$

Proof;

By the definition of central graph, < U_j; j=1to m+n > form a clique of order m+n. The vertex V_1 is adjacent to { U_j; j=m+1 to m+n} and V_2 is adjacent to { U_i; i=1 to m}. Now, consider the coloring C' of C[$B_{m,n}$] as follows.

Assign C_i to U_i ,i = 1 to m+n. Assign C_1 to V_{12}, C_2 to V_1 and C_{m+1} to V_2.Assign C_r to $W_{1,i}$(i=1 to m-1) where r = i+2(mod m) and C_1 to $W_{1,m}$. Similarly, assign C_s to $W_{2,m+j}$(j= 1 to n-1) where s = m+j+1(mod m+n) and assign C_{m+2} to $W_{2,m+n}$.

Since < U_i; i =1 to m+n > form a complete graph, it contains no bicolored cycle. So, if C[$B_{m,n}$] has any bicolored cycle, then that cycle must contain the path either $U_2W_{12}V_1$ or $U_{m+1}W_{2,m+1}V_2$.(All other paths connecting U's and V's are colored with 3 colors).

Consider the path $U_2W_{12}V_1$.

Case (i)

When m=3, W_{12} has color C_1 in C' and so we get only the bicolored (C_1-C_2) path ,namely $U_1U_2W_{12}V_1$,none of the other adjacent vertices of V_1 has color C_1. So, C[$B_{m,n}$] has only bicolored path,but not cycle.

Case (ii)

When $m \geq 4$,W_{12} has color C_4. In this case, we get only the bicolored (C_2-C_4) path $U_4U_2W_{12}V_1$ as W_{14} has color different from C_2 and C_4.

In both cases,,C[$B_{m,n}$] has only bicolored (C_1-C_2) path,but not cycle.

Next, consider the path $U_{m+1}W_{2,m+1}V_2$. $W_{2,m+1}$ has color C_{m+n}. So, the path $U_{m+2}U_{m+1}W_{2,m+1}V_2W_{2,m+n}$ become a bicolored path,not cycle as U_{m+n} has color C_{m+n}. As the other adjacent vertices of V_2 have colors different from C_{m+n}, C[$B_{m,n}$] has no bicolored cycle.

Therefore, the coloring C' is acyclic and hence

$$A (C[B_{m,n}]) = m+n \ , \ m, n \geq 3.$$

B. *Star coloring of C[Bm,n]*

The coloring C' discussed in sec 3.1 is not a star coloring as C[$B_{m,n}$] has the following path of length 3,(i) and (iii) or (ii) and (iii).

(i) the bicolored (C_1-C_2) path $U_1U_2W_{12}V_1$ (when m=3).
(ii) the bicolored (C_2-C_4) path $U_4U_2W_{12}V_1$(when $m \geq 4$)
(iii) the bicolored (C_{m+1}-C_{m+2}) path $U_{m+2}U_{m+1}W_{2,m+1}V_2$.

If we assign any C_k ($1 \leq k \leq m+n$, $k \neq 2$) to W_{12} or C_k ($1 \leq k \leq m+n$, $k \neq m+1$) to $W_{2,m+1}$, then $U_kU_2W_{12}V_1$ and $U_kU_{m+1}W_{2,m+1}V_2$ will form respectively, the bicolored (C_2-C_k) and (C_{m+1}-C_k) path, each of length 3. So, assign new color C_{m+n+1} to W_{12} and $W_{2,m+1}$. This new coloring of C[$B_{m,n}$] is a star coloring and hence we have the following theorem.

ii. *Theorem*

The star chromatic number of C[$B_{m,n}$] is

$$X_S (C[B_{m,n}]) = m+n+1$$

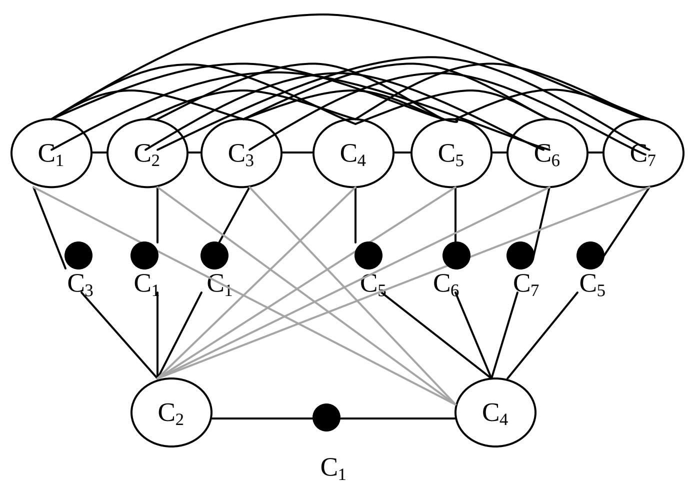

Fig. 2: a(C[$B_{3,4}$]) = 7

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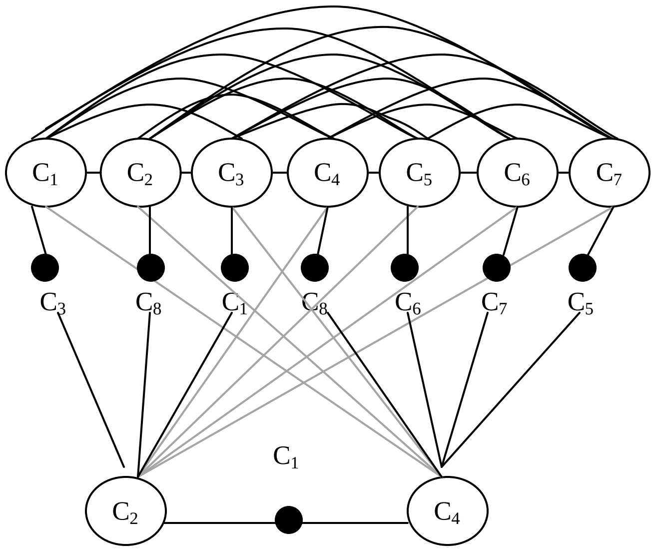

Fig. 3: X_S ($C[B_{3,4}]$) = 8

IV. ACYCLIC AND STAR COLORING OF $T[B_{M,N}]$

A. *Acyclic coloring of $T[B_{m,n}]$*

i. Theorem

The acyclic chromatic number of $T[B_{m,n}]$ is

$$a(T[B_{m,n}]) = \max(m,n) +2 \quad \text{when } m \neq n$$
$$= n + 2 \quad \text{when } m= n.$$

Proof:

By the definition of total graph, $< V_1,V_{12},U_i$; i=1 to m > form a clique of order m+2. Similarly, $< V_2,V_{12},U_j$; j= m+1 to m+n > form a clique of order n+2. V_1 and V_2 are adjacent in$T[B_{m,n}]$. V_1 is adjacent to $\{U_1,U_2,\ldots,U_m\}$ and V_2 is adjacent to $\{U_{m+1},U_{m+2},\ldots,U_{m+n}\}$.

Now, consider the coloring C" of $T[B_{m,n}]$ as follows.

Assign C_1 to V_{12},C_2 to V_2 and C_3 to V_1. Assign C_3 to $W_{2,m+1}$ and C_{i+2} to $W_{1,i}$(i =2 to m). Assign C_3 to $W_{2,m+1}$ and C_{j+2} to $W_{2,m+j}$ (j=2 to n). Next, assign C_{i+3} to U_i, $1 \leq i \leq m-1$ and C_2 to U_m. Assign C_{i+3} to U_{m+i}, $1\leq i \leq n-1$ and C_3 to U_{m+n}. So, we need m+2 colors for coloring $<V_1,V_{12},U_i$; i =1 to m > and n+2 colors for $< V_2,V_{12},U_j$; j= m+1 to m+n >.

Now, we show that C" is acyclic.

Case (i) Since V_{12} is the only vertex with color C_1 $T[B_{m,n}]$ has no bicolored (C_1-C_i) ($2\leq i \leq \max(m,n)+2$) cycle.

Case (ii) Consider C_2 and C_3. The induced subgraph of these color classes contains only the bicolored paths $W_{11}V_1, V_2, W_{2,m+1}$ and U_m V_1 V_2 U_{m+n}. Hence, there is no bicolored (C_2-C_3) cycle in $T[B_{m,n}]$.

Case (iii) Consider C_2 and C_j ($4\leq j\leq \max(m,n) +2$).

The induced subgraph of these color classes contains the bicolored path U_{j-3} V_1 $W_{1, j-2}$ and $W_{2,m+1}$ $W_{2,m+j-2}$. So, $T[B_{m,n}]$ has no bicolored (C_2- C_j)cycle.

Case (iv) Consider C_3 and C_j ($4\leq j \leq \max(m,n)+2$).

By the same argument as in case (iii), $T[B_{m,n}]$ has no bicolored (C_3- C_j) cycle.

Case (v) Consider C_i and C_j. ($4\leq i, j \leq \max(m,n)+2$) .

The induced subgraph of these color classes contain bicolored path of length _atmost 2.(If we consider C_4 and C_5 in fig (4), then the induced sub graph contains the path U_1 W_{11} W_{12} and U_6 W_{26} W_{27}. Hence, $T[B_{m,n}]$ has no bicolored (C_i- C_j) cycle.

Thus, the coloring C" is acyclic and therefore

$$a\ (T[B_{m,n}] = \max(m,n)+2 \quad \text{when } m\neq n$$
$$= n+2 \quad \text{when } m=n$$

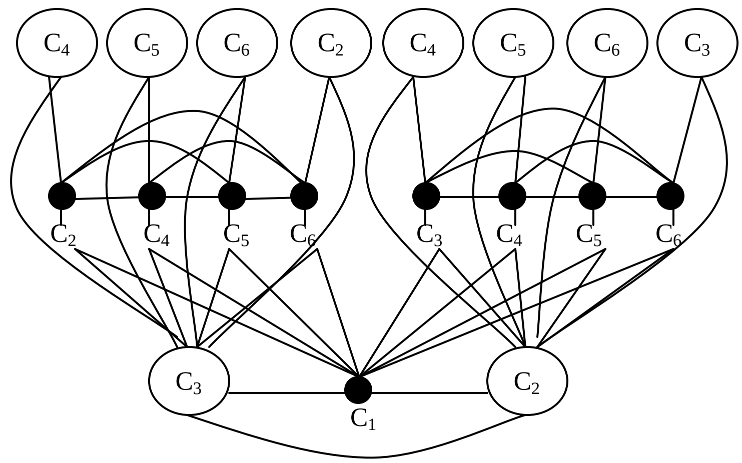

Fig. 4: a($T[B_{4,4}]$) = 6

The coloring C" is not a star coloring as W_{11} V_1 V_2 W_{21} and U_m V_1 V_2 U_{m+n} are bicolored (C_2-C_3) path of length 3. In the next theorem, we have proved that the star chromatic number and the acyclic chromatic number of $T[B_{m,n}]$ are equal when m≠n and X_S ($T[B_{m,n}]$ is one greater than a($T[B_{m,n}]$)) in the case of m=n.

B. *Star Coloring of T[Bm,n]*

ii. Theorem

For any Bistar graph $B_{m,n}$,

$$X_S\,[T(B_{m,n})] = \max(m,n)+2,\ m\neq n,\ m>2$$
$$= n+3,\ m=n,\ n>2$$

Proof:

Consider the coloring C" that we discussed in sec 4.1. In case (i), we show that the acyclic and star chromatic number are equal by making small changes in C". In case (ii), we show that the star chromatic number is one greater than the acyclic chrmotic number.

Case (i) m≠n

Let us assume that m<n. In the coloring C", we make the following changes without affecting the number of colors used. Assign C_{i+3} to $W_{1,i}$ (i=1 to m) and C_{i+4} to U_i(i=1 to m-1) and C_4 to U_m. The induced subgraph of any two color classes is star and hence it is a star coloring.

Therefore, X_S ($T[B_{m,n}]$) = n+2, m≤n, m>2.

When n<m, the result becomes X_S ($T[B_{m,n}]$) = m+2,n≤m, n>2.

Case (ii) m=n.

In the coloring C", we make the following changes to make it as a star coloring. Assign C_{i+3} to $W_{1,i}$ ($1\leq i \leq n$) and C_{i+4} to U_i($1\leq I \leq n-1$) and C_4 to U_n. The induced subgraph of any two color classes in the new coloring is a star.

Therefore,X_S ($T[B_{m,n}]$) = n+3 for n>2.

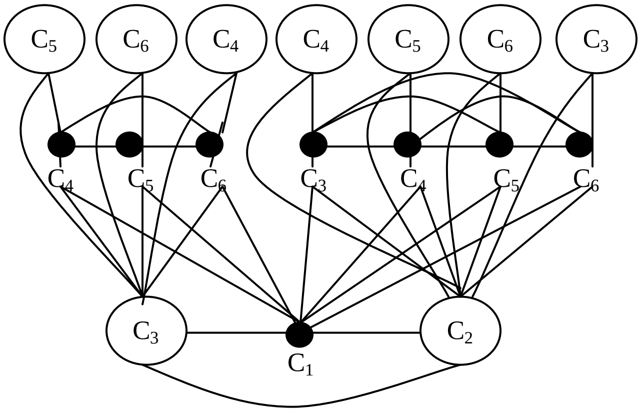

Fig. 5: a(T[$B_{3,4}$]) = 6 = X_S (T[$B_{3,4}$])

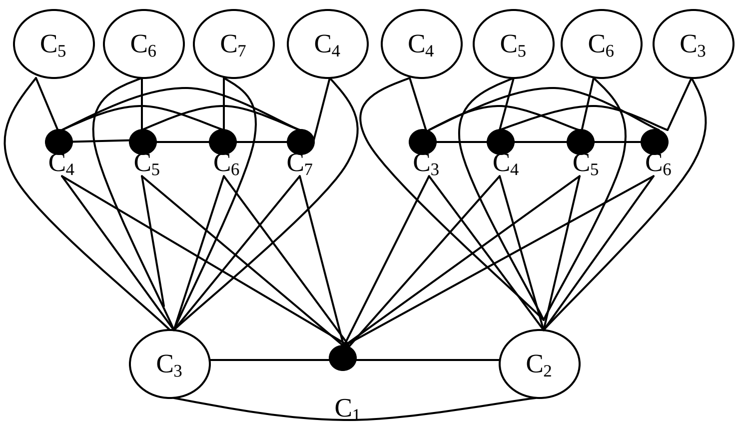

Fig. 6: X_S (T[$B_{4,4}$]) = 7

V. CONCLUSION

In this work, we have shown the relationship between the star and acyclic chromatic number of some Bistar graph families. We are working to find the relationship between these two chromatic numbers for certain families of graphs.

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