ISSN 2250-3153

APPROXIMATION OF FUNCTION BELONGING TO W (L_p, $\xi(t)$) CLASS BY (E, q) (N, p_n) MEANS OF ITS FOURIER SERIES

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Abstract- The present paper deal with approximation of a function belonging to the W(L_p, ξ (t)) class by product summability method. Here product of Euler (E,q) summability method and Nörlund (N, p_n) method has been taken. A new estimate on degree of approximation of a function belonging to W(L_p, ξ (t)) class has been determined by (E, q) (N, p_n) summability of a Fourier series.

Mathematics Subject Classification: 42B05, 42B08

Index Terms- Degree of approximation, W(L_p, ξ (t)) class, (E, q) (N, p_n) summability, Fourier series, Conjugate series of a Fourier series

I. DEFINITION AND NOTATIONS

Let f be a 2π-periodic function in L[-π, π]. The Fourier series associated with f at a point x is defined by

$$f(x) \approx \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n \cos nx + b_n \sin nx\right) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} A_n(x) \quad (1.1)$$

With partial sum s_n (f; x)
where

$$\left.\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx \; dx \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx \; dx \end{aligned}\right\} \quad \text{for } n = 1.2.3\ldots\ldots\ldots$$

and

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx$$

The L_p norm is defined by

$$\|f\|_p = \left\{\int_0^{2\pi} \left|f(x)\right|^p dx\right\}^{1/p}, \qquad p \ge 1$$

and the degree of approximation E_n (f) is given by (Zygmund [6]).

$$E_n(f) = \min_{T_n}\left\|f(x) - T_n(x)\right\|_p,$$

in term of n, where T_n (x) is a trigonometric polynomial of degree n.

A function $f \in Lip\alpha$ if

$$\left|f(x+t)-f(x)\right| = O\left(|t|^{\alpha}\right) \quad \text{for } 0<\alpha\le 1,$$

and $f(x)\in Lip(\alpha, p)$ for $0\le x\le 2\pi$ if.

$$\left(\int_0^{2\pi}\left|f(x+t)-f(x)\right|^p dx\right)^{1/p} = O\left(|t|^{\alpha}\right), \quad 0<\alpha\le 1,\ p\ge 1$$ (Mc Fadden [3])

Given a positive increasing function $\xi(t)$, $p\ge 1$, $f\in Lip(\xi(t), p)$ if

$$\left(\int_0^{2\pi}\left|f(x+t)-f(x)\right|^p dx\right)^{1/p} = O\left(\xi(t)\right)$$

and $f\in W\left(L_p, \xi(t)\right)$ if

$$\left(\int_0^{2\pi}\left|\left[f(x+t)-f(x)\right]\sin^{\beta} x\right|^p dx\right)^{1/p} = O\left(\xi(t)\right), \quad \beta\ge 0$$

We observe that

$$W\left(L_p, \xi(t)\right)\xrightarrow{\beta=0} Lip\left(\xi(t), p\right)\xrightarrow{\xi(t)=t^{\alpha}} Lip(\alpha, p)\xrightarrow{p\to\infty} Lip\alpha$$

We write

$$\phi(t) = \frac{1}{2}\left\{f(x+t)-f(x-t)-2f(x)\right\}$$

Let $\sum_{n=0}^{\infty} u_n$ be an infinite series with sequence of n[th] partial sum $s_n = \sum_{k=0}^{n} u_k$ and a sequence $\{p_n\}$ of real constant such that $P_n = \sum_{n=0}^{\infty} p_k \ne 0$, $P_{-1} = 0 = p_{-1}$

The Nörlund means of the sequence $\{s_n\}$ is given by (Hardy [2])

$$t_n^N = \frac{1}{P_n}\sum_{k=0}^{n} p_{n-k} s_k$$

If $t_n^N \to s$ as $n\to\infty$, then the sequence $\{s_n\}$ is summable to s by Nörlund method.
The Euler means (E, q) is given by (Hardy [2])

$$E_n^q = \frac{1}{(1+q)^n}\sum_{k=0}^{n}\binom{n}{k} q^{n-k} s_k ,$$

If $E_n^q \to s$ as $n\to\infty$, then the sequence $\{s_n\}$ is summable to s by Euler method. The (E, q) transform of the (N, p_n) transform define the (E, q) (N, p_n) transform of the partial sum $\{s_n\}$ of series $\sum_{n=0}^{\infty} u_n$. The (E, q) (N, p_n) means defines a sequence $\left\{t_n^{E^q N}\right\}$ by

$$t_n^{E^q N} = \frac{1}{(1+q)^n}\sum_{k=0}^{n}\binom{n}{k} q^{n-k} \quad \frac{1}{P_k}\sum_{r=0}^{k} p_{k-r} \ s_r$$

If $t_n^{E^q N} \to s$ as $n\to\infty$, then the sequence $\{s_n\}$ is said to summable by (E, q) (N, p_n) method to s.

$\tau = \left[\frac{1}{t}\right]$, where τ denotes the greatest integer not greater than $\left[\frac{1}{t}\right]$.

Particular Cases : Two particular cases of (E, q) (N, p_n) means are :

1) (E, q) (C, 1) if $p_n = 1$, $\forall n$
2) (E, q) (C, δ) if $p_n = \binom{n+\delta-1}{\delta-1}$, $\delta > 0$

The following theorems are due to Binod Prasad Dhakal [1].

Theorem A : If $f : R \to R$ is 2π – periodic function, Lebesgue integrable on [-π , π] and Lip(α, p) class function for $\frac{1}{p} < \alpha \le 1$, $p \ge 1$, then the degree of approximation of *f* by the (E,1) (N, p_n) mean of its Fourier series is given by :

$$\left\| t_n^{EN}(x) - f(x) \right\|_p = O\left(\frac{1}{(n+1)^{\alpha - 1/p}} \right),$$

Where, $$t_n^{EN} = \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} \frac{1}{P_k} \sum_{r=0}^{k} p_{k-r}\, s_r$$ is (E, 1) (N, p_n) means of Fourier series (1.1).

II. MAIN THEOREM

The degree of approximation of function belonging to the Lipschitz class by (E, q) (C, 1) and by (E, 1) (N, p_n) mean has discussed by a number of researchers like S.K. Tiwari and Chandrashekhar Bariwal [5] and Binod Prasad Dhakal [1]. But till now no work seem to have been done to obtain the degree of approximation of the function belonging to $W(L_p, \xi(t))$ class by (E, q) (N, p_n) product mean of its Fourier series.

In an attempt to make study in this direction, one theorem on the degree of approximation of function of $W(L_p, \xi(t))$ class by product summability mean of the form (E, q) (N, p_n) has been determined as following.

Theorem : If $f : R \to R$ is 2π – periodic, Lebesgue integrable [-π, π] and belonging to the class $W(L_p, \xi(t))$, $p \ge 1$ by using $t_n^{E^qN}(x)$ on its Fourier series (1.1) is given by.

(2.1) $$\left\| t_n^{E^qN} - f \right\|_p = O\left((n+1)^{\beta + 1/p} \xi\left(\frac{1}{n+1} \right) \right)$$

Provided $\xi(t)$ satisfies the following conditions :

(2.2) $$\left\{ \int_0^{\pi/(n+1)} \left(\frac{t|\phi(t)|}{\xi(t)} \right)^P \sin^{\beta p} t \;\; dt \right\}^{1/p} = O\left(\frac{1}{n+1} \right)$$

and

(2.3) $$\left\{ \int_{\pi/(n+1)}^{\pi} \left(\frac{t^{-\delta} |\phi(t)| \sin^{\beta} t}{\xi(t)} \right)^P dt \right\}^{1/p} = O\left\{ (n+1)^{\delta} \right\}$$

Where δ is an arbitrary number such that $q(1-\delta)-1>0$, conditions (2.2) and (2.3) hold uniformly in *x* and where $\frac{1}{p}+\frac{1}{q}=1$ such that $1\le p\le\infty$.

and $t_n^{E^q N}=\frac{1}{(1+q)^n}\sum_{k=0}^{n}\binom{n}{k}q^{n-k}.\frac{1}{P_k}\sum_{r=0}^{k}p_{k-r}$ s_r is (E,q) (N,p_n) means of Fourier series (1.1).

Lemma 1 : If

$$\left(E^qN\right)_n(t)=\frac{1}{2\pi(1+q)^n}\sum_{k=0}^{n}\binom{n}{k}q^{n-k}.\frac{1}{P_k}\sum_{r=0}^{k}p_{k-r}\,\frac{\sin\left(r+\frac{1}{2}\right)t}{\sin \frac{t}{2}}$$

Then $\left(E^qN\right)_n(t)=O(n+1)$ for $0<t\le\frac{\pi}{n+1}$

Proof :

$$\left|\left(E^qN\right)_n(t)\right|=\left|\frac{1}{2\pi(1+q)^n}\sum_{k=0}^{n}\binom{n}{k}\frac{q^{n-k}}{P_k}\sum_{r=0}^{k}p_{k-r}\,\frac{\sin\left(r+\frac{1}{2}\right)t}{\sin \frac{t}{2}}\right|$$

$$\le\frac{1}{2\pi(1+q)^n}\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\,\frac{1}{P_k}\sum_{r=0}^{k}p_{k-r}\,\frac{(2r+1)\sin \frac{t}{2}}{\sin \frac{t}{2}}$$

$$\le\frac{1}{2\pi(1+q)^n}\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\,\frac{(2k+1)}{P_k}\sum_{r=0}^{k}p_{k-r}\qquad \because \sum_{k=0}^{n}p_{n-k}=P_n$$

$$=\frac{1}{2\pi(1+q)^n}\sum_{k=0}^{n}\binom{n}{k}q^{n-k}(2k+1)$$

$$=O(n+1)\qquad \because \sum_{k=0}^{n}\binom{n}{k}q^{n-k}=(1+q)^n$$

Lemma 2 : If $\left(E^qN\right)_n(t)$ is given as in lemma 1, then

$\left(E^qN\right)_n(t)=O\left(\frac{1}{t}\right)$ for $\frac{\pi}{n+1}\le t\le\pi$ and $|\sin kt|\le 1$, $\sin \frac{t}{2}\ge \frac{t}{\pi}$

Proof :

$$\left|\left(E^qN\right)_n(t)\right|=\left|\frac{1}{2\pi(1+q)^n}\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\,\frac{1}{P_k}\sum_{r=0}^{k}p_{k-r}\,\frac{\sin\left(r+\frac{1}{2}\right)t}{\sin \frac{t}{2}}\right|$$

$$\le\frac{1}{2\pi(1+q)^n}\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\,\frac{1}{P_k}\sum_{r=0}^{k}p_{k-r}\,\frac{\left|\sin\left(r+\frac{1}{2}\right)t\right|}{\left|\sin \frac{t}{2}\right|}$$

$$\le\frac{1}{2(1+q)^n\,t}\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\,\frac{1}{P_k}\sum_{r=0}^{k}p_{k-r}$$

$$= \mathrm{O}\left(\frac{1}{t}\right) \qquad \left[\mathrm{Q}\sum_{k=0}^{n} p_{n-k} = P_n \,,\ \sum_{k=0}^{n}\binom{n}{k} q^{n-k} = (1+q)^n\right]$$

III. PROOF OF THEOREM

Following Titchmarsh [4], the n[th] partial sum $s_n(x)$ of the Fourier series is given by

$$s_n(x) - f(x) = \frac{1}{2\pi}\int_0^{\pi} \frac{\phi(t)\sin\left(n+\frac{1}{2}\right)t}{\sin \frac{t}{2}}\,dt \tag{3.1}$$

The (N, p_n) transform of the sequence $\{s_n(x)\}$ is given by

$$t_n^N(x) - f(x) = \frac{1}{2\pi P_n}\int_0^{\pi}\phi(t)\sum_{k=0}^{n} p_{k-r}\,\frac{\sin\left(n+\frac{1}{2}\right)t}{\sin\frac{t}{2}}\,dt \tag{3.2}$$

The (E, q) transform of $\{t_n^N(x)\}$ is given by

$$t_n^{E^qN} - f(x) = \frac{1}{2\pi(1+q)^n}\int_0^{\pi}\frac{\phi(t)}{\sin\frac{t}{2}}\sum_{k=0}^{n}\binom{n}{k}q^{n-k}.\frac{1}{P_k}\sum_{r=0}^{k}p_{k-r}\sin\left(r+\frac{1}{2}\right)t\,dt \qquad = \int_0^{\pi}\phi(t)\left(E^qN\right)_n(t)\,dt$$

$$= \int_0^{\pi/n+1}\phi(t)\left(E^qN\right)_n(t)\,dt + \int_{\pi/n+1}^{\pi}\phi(t)\left(E^qN\right)_n(t)\,dt$$

$$= I_1 + I_2 \qquad \text{(say)} \tag{3.3}$$

Applying Hölder's inequality,

$$\left|I_1\right| \le \left[\int_0^{\pi/n+1}\left\{\frac{t\left|\phi(t)\right|\sin^{\beta} t}{\xi(t)}\right\}^p dt\right]^{1/p}\left[\int_0^{\pi/n+1}\left\{\frac{\xi(t)\left|\left(E^qN\right)_n(t)\right|}{t\sin^{\beta} t}\right\}^q dt\right]^{1/q}$$

$$= \mathrm{O}\left(\frac{\pi}{n+1}\right)\left[\int_0^{\pi/n+1}\left\{\frac{(n+1)\xi(t)}{t^{1+\beta}}\right\}^q dt\right]^{1/q}$$

$$= \mathrm{O}\left(\frac{1}{n+1}\right).\mathrm{O}(n+1)\left[\int_0^{\pi/n+1}\left(\frac{\xi(t)}{t^{1+\beta}}\right)^q dt\right]^{1/q}$$

$$= \mathrm{O}\left(\xi\left(\frac{1}{n+1}\right)\right)\left[\int_0^{\pi/n+1} t^{-(\beta+1)q}\,dt\right]^{1/q}$$

$$= \mathrm{O}\left(\xi\left(\frac{1}{n+1}\right)\right)\left[t^{-(\beta+1)+\frac{1}{q}}\right]_0^{\pi/n+1}$$

$$= \mathrm{O}\left(\xi\left(\frac{1}{n+1}\right).(n+1)^{\beta+1-\frac{1}{q}}\right)$$

$$= O\left((n+1)^{\beta+1/p} . \xi\left(\frac{1}{n+1} \right) \right)$$

(3.4)

Now,

$$|I_2| \le \left[\int_{\pi/n+1}^{\pi} \left\{ \frac{t^{-\delta} |\phi(t)| \sin^{\beta} t}{\xi(t)} \right\}^p dt \right]^{1/p} . \left[\int_{\pi/n+1}^{\pi} \left\{ \frac{(E^q N)_n (t) . \xi(t)}{t^{-\delta} \sin^{\beta} t} \right\}^q dt \right]^{1/q}$$

$$= O(n+1)^{\delta} \left[\int_{\pi/n+1}^{\pi} \left\{ \frac{\xi(t)}{t^{-\delta+\beta+1}} \right\}^q dt \right]^{1/q}$$

$$= O(n+1)^{\delta} . \left[\int_{1/\pi}^{n+1/\pi} \left\{ \frac{\xi\left(1/y \right)}{y^{-(\beta+1)+\delta}} \right\}^q \frac{dy}{y^2} \right]^{1/q}$$

$$= O\left\{ (n+1)^{\delta} \xi\left(\frac{1}{n+1} \right) \right\} \left[\left(y^{(\beta-\delta+1)q-1} \right)^{1/q} \right]_{1/\pi}^{n+1/\pi}$$

$$= O\left\{ (n+1)^{\delta} \xi\left(\frac{1}{n+1} \right) \right\} (n+1)^{\beta-\delta+1-1/q}$$

$$= O\left\{ \xi\left(\frac{1}{n+1} \right) . (n+1)^{\beta+1-1/q} \right\}$$

$$= O\left\{ \xi\left(\frac{1}{n+1} \right) . (n+1)^{\beta+1/p} \right\}$$

$$= O\left\{ (n+1)^{\beta+1/p} \xi\left(\frac{1}{n+1} \right) \right\} \qquad \text{since } \frac{1}{p} + \frac{1}{q} = 1$$

(3.5)

Now combining (3.3), (3.4) & (3.5), we get

$$\left| t_n^{E^q N}(x) - f(x) \right| = O\left\{ (n+1)^{\beta+1/p} \xi\left(\frac{1}{n+1} \right) \right\}$$

$$\left\| t_n^{E^q N}(x) - f(x) \right\|_p = \left\{ \int_0^{2\pi} O\left\{ (n+1)^{\beta+1/p} \xi\left(\frac{1}{n+1} \right) \right\}^p dx \right\}^{1/p}$$

$$= O\left\{ (n+1)^{\beta+1/p} \xi\left(\frac{1}{n+1} \right) \right\} \left[\left\{ \int_0^{2\pi} dx \right\}^{1/p} \right]$$

$$= O\left\{ (n+1)^{\beta+1/p} \xi\left(\frac{1}{n+1} \right) \right\}$$

This completes the proof of the main theorem.

IV. COROLLARIES

Following corollaries can be derived from our main theorem.

Corollary – 1 : β = 0 and $\xi(t) = t^{\alpha}$ then the degree of approximation of a function $f \in W(L_p, \xi(t))$, $0 < \alpha \le 1$ is given by

$$\left\| t_n^{E^q N}(x) - f(x) \right\| = O\left\{ \frac{1}{(n+1)^{\alpha - 1/p}} \right\}$$

Which is reduces to the Theorem A due to Binod Prasad Dhakal

Corollary – 2 : If $p \to \infty$ in corollary 1, then for $0 < \alpha < 1$.

$$\left\| t_n^{E^q N}(x) - f(x) \right\|_\infty = \underset{-\pi \le x \le \pi}{sub} \left| t_n^{E^q N}(x) - f(x) \right| = O\left(\frac{1}{(n+1)^{\alpha}} \right), \quad \text{for } 0 < \alpha < 1$$

Where

$$t_n^{E^q N} = \frac{1}{(1+q)^n} \sum_{k=0}^{n} \binom{n}{k} q^{n-k} \frac{1}{P_k} \sum_{r=O}^{k} p_{k-r} \, s_r$$

in the (E, q) (N, p_n) means of Fourier series (1.1).

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