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EXISTENCE THEORY OF RANDOM DIFFERENTIAL EQUATIONS

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Abstract
In this paper, the existence result of nonlinear first order ordinary random differential equations is proved using random fixed point theorem.

Index Terms: Random differential equation, multi-valued function, initial value problem
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I. STATEMENT OF THE PROBLEM

Let R denote the real line and R_+, the set of nonnegative real numbers. Let C(R_+, R) denote the class of real-valued functions defined and continuous on R_+. Given a measurable space (Ω, A) and a measurable function $x:\Omega \to C(R_+, R)$, consider the initial value problem of nonlinear first order ordinary random differential equations (RDE)

$$\begin{aligned} x'(t,\omega) - k(t,\omega)x(t,\omega) &= f(t, x(t,\omega),\omega) \quad a.\ e.\ t \in R_+ \\ x(0,\omega) &= q(\omega) \end{aligned} \tag{1.1}$$

forall $\omega \in \Omega$, where $k: R_+ \times \Omega \to R_+$, $q:\Omega \to R$ *and* $f: R_+ \times R \times \Omega \to R$.By a random solution of the RDE (1.1) I mean a measurable function $x:\Omega \to AC(R_+, R)$ that satisfies the equations in (1.1), where $AC(R_+, R)$ is the space of absolutely continuous real-valued functions defined on R_+.

The initial value problems of ordinary differential equations have been studied in the literature on bounded as well as unbounded internals of the real line for different aspects of the solution. See for example, Burton and Furumochi [2], Dhage [3],. Similarly, the initial value problem of random differential equations have also been discussed in the literature for existence theorems on bounded intervals, however, the study of such random equations has not been made on unbounded intervals of the real line for any aspects of the random solutions. Some results along this lines appear in Itoh [5], Bharucha-Reid [1] and Dhage [4].

Therefore, nonlinear random differential equations on unbounded intervals need to pay attention to the existence of the random solutions. The present paper proposes to discuss the existence results for random differential equations (1.1) on the right half R_+ of the real line R. The classical fixed point theory, in particular, random version of Schauder's fixed point theorem will be employed to prove the main result of this paper. I claim that the results of this paper is new to the literature on random differential equations.

II. AUXILIARY RESULTS

The following theorem is often used in the study of nonlinear discontinuous random differential equations. I also need this result in the subsequent part of this paper.

Theorem 2.1 (Itoh[5]). *Let X be a non-empty, closed convex bounded subset of the separable Banach space E and let* $Q:\Omega \times X \to X$ *be a compact and continuous random operator. Then the random equation Q(* ω *)x = x has a random solution, that is there is a measurable function* $\xi:\Omega \to X$ *such that* $Q(\omega)\xi(\omega) = \xi(\omega)$ *for all* $\omega \in \Omega$.

Theorem2.2(Carathéodory). *Let* $Q:\ \Omega \times E \to E$ *be a mapping such that Q(·, x) is measurable for all x* $\in$ *E and Q(* ω *, ·) is continuous for all* $\omega \in \Omega$ *. Then the map (* ω *, x)* $\to$ *Q(* ω *, x) is jointly measurable.*

III. EXISTENCE RESULTS

I need the following definition in the sequel

Definition 4.1. A function $f : R_+ \times R \times \Omega \to R$ is called **random Carathéodory**if i) the map $\omega \to f(t,x,\omega)$ is measurable for all $t \in R_+$ and $x \in R$ and (ii) the map $(t,x) \to f(t,x,\omega)$ is jointly continuous for all $\omega \in \Omega$ Furthermore, a random Carathéodory function $f : R_+ \times R \times \Omega \to R$ is called **random L^1 - Carathéodory,** if there exists a function $h \in L^1(R_+, R)$ such that

$$\left| f(t,x,\omega) \right| \le h(t) \quad a.e.\ t \in R_+$$

for all $\omega \in \Omega$ and $x \in R$. The function h is the growth function of f on $R_+ \times R \times \Omega$.

I consider the following set of hypotheses follows**:**

(H_0) The function $k : R_+ \times R \to R$ is measurable and bounded. Moreover, for each $\omega \in \Omega$,

$$\lim_{t \to \infty} K(t,\omega) = \infty$$

Where $K(t,\omega) = \int_0^t k(s,\omega)ds$.

(H_1) The function $q : \Omega \to R$ is measurable and bounded. Moreover,

$$\operatorname{ess}\sup_{\omega \in \Omega} \left| q(\omega) \right| = c_1$$

for some real number c_1>0.

(H_2) The function f is random L^1-Carathéodory with growth function h on R_+.Moreover,

$$\lim_{t \to \infty} e^{-K(t,\omega)} \int_0^t e^{k(s,\omega)} h(s)ds = 0$$

forall $\omega \in \Omega$.

Remark 4.2. If the hypothesis (H_2) holds, then the function $w : R_+ \times \Omega \to R_+$ defined by $w(t,\omega) = e^{-K(t,\omega)} \int_0^t e^{k(s,\omega)} h(s)ds$ is continuous and the number

$$w = \sup_{t \ge 0} w(t,\omega) = \sup_{t \ge 0} e^{-K(t,\omega)} \int_0^t e^{k(s,\omega)} h(s)ds$$

exists for all $\omega \in \Omega$. Hypothesis (H_2) has been considered in a number of papers in the literature. See for example, Dhage [3], Burton and Furumochi [2] and the references therein.

IV. MAIN RESULT

Theorem 4.3. *Assume that the hypotheses* (H_0) *through* (H_2) *hold. Then the RDE(1.1) admits a random solution..*

Proof. Now RDE (1.1) is equivalent to the random equation

$$x(t,\omega) = q(\omega)e^{K(t,\omega)} + e^{K(t,\omega)} \int_0^t e^{-K(s,\omega)} f(s, x(s,\omega), \omega)ds \tag{4.1}$$

for all $t \in R_+$ and $\omega \in \Omega$.

Set E = BC (R_+, R). For a given function $x:\Omega \to E$, define a mapping Q on $\Omega \times E$ by

$$Q(\omega)x(t,\omega) = q(\omega)e^{K(t,\omega)} + e^{K(t,\omega)}\int_0^t e^{-K(s,\omega)} f(s, x(s,\omega),\omega)ds \tag{4.2}$$

for all $t \in R_+$ and $\omega \in \Omega$. For the sake of convenience, I write $Q(\omega)x(t,\omega) = Q(\omega)x(t)$ omitting the double appearance of ω merge it into $Q(\omega)$.

Clearly, Q defines a mapping $Q:\Omega \times E \to E$. To see this, let $x \in E$ be arbitrary. Then for each $\omega \in \Omega$, the continuity of map $t \to Q(\omega)x(t)$ follows from the fact that exponential $e^{K(t,\omega)}$ and the indefinite integral

$$\int_0^t f(s, x(s,\omega),\omega)ds$$

are continuous functions of t on R_+. Next, I show that the function $Q(\omega)x : R_+ \to R$ is bounded for each $\omega \in \Omega$. Now by hypotheses (H_1) and (H_2) ,

$$\begin{aligned} |Q(\omega)x(t)| &\le |q(\omega)|e^{K(t,\omega)} + e^{K(t,\omega)}\int_0^t e^{-K(s,\omega)} |f(s, x(s,\omega),\omega)| ds \\ &\le c_1 e^{K(t,\omega)} + e^{K(t,\omega)}\int_0^t e^{-K(s,\omega)} h(s)ds \\ &\le c_1 + W \end{aligned} \tag{4.3}$$

for all $\omega \in \Omega$. As a result, $Q:\Omega \times E \to E$.

Define a closed ball $\bar{B}_r(0)$ in the Banach space E centered at origin of radius r = $c_1 + W$ from (4.3),

$$|Q(\omega)x| \le c_1 + W$$

for all $\omega \in \Omega$ and $x \in E$. Hence $Q:\Omega \times E \to \bar{B}_r(0)$, and in particular, Q defines a map $Q:\Omega \times \bar{B}_r(0) \to \bar{B}_r(0)$. Now I show that Q satisfies all the conditions of Theorem 2.1 with $X = \bar{B}_r(0)$.

Firstly, I show that Q is a random operator on $\Omega \times \bar{B}_r(0)$ *in to* $\bar{B}_r(0)$. By hypothesis (H_2), the map $\omega \to f(t,x,\omega)$ is measurable by the Carathéodory theorem. Since a continuous function is measurable, the map $t \to e^{K(t,\omega)}$ is measurable and so the product $t \to e^{K(t,\omega)} f(t, x(t,\omega),\omega)$ is measurable in ω for all $t \in R_+$ and $x \in R$. Since the integral is a limit of the finite sum of measurable functions, I have that the function $\omega \to \int_0^t e^{-K(s,\omega)} f(s, x(s,\omega),\omega)ds$ is measurable. Similarly, the map

$$\omega \to q(\omega)e^{K(t,\omega)} + e^{K(t,\omega)}\int_0^t e^{-K(s,\omega)} f(s, x(s,\omega),\omega)ds$$

is measurable for all $t \in R_+$. Consequently, the map $\omega \to Q(\omega)x$ is measurable for all $x \in E$ and that Q is a random operator on $\Omega \times \bar{B}_r(0)$.

Secondly, I show that the random operator $Q(\omega)$ is continuous on $\bar{B}_r(0)$. By hypothesis (H_2) $\lim_{t\to\infty} w(t,\omega) = \lim_{t\to\infty} e^{K(t,\omega)}\int_0^t e^{-K(s,\omega)} h(s)ds = 0$, there is a real number T > 0 such that $w(t) < \frac{\varepsilon}{4}$ for all t $\ge$T.

I show that the continuity of the random operator $Q(\omega)$ in the following two cases:

Case I. Let $t \in [0,T]$ and let $\{x_n\}$ be a sequence of points in $\bar{B}_r(0)$ such that $x_n \to x$ as $n \to \infty$. Then, by the dominated convergence theorem,

$$\lim_{n\to\infty} Q(\omega)x_n(t) = \lim_{n\to\infty}\left(q(\omega)e^{K(t,\omega)} + e^{K(t,\omega)}\int_0^t e^{-K(s,\omega)} f(s, x_n(s,\omega),\omega)ds \right)$$
$$= q(\omega)e^{K(t,\omega)} + \lim_{n\to\infty}\left(e^{K(t,\omega)}\int_0^t e^{-K(s,\omega)} f(s, x_n(s,\omega),\omega)ds \right)$$
$$= q(\omega)e^{K(t,\omega)} + e^{K(t,\omega)}\int_0^t \lim_{n\to\infty}\left(e^{-K(s,\omega)} f(s, x_n(s,\omega),\omega)\right)ds$$
$$= q(\omega)e^{K(t,\omega)} + e^{K(t,\omega)}\int_0^t \left(e^{-K(s,\omega)} f(s, x(s,\omega),\omega)\right)ds$$
$$= Q(\omega)x(t)$$

for all $t \in [0,T]$ and $\omega \in \Omega$.

Case II. Suppose that t $\geq$T . Then ,

$$|Q(\omega)x_n(t) - Q(\omega)x(t)| = \left| e^{K(t,\omega)}\int_0^t e^{-K(s,\omega)} f(s, x_n(s,\omega),\omega)ds - e^{K(t,\omega)}\int_0^t e^{-K(s,\omega)} f(s, x(s,\omega),\omega)ds \right|$$
$$\leq \left| e^{K(t,\omega)}\int_0^t e^{-K(s,\omega)} f(s, x_n(s,\omega),\omega)ds \right| + \left| e^{K(t,\omega)}\int_0^t e^{-K(s,\omega)} f(s, x(s,\omega),\omega)ds \right|$$
$$\leq \left| e^{K(t,\omega)}\int_0^t e^{-K(s,\omega)} f(s, x_n(s,\omega),\omega)ds \right| + \left| e^{K(t,\omega)}\int_0^t e^{-K(s,\omega)} f(s, x(s,\omega),\omega)ds \right|$$
$$\leq 2\omega(t) < \varepsilon$$

for all $t \geq T$ and $\omega \in \Omega$. Since ε is arbitrary, one has $\lim_{n\to\infty} Q(\omega)x_n(t) = Q(\omega)x(t)$ all t $\geq$T and $\omega \in \Omega$. Now combining the Case I with Case II, I conclude that $Q(\omega)$ is a pointwise continuous random operator on $\bar{B}_r(0)$ into itself. Further, it is shown below that the family of functions $\{Q(\omega)x_n\}$ is an equicontinuous set in E for a fixed $\omega \in \Omega$. Hence, the above convergence is uniform on R_+ and consequently, $Q(\omega)$ is a continuous random operator on $\bar{B}_r(0)$ into itself.

Next, I show that $Q(\omega)$ is a compact random operator on $\bar{B}_r(0)$. Let $\omega \in \Omega$ be fixed and consider a sequence $\{Q(\omega)x_n\}$ of points in $\bar{B}_r(0)$. To finish, it is enough to show that the sequence $\{Q(\omega)x_n\}$ has a Cauchy subsequence for each $\omega \in \Omega$. Clearly $\{Q(\omega)x_n\}$ is a uniformly bounded subset of $\bar{B}_r(0)$.

I show that it is an equi-continuous sequence of functions on R_+.

Let ε > 0 be given. Since $\lim_{t\to\infty} w(t,\omega) = 0$, there exists a real number T > 0 such for t < $\frac{\varepsilon}{4}$ for t $\geq$ T. I consider the following three cases:

Case I. Let $t_1, t_2 \in [0,T]$ .Then ,

$$|Q(\omega)x_n(t_1) - Q(\omega)x(t_2)| \leq$$
$$\left| q(\omega)e^{K(t_1,\omega)} + e^{K(t_1,\omega)}\int_0^{t_1} e^{-K(s,\omega)} f(s, x_n(s,\omega),\omega)ds - q(\omega)e^{K(t_2,\omega)} - e^{K(t_2,\omega)}\int_0^{t_2} e^{-K(s,\omega)} f(s, x(s,\omega),\omega)ds \right|$$
$$\leq |q(\omega)|\left|e^{K(t_1,\omega)} - e^{K(t_2,\omega)}\right| +$$
$$\left| e^{K(t_1,\omega)}\int_0^{t_1} e^{-K(s,\omega)} f(s, x_n(s,\omega),\omega)ds - e^{K(t_2,\omega)}\int_0^{t_2} e^{-K(s,\omega)} f(s, x(s,\omega),\omega)ds \right|$$

$$\leq |q(\omega)| \left| e^{K(t_1,\omega)} - e^{K(t_2,\omega)} \right| +$$
$$\left| e^{K(t_1,\omega)} \int_0^{t_1} e^{-K(s,\omega)} f(s, x_n(s,\omega), \omega)\, ds - e^{K(t_2,\omega)} \int_0^{t_1} e^{-K(s,\omega)} f(s, x(s,\omega), \omega)\, ds \right| +$$
$$\left| e^{K(t_2,\omega)} \int_0^{t_1} e^{-K(s,\omega)} f(s, x_n(s,\omega), \omega)\, ds - e^{K(t_2,\omega)} \int_0^{t_2} e^{-K(s,\omega)} f(s, x(s,\omega), \omega)\, ds \right|$$
$$\leq |q(\omega)| \left| e^{K(t_1,\omega)} - e^{K(t_2,\omega)} \right| + \left| e^{K(t_1,\omega)} - e^{K(t_2,\omega)} \right| \left| \int_0^{t_1} e^{-K(s,\omega)} f(s, x_n(s,\omega), \omega)\, ds \right| +$$
$$e^{K(t_2,\omega)} \left| \int_0^{t_1} e^{-K(s,\omega)} f(s, x_n(s,\omega), \omega)\, ds - \int_0^{t_2} e^{-K(s,\omega)} f(s, x(s,\omega), \omega)\, ds \right|$$
$$\leq \left[c_1 + e^{k_1} \|h\|_{L^1} \right] \left| e^{K(t_1,\omega)} - e^{K(t_2,\omega)} \right| + |p(t_1) - p(t_2)|$$

for all n $\in$ N, where k_1 is the bound for $k(t,\omega)$ and $p(t) = \int_0^t e^{-K(s,\omega)} h(s)\, ds$. Since the functions $e^{K(t,\omega)}$ and $p(t)$ are continuous on $[0,T]$, they are uniformly continuous there. Hence, $|Q(\omega) x_n(t_1) - Q(\omega) x(t_2)| \to 0$ as $t_1 \to t_2$ uniformly for all $t_1, t_2 \in [0,T]$ and for all n $\in$ N.

Case II. If $t_1, t_2 \in [T, \infty]$, then

$$|Q(\omega) x_n(t_1) - Q(\omega) x(t_2)| \leq$$
$$\left| e^{K(t_1,\omega)} \int_0^{t_1} e^{-K(s,\omega)} f(s, x_n(s,\omega), \omega)\, ds - e^{K(t_2,\omega)} \int_0^{t_2} e^{-K(s,\omega)} f(s, x(s,\omega), \omega)\, ds \right|$$

$$\leq e^{K(t_1,\omega)} \int_0^{t_1} e^{-K(s,\omega)} |f(s, x_n(s,\omega), \omega)|\, ds + e^{K(t_2,\omega)} \int_0^{t_2} e^{-K(s,\omega)} |f(s, x(s,\omega), \omega)|\, ds \leq w(t_1) + w(t_2)$$
$$\leq \frac{\varepsilon}{4} + \frac{\varepsilon}{4} < \varepsilon .$$

Since ε is arbitrary, one has

$$|Q(\omega) x_n(t_1) - Q(\omega) x(t_2)| \to 0 \quad as \quad t_1 \to t_2$$

uniformly for all n $\in$ N.

Case III. If $t_1 < T < t_2$, then

$$|Q(\omega) x_n(t_1) - Q(\omega) x_n(t_2)| \leq |Q(\omega) x_n(t_1) - Q(\omega) x_n(T)| + |Q(\omega) x_n(T) - Q(\omega) x_n(t_2)|$$

$As \quad t_1 \to t_2, t_1 \to T \;and\; t_2 \to T$, and so

$$|Q(\omega)\, x_n(t_1) - Q(\omega) x_n(T)| \to 0$$

and

$$|Q(\omega) x_n(T) - Q(\omega) x_n(T)| \to 0$$

as $t_1 \to t_2$ uniformly for all n $\in$ N.

Hence

$$|Q(\omega) x_n(t_1) - Q(\omega) x(t_2)| \to 0 \quad as\ t_1 \to t_2$$

uniformly for all $t_1 < T \;and\; t_2 > T$ and for all n $\in$ N. Thus, in all three cases

$$|Q(\omega) x_n(t_1) - Q(\omega) x_n(t_2)| \to 0 \quad as\ t_1 \to t_2$$

uniformly for all $t_1, t_2 \in R_+$ and for all n $\in$ N.

This shows that $\{Q(\omega)x_n\}$ is a equicontinuous sequence in X. Now an application of Arzelá-Ascoli theorem yields that $\{Q(\omega)x_n\}$ has a uniformly convergent subsequence on the compact subset [0, T] of R. Without loss of generality, call the subsequence to be the sequence itself. I show that $\{Q(\omega)x_n\}$ is Cauchy in X. Now shows that $|Q(\omega)x_n(t) - Q(\omega)x(t)| \to 0 \ as\ n \to \infty$ for all t$\in$ [0, T]. Then for given ε>0 there exist $n_0 \in$ N such that

$$\sup_{0\le p\le T} e^{k_1 p} \int_0^p e^{-K(s,\omega)} \left| f(s, x_m(s,\omega),\omega) - f(s, x_n(s,\omega),\omega) \right| ds < \frac{\varepsilon}{2}$$

for all m , n $\ge n_0$. Therefore, if m , n $\ge n_0$, then we have

$$\begin{aligned} \left| Q(\omega)x_m - Q(\omega)x_n \right| &= \\ &\sup_{0\le t<\infty} \left| e^{K(t,\omega)} \int_0^t e^{-K(s,\omega)} \left| f(s, x_m(s,\omega),\omega) - f(s, x_n(s,\omega),\omega) \right| ds \right| \\ &\le \sup_{0\le p\le T} \left| e^{K(p,\omega)} \int_0^p e^{-K(s,\omega)} \left| f(s, x_m(s,\omega),\omega) - f(s, x_n(s,\omega),\omega) \right| ds \right| + \\ &\quad \sup_{p\ge T} e^{K(p,\omega)} \int_0^p e^{-K(s,\omega)} \left| f(s, x_m(s,\omega),\omega) + f(s, x_n(s,\omega),\omega) \right| ds \\ &< \varepsilon \end{aligned}$$

This shows that $\{Q(\omega)x_n\} \subset Q(\omega)(\bar{B}_r(0)) \subset (\bar{B}_r(0))$ is Cauchy. Since X is complete, $\{Q(\omega)x_n\}$ converges to a point in X. As $Q(\omega)(\bar{B}_r(0))$ is closed, the sequence $\{Q(\omega)x_n\}$ converges to a point in $Q(\omega)(\bar{B}_r(0))$. Hence $Q(\omega)(\bar{B}_r(0))$ is relatively compact for each $\omega \in \Omega$ and consequently Q is a continuous and compact random operator on $\Omega \times (\bar{B}_r(0))$. Now an application of Theorem 2.1 to the operator $Q(\omega)$ on $(\bar{B}_r(0))$ yields that Q has a fixed point in $(\bar{B}_r(0))$ which further implies that the RDE (1.1) random solution on R_+.

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