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ON SOME FRACTIONAL DERIVATIVES

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Abstract: The subject of fractional calculus and its applications (that is, calculus of integrals and derivatives of any arbitrary real or complex order) has gained considerable popularity and importance during the past three decades or so, due mainly to its demonstrated applications in numerous seemingly diverse and widespread fields of science and engineering.

The fractional derivatives (and fractional integration) of special functions of one and more variables is important, such as in the evaluation of series and integrals, the derivation of generating functions and the solution of differential and integral equations.

Looking into the requirement and importance of fractional calculus in various branches, in this paper we establish some new fractional derivatives involving H-function and other functions, which will be useful to analysis the various problems in different fields.

1. Introduction:

H-function of one variable which is introduced by Fox [1, p.408], will be represented as follows:

$$H_{p,q}^{m,n}\left[x\Big|{}_{(b_j,\,\beta_j)_{1,q}}^{(a_j,\,\alpha_j)_{1,p}}\right] = (1/2\pi i)\int_L \theta(s)\, x^s\, ds \qquad (1)$$

where $i = \sqrt{(-1)}$,

$$\theta(s) = \frac{\prod_{j=1}^{m}\Gamma(b_j-\beta_j s)\ \prod_{j=1}^{n}\Gamma(1-a_j+\alpha_j s)}{\prod_{j=m+1}^{q}\Gamma(1-b_j+\beta_j s)\ \prod_{j=n+1}^{p}\Gamma(a_j-\alpha_j s)}$$

x is not equal to zero and an empty product is interpreted as unity; p, q, m, n are integers satisfying $1 \le m \le q$, $0 \le n \le p$, α_j ($j = 1, \ldots, p$), β_j ($j = 1, \ldots, q$) are positive numbers and a_j ($j = 1, \ldots, q$) are complex numbers. L is a suitable contour of Barnes type such that poles of $\Gamma(b_j - \beta_j s)$ ($j = 1, \ldots, m$) lie to the right and poles of $\Gamma(1 - a_j + \alpha_j s)$ ($j = 1, \ldots, n$) to the left of L. These assumptions for the H-function will be adhered to throughout this paper.

According to Braakasma

$$H_{p,q}^{m,n}\left[x\Big|{}_{(b_j,\,\beta_j)_{1,q}}^{(a_j,\,\alpha_j)_{1,p}}\right] = O\left(|x|^{\alpha}\right) \text{ for small } x,$$

where $\sum_{j=1}^{p}\alpha_j - \sum_{j=1}^{q}\beta_j \le 0$ and $\alpha = \min R(b_h/\beta_h)$ ($h = 1, .., k$)

and

$$H_{p,q}^{m,n}\left[x\Big|{}_{(b_j,\,\beta_j)_{1,q}}^{(a_j,\,\alpha_j)_{1,p}}\right] = O\left(|x|^{\beta}\right) \text{ for large } x,$$

where
$$\sum_{j=1}^{n} \alpha_j - \sum_{j=n+1}^{p} \alpha_j + \sum_{j=1}^{m} \beta_j - \sum_{j=m+1}^{q} \beta_j \equiv A > 0, \quad (2)$$

$$\sum_{j=1}^{p} \alpha_j - \sum_{j=1}^{q} \beta_j < 0$$

$|\arg x| < \frac{1}{2} A\pi$ and $\beta = \max R[(a_j - 1)/\alpha_j]$ $(j = 1, .., n)$

In the present investigation we require the following formula:

2. Fractional Derivatives Involving H-function:

In this section, we evaluate following some fractional derivatives involving H-function:

$$D_x^{\mu}\{x^k(x^{\nu_1}+a)^{\lambda}(b-x^{\nu_2})^{-\delta}\, H\left[zx^{-\rho_1}(x^{\nu_1}+a)^{\sigma_1}(b-x^{\nu_2})^{-\delta_1}\right]\}$$

$$= a^{\lambda}b^{-\delta}x^{k-\mu}\sum_{r,s=0}^{\infty}\frac{(x^{\nu_1}/a)^r(x^{\nu_2}/b)^s}{r!\, s!}\times$$

$$H_{p+3,q+3}^{m+1,n+2}\left[zx^{-\rho_1}a^{\sigma_1}b^{-\delta_1}\left|{}^{(-\lambda,\sigma_1),(1-\delta-s,\delta_1),(a_j,\alpha_j)_{1,p},(1+k-\mu+r\nu_1+s\nu_2,\rho_1)}_{(1+k+r\nu_1+s\nu_2,\rho_1),(b_j,\beta_j)_{1,q},(-\lambda+r,\sigma_1),(1-\delta,\delta_1)}\right.\right], \quad (3)$$

provided (in addition to the appropriate convergence and existence conditions) that
$\min\{\nu_1, \nu_2, \rho_1, \sigma_1, \delta_1\} > 0$, $\max\{|\arg(x^{\nu_1}/a)|, |\arg(x^{\nu_2}/b)|\} < \pi$.

$${}_xD_{\infty}^{\mu}\{x^k(x^{\nu_1}+a)^{\lambda}(b-x^{\nu_2})^{-\delta}\, H\left[zx^{\rho_1}(x^{\nu_1}+a)^{-\sigma_1}(b-x^{\nu_2})^{-\delta_1}\right]\}$$

$$= a^{\lambda}b^{-\delta}x^{k-\mu}(-1)^{\mu}\sum_{r,s=0}^{\infty}\frac{(x^{\nu_1}/a)^r(x^{\nu_2}/b)^s}{r!\, s!}\times$$

$$H_{p+3,q+3}^{m+2,n+1}\left[zx^{\rho_1}a^{-\sigma_1}b^{-\delta_1}\left|{}^{(1-\delta-s,\delta_1),(a_j,\alpha_j)_{1,p}(1+\lambda-r,\sigma_1),(-k-r\nu_1-s\nu_2,\rho_1)}_{(1+\lambda,\sigma_1),(b_j,\beta_j)_{1,q},(-k+\mu-r\nu_1-s\nu_2,\rho_1),(1-\delta,\delta_1)}\right.\right], \quad (4)$$

provided (in addition to the appropriate convergence and existence conditions) that
$\min\{\nu_1, \nu_2, \rho_1, \sigma_1, \delta_1\} > 0$, $\max\{|\arg(x^{\nu_1}/a)|, |\arg(x^{\nu_2}/b)|\} < \pi$.

Proof:

To prove of (3), we first replace the H-function occurring on the L.H.S. by its Mellin-Barnes contour integrals, collected the powers of x, $(x^{\nu_1}+a)$ and $(b-x^{\nu_2})$ and apply binomial expansion:

$$(x+\xi)^{\lambda} = \xi^{\lambda} \sum_{r=0}^{\infty} \binom{\lambda}{r} \left(\frac{x}{\xi}\right)^{r}; \left|\frac{x}{\xi}\right| < 1, \quad (5)$$

we then apply the formula [2, p.67 eq.(4.4.4)]:

$$D_x^{\mu}\left(x^{\lambda}\right) = \frac{\Gamma(1+\lambda)}{\Gamma(1+\lambda-\mu)} . x^{\lambda-\mu}; \ (\mathrm{Re}(\lambda) > -1), \quad (6)$$

and interpret the resulting Mellin-Barnes contour integrals as a H-function, we shall arrive at (3).

On similar manner as given above and using formula Shrivastava [3, (2.7.8), p.21]:

$${}_{z}D_{\infty}^{q} z^{-\lambda} = \frac{(-1)^{q}\Gamma(\lambda+q)}{\Gamma(\lambda)} z^{-\lambda-q}, \ q \text{ arbitrary}, \quad (7)$$

we can easily derive the results from (4).

References

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