ISSN 2250-3153

Numerical Solution of Multidimensional Integral by Using Newton Kote's Formula

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Abstract- In many different fields such as Physics, Geostatistics, Psychometrics, & finance among others the estimation process of parameters of a model involve the calculation of an integral. Although in principle, unfortunately the resulting (multidimensional) integrals usually does not have an analytic solution and therefore it is necessary to use numerical methods to approximate it. In this paper we mainly focus on the numerical solution of Weddle's rule by using Newton Kote's formula.

Index Terms- Numerical Integration, Multidimensional integral, Weddle's rule, Newton Kote's formula

I. Introduction

Numerical integration is the process of computing the value of definite integral from a set of numerical values of the integrand. The process is sometimes referred as mechanical quadrature. The approximate values of the definite integral which are either impossible to be computed analytically or for which anti derivative is too complex to be used. The inverse process to differentiation in calculus is represented by

$$I = \int_a^b f(x)dx \qquad \text{.....1.1.1}$$

Which means that the integral of a function f(x) with respect to the independent variable x evaluated between the limits x = a to x = b. Equation 1.1.1 also corresponds to the area under the curve of f(x) between x = a and x = b.

1.2] Need for numerical integration

There are several reasons for carrying out numerical integration. The integrand f(x) may be known only at certain points such as obtained by sampling. Some embedded system and other computer application may need numerical integration.

A formula for the integrand may be known but it may be difficult or impossible to find out an anti derivative which is elementary function. An example of such an integrand is f(x) = e^{-x^2} the anti derivative of which can not be written in elementary form.

It may be possible to find an anti derivative symbolically but it may be easier to compute a numerical approximation than to compute the anti derivative. That may be the case if the anti derivative is given as an infinite series or product or if its evaluation requires a special function which is not available.

1.3] Newton Kote's formula

Suppose f(x) is given for equidistant values of x say a = x_0, x_0+h, x_0+2h,….., x_0+nh = b. Let the range of integration (a,b) is divided into n equal parts each of width h so that b-a = nh

By using fundamental theorem of numerical analysis it has been proved the Newton Kote's formula which is as follows.

$$I = nh[f(x_0) + \frac{n}{2}\Delta f(x_0) + \frac{(2n^2-3n)}{12}\Delta^2 f(x_0) + \frac{(n^3-4n^2+4n)}{24}\Delta^3 f(x_0) + (\frac{n^4}{5} - \frac{3}{2}n^3 + \frac{11}{3}n^2 - 3n) \frac{\Delta^4 f(x_0)}{4!} + \left(\frac{n^5}{6} - 2n^4 + \frac{35}{4}n^3 - \frac{50}{3}n^2 + 12n\right)\frac{\Delta^5 f(x_0)}{5!} + \left(\frac{n^6}{7} - \frac{15}{6}n^5 + 17n^4 - \frac{225}{4}n^3 + \frac{274}{3}n^2 - 60n\right)\frac{\Delta^6 f(x_0)}{6!} + \dots +$$

upto (n+1) terms]

This can be written in simplified form as follows

$$I = nh[f(x_0) + \frac{n}{2}\Delta f(x_0) + \frac{n(2n-3)}{12}\Delta^2 f(x_0) + \frac{n(n-2)^2}{24}\Delta^3 f(x_0) + (\frac{n^4}{5} - \frac{3}{2}n^3 + \frac{11}{3}n^2 - 3n) \frac{\Delta^4 f(x_0)}{4!} + \left(\frac{n^5}{6} - 2n^4 + \frac{35}{4}n^3 - \frac{50}{3}n^2 + 12n\right)\frac{\Delta^5 f(x_0)}{5!} + \left(\frac{n^6}{7} - \frac{15}{6}n^5 + 17n^4 - \frac{225}{4}n^3 + \frac{274}{3}n^2 - 60n\right)\frac{\Delta^6 f(x_0)}{6!} + \dots +$$

upto (n+1) terms] …. 1.3.1

The equation 1.3.1 is referred as Newton Kote's formula.

1.4] Weddle's rule

Let the values of a function f(x) be tabulated at points x_i equally spaced by h= x_{i+1}- x_i so that $f_1=f(x_1), f_2=f(x_2), \ldots$. Then Weddle's rule approximating the integral of f(x) is given by putting n = 6 in the Newton Kote's formula. For Weddle's rule the number of subinterval should be taken as multiple of 6.

Since we take n = 6 means f(x) can be approximated by a polynomial of 6^{th} degree so that seventh and higher order differences are vanishes in the Newton Kote's formula. This is illustrated in the following figure

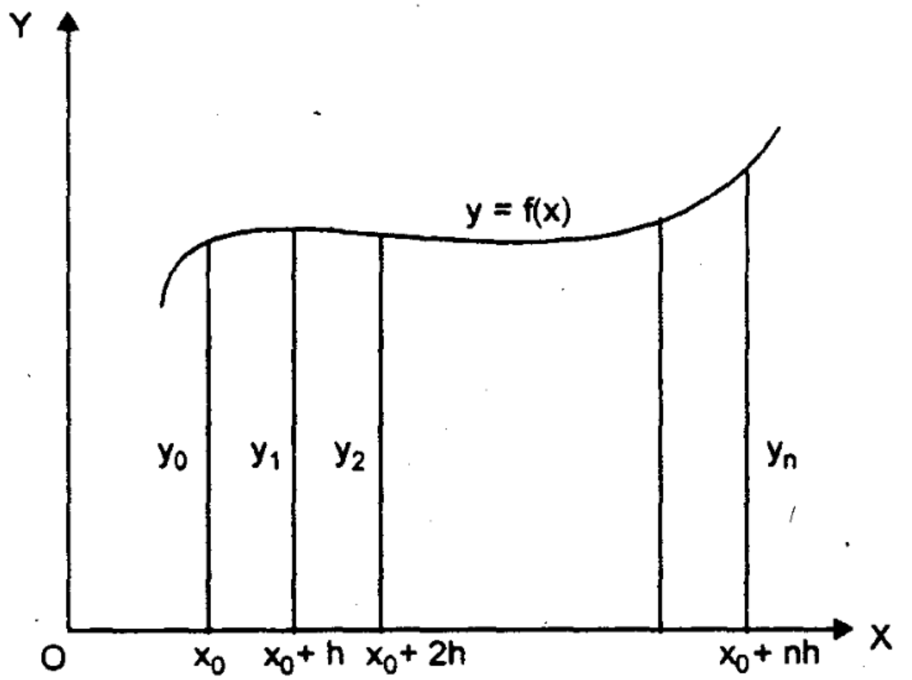

Fig 1.4.1: Illustration of Weddle's rule

To find the area we consider the integral as follows

$$I = \int_a^b f(x)\,dx = \int_{x_0}^{x_6} f(x)\,dx = \int_a^{a+6h} f(x)\,dx \quad \text{where } x_i = a+ih \text{ and } h = \frac{(b-a)}{6}$$

Substituting n = 6 in the above equation we get

$$I = 6h\left[f(a) + 3\Delta f(a) + \frac{9}{2}\Delta^2 f(a) + 4\Delta^3 f(a) + \frac{1}{24}\times\frac{246}{5}\Delta^4 f(a) + \frac{1}{120}\times 66\Delta^5 f(a) + \frac{1}{720}\times\frac{246}{7}\Delta^6 f(a)\right]$$

$$I = \frac{6h}{20}[\, 20 f(a) + 60 \{f(a+h)-f(a)\} + 90 \{f(a+2h)-2f(a+h)+f(a)\} + 80 \{f(a+3h)-3f(a+2h)+3f(a+h)-f(a)\}$$
$$+ 41 \{ f(a+4h)-4f(a+3h)+6 f(a+2h) -4f(a+h)+f(a)\} + 11 \{ f(a+5h)-5 f(a+4h) +10 f(a+3h) -10 f(a+2h)+ 5 f(a+h) - f(a) \} + \{ f(a+6h) - 6f(a+5h) +15 f(a+4h)- 20 f(a+3h) +15 f(a+2h) - 6 f(a+h) + f(a) \} \,]$$

After simplifying we get Weddle's rule as follows

$$I = \frac{3h}{10}[\, f(a) +5 f(a+h) + f(a+2h) + 6 f(a+3h) + f(a+4h) +5 f(a+5h) + f(a+6h) \,]$$

1.5] Composite Weddle's rule

$$I = \int_{x_0}^{x_0+nh} f(x)\,dx = \int_{x_0}^{x_0+6h} f(x)\,dx + \int_{x_0+6h}^{x_0+12h} f(x)\,dx + \int_{x_0+12h}^{x_0+18h} f(x)\,dx + \ldots + \int_{x_0+(n-6)h}^{x_0+nh} f(x)\,dx$$

$$= \int_{a}^{a+6h} f(x)\,dx + \int_{a+6h}^{a+12h} f(x)\,dx + \int_{a+12h}^{a+18h} f(x)\,dx + \ldots + \int_{a+(n-6)h}^{a+nh} f(x)\,dx$$

$$= \frac{3h}{10}\left[f_a + 5f_{a+h} + f_{a+2h} + 6f_{a+3h} + f_{a+4h} + 5f_{a+5h} + f_{a+6h} \right]$$

$$+ \frac{3h}{10}\left[f_{a+6h} + 5f_{a+7h} + f_{a+8h} + 6f_{a+9h} + f_{a+10h} + 5f_{a+11h} + f_{a+12h} \right]$$

$$+ \frac{3h}{10}\left[f_{a+12h} + 5f_{a+13h} + f_{a+14h} + 6f_{a+15h} + f_{a+16h} + 5f_{a+17h} + f_{a+18h} \right]$$

$$+ \cdots + \frac{3h}{10}\left[f_{a+(n-6)h} + 5f_{a+(n-5)h} + f_{a+(n-4)h} + 6f_{a+(n-3)h} + f_{a+(n-2)h} + 5f_{a+(n-1)h} + f_{a+nh} \right]$$

$$= \frac{3h}{10}\left[f_a + f_n + 5\{ f_{a+h} + f_{a+5h} + f_{a+7h} + f_{a+11h} + \cdots + f_{a+(n-5)h} + f_{a+(n-1)h} \} \right.$$

$$+ \{ f_{a+2h} + f_{a+4h} + f_{a+8h} + f_{a+10h} + \cdots + f_{a+(n-4)h} + f_{a+(n-2)h} \}$$

$$+ 6\{ f_{a+3h} + f_{a+9h} + f_{a+15h} + \cdots + f_{a+(n-3)h} \}$$

$$\left. + 2\{ f_{a+6h} + f_{a+12h} + f_{a+18h} + \cdots + f_{a+(n-6)h} \} \right] \quad \ldots\ldots 1.5.1$$

Equation 1.5.1 yields to give composite Weddle's rule. Weddle's rule is known to yield the most accurate results for hand calculators and those computers without preprogrammed routines.

1.6] Application of Numerical Integration

Numerical integration has always been useful in biostatistics to evaluate distribution function and other quantities. Emphasis in recent years on Bayesian and Empirical Bayesian method and on mixture models has greatly increased the importance of numerical integration for computing likelihood and posterior distributions and associated moments and derivatives. Many recent statistical methods are dependant especially on multiple integration possibly in very high dimension.

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