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Some approximation results on Basakakov type special class of positive linear operators

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Abstract- Recently S. P. Singh et. al. (Proceeding of The Mathematical Society BHU Vol. 24 (2008) 1-9) introduced special class of positive linear operators and studied some approximation results on it. Which are modified operators given by Deo N. et. al. (Appl. Maths. Compt., 201 (2008),604-612.).We shall study some approximation results on it.

I. INTRODUCTION

Recently S. P. Singh et. al. (Proceeding of The Mathematical Society BHU Vol. 24 (2008) 1-9) introduced a sequence of positive linear operators $\{A_n f\}$ which are defined as,

$$(A_n f)(x) = n\left(1+\frac{1}{n}\right)^2 \sum_{k=0}^{\infty} \binom{k+\frac{n}{c}-1}{k} \frac{(cx)^k}{(1+cx)^{\frac{n}{c}+k}} \ldots \int_0^{\frac{n}{n+1}} p_{n,k}(t) f(t)\,dt \quad \ldots (1.1)$$

$$\text{where } p_{n,k}(t) = \left(1+\frac{1}{n}\right)^n \binom{n}{k} t^k \left(\frac{n}{n+1}-t\right)^{n-k}; \quad c>0 \text{ and } \quad \text{for } t \in \left[0,\frac{n}{n+1}\right].$$

we studied some approximation results on it.

Again S. P. Singh et. al. [5] introduced a sequence of positive linear operators $\{A_{n,x} f\}$ which are defined as,

$$(A_{n,x} f)(x) = n\left(1+\frac{1}{n}\right)^2 \sum_{k=0}^{\infty} \binom{k+\frac{n}{c}-1}{k} \frac{(ct)^k}{(1+ct)^{\frac{n}{c}+k}} \ldots \int_0^{\frac{n}{n+1}} p_{n,k}(y) f(x+y)\,dy \quad \ldots (1.2)$$

$$\text{where } t,x \in \left[0,\frac{n}{n+1}\right] \text{ and } x \text{ is fixed.}$$

and studied some approximation results on it.

Deo N.et.al. [1] introduced a new Bernstein type special operators $\{V_n f\}$ defined as,

$$(V_n f)(x) = \sum_{k=0}^{n} p_{n,k}(x) f\left(\frac{k}{n}\right) \quad \ldots\ldots (1.3)$$

$$\text{where } p_{n,k}(x) = \left(1+\frac{1}{n}\right)^n \binom{n}{k} x^k \left(\frac{n}{n+1}-x\right)^{n-k}; \quad \text{for } 0 \le x \le \frac{n}{n+1}$$

Again Deo N.et.al.[1] gave the integral modification of the operators (1.3) which are defined as ,

$$(L_n f)(x) = n\left(1+\frac{1}{n}\right)^2 \sum_{k=0}^{n} p_{n,k}(x) \int_0^{\frac{n}{n+1}} p_{n,k}(t) f(t)\,dt \quad \ldots (1.4)$$

and prove some approximation results on the operators (1.4).

Singh S.P. [4] studied some approximation results on a sequence of Szȧsz type operators defined as,

$$(S_{n,x} f)(t) = \sum_{k=0}^{\infty} b_{n,k}(t) f\left(x+\frac{k}{n}\right); \quad \ldots\ldots (1.5)$$

$$\text{where } b_{n,k}(t) = e^{-nt}\frac{(nt)^k}{k!}; \quad x \in [0,\infty) \text{ is fixed.}$$

which map the space of bounded continuous functions $C_B[0,\infty)$ into itself following [3].

Kasana H.S. et. el. [2] obtained a sequence of modified Szȃsz operators for integrable function on $[0,\infty)$ defined as,

$$(M_{n,x} f)(t) \equiv M_{n,x}(f(y);t) = n\sum_{k=0}^{\infty} b_{n,k}(t) \int_0^{\infty} b_{n,k}(y) f(x+y)\,dy \quad .(1.6)$$

$$\text{where } t,x \in [0,\infty) \text{ and } x \text{ is fixed.}$$

In this paper if we put $c=1$ in the operators (1.1) and (1.2), it gives the new modified operators $\{A_n^* f\}$ and $\{A_{n,x}^* f\}$ as follows:

$$(A_n^* f)(x) = n\left(1+\frac{1}{n}\right)^2 \sum_{k=0}^{\infty} \binom{n+k-1}{k} \frac{x^k}{(1+x)^{n+k}} \ldots \int_0^{\frac{n}{n+1}} p_{n,k}(t) f(t)\,dt \quad \ldots (1.7)$$

and

$$(A_{n,x}^* f)(x) = n\left(1+\frac{1}{n}\right)^2 \sum_{k=0}^{\infty} \binom{n+k-1}{k} \frac{t^k}{(1+t)^{n+k}} \ldots \int_0^{\frac{n}{n+1}} p_{n,k}(y) f(x+y)\,dy \quad \ldots (1.8)$$

$$\text{where } t,x \in \left[0,\frac{n}{n+1}\right] \text{ and } x \text{ is fixed.}$$

We shall study some approximation results on the operators (1.7) and (1.8).

II. BASIC RESULTS-I

In order to prove our main result, the following basic results are needed.

1. $\sum_{k=0}^{\infty} \binom{n+k-1}{k} \frac{x^k}{(1+x)^{n+k}} k = nx \quad \ldots (2.1)$
2. $\sum_{k=0}^{\infty} \binom{n+k-1}{k} \frac{x^k}{(1+x)^{n+k}} k^2 = n^2x^2 + nx(1+x) \quad \ldots\ldots (2.2)$

3. $\sum_{k=0}^{\infty}\binom{n+k-1}{k}\frac{x^k}{(1+x)^{n+k}}k^3$
$= n^3x^3 + 3n^2x^2(1+x) + nx(1+2x)(1+x)$ ......(2.3)

4. $\sum_{k=0}^{\infty}\binom{n+k-1}{k}\frac{x^k}{(1+x)^{n+k}}k^4$
$= n^4x^4 + 6n^3x^3(1+x) + 4n^2x^2(2+5x+2x^2)$
$+nx(1+6x+6x^2)$ ……… (2.4)

III. PROOF OF BASIC RESULTS-I

We know that

$$(1+x)^n = \sum_{k=0}^{\infty}\binom{n+k-1}{k}\left(\frac{x}{1+x}\right)^k \quad \text{….} (2.5)$$

Differentiating with respect to x, we get

$$n(1+x)^{n-1} = \sum_{k=0}^{\infty}\binom{n+k-1}{k} k x^{k-1}(1+x)^{-k} + \sum_{k=0}^{\infty}\binom{n+k-1}{k} x^k(-k)(1+x)^{-k-1}$$

Multiplying x both sides, we get

$$nx(1+x)^{n-1} = \sum_{k=0}^{\infty}\binom{n+k-1}{k}\left(\frac{x}{1+x}\right)^k k\left[1-\frac{x}{1+x}\right]$$

$$nx(1+x)^{n-1} = \sum_{k=0}^{\infty}\binom{n+k-1}{k}\left(\frac{x}{1+x}\right)^k k\left(\frac{1}{1+x}\right)$$

$$nx(1+x)^{n} = \sum_{k=0}^{\infty}\binom{n+k-1}{k}\left(\frac{x}{1+x}\right)^k k \quad \text{…} (2.6)$$

$$nx = \sum_{k=0}^{\infty}\binom{n+k-1}{k}\frac{x^k}{(1+x)^{n+k}}k$$

This completes the proof of (2.1).

Again differentiating (2.6) with respect to x, we get

$$n^2x(1+x)^{n-1} + n(1+x)^n = \sum_{k=0}^{\infty}\binom{n+k-1}{k}k^2x^{k-1}(1+x)^{-k} + \sum_{k=0}^{\infty}\binom{n+k-1}{k}x^k k(-k)(1+x)^{-k-1}$$

Multiplying x both sides, we get

$$n^2x^2(1+x)^{n-1} + nx(1+x)^n = \sum_{k=0}^{\infty}\binom{n+k-1}{k}k^2x^k(1+x)^{-k} - \sum_{k=0}^{\infty}\binom{n+k-1}{k}x^{k+1}k^2(1+x)^{-k-1}$$

$$[n^2x^2 + nx(1+x)](1+x)^{n-1} = \sum_{k=0}^{\infty}\binom{n+k-1}{k}k^2x^k(1+x)^{-k}\left[1-\frac{x}{1+x}\right]$$

$$[n^2x^2 + nx(1+x)](1+x)^{n} = \sum_{k=0}^{\infty}\binom{n+k-1}{k}k^2x^k(1+x)^{-k} \quad \text{… ….} (2.7)$$

$$n^2x^2 + nx(1+x) = \sum_{k=0}^{\infty}\binom{n+k-1}{k}\frac{x^k}{(1+x)^{n+k}}k^2$$

This completes the proof of (2.2).

In the same way after differentiations and calculations, we get required result s (2.3) and (2.4).

IV. BASIC RESULTS-II

1. $(A_n^*1)(x) = 1$ ……………… (2.8)
2. $(A_n^*t)(x) \to x \quad as \quad n \to \infty.$ … (2.9)
3. $(A_n^*t^2)(x) \to x^2 \quad as \quad n \to \infty.$ ……… (2.10)
4. $(A_n^*t^3)(x) = \dfrac{n^3[n^3x^3 + 3n^2x^2(1+x) + nx(18+9x+2x^2)+6]}{(n+1)^3\ (n+2)\ (n+3)\ (n+4)}$ ……… (2.11)
5. $(A_n^*t^4)(x) = \dfrac{n^4[n^4x^4 + 2n^3x^3(8+3x) + n^2x^2(73+50x+3x^2)]}{(n+1)^4\ (n+2)\ (n+3)\ (n+4)\ (n+5)} + \dfrac{nx(96+71x+26x^2)+24}{(n+1)^4\ (n+2)\ (n+3)\ (n+4)\ (n+5)}$ ……… (2.12)
6. $(A_n^*(t-x))(x) = \dfrac{n[1-3x]-2x}{(n+1)\ (n+2)}$ ………… (2.13)
7. $(A_n^*(t-x)^2)(x) = \dfrac{2xn^3 + n^2[11x^2-8x+2] + n[17x^2-6x] + 6x^2}{(n+1)^2\ (n+2)\ (n+3)}$ ….……… (2.14)
8. $(A_n^*(t-x)^3)(x) = o\left(\dfrac{1}{n}\right).$ ……….(2.15)
9. $(A_n^*(t-x)^4)(x) = o\left(\dfrac{1}{n^2}\right).$ … ….(2.16)

V. PROOF OF BASIC RESULTS-II.

By putting $f(t) = 1$ in equation (1.7), we get

$$(A_n^*1)(x) = n\left(1+\frac{1}{n}\right)^2\sum_{k=0}^{\infty}\binom{n+k-1}{k}\frac{x^k}{(1+x)^{n+k}}$$

$$\text{….}\int_0^{\frac{n}{n+1}}\left(\frac{n+1}{n}\right)^n\binom{n}{k}t^k\left(\frac{n}{n+1}-t\right)^{n-k}1dt$$

$$= n\left(1+\frac{1}{n}\right)^2\sum_{k=0}^{\infty}\binom{n+k-1}{k}\frac{x^k}{(1+x)^{n+k}}\frac{1}{n}\left(\frac{n}{n+1}\right)^2$$

$$= \sum_{k=0}^{\infty}\binom{n+k-1}{k}\frac{x^k}{(1+x)^{n+k}} = 1.$$

This completes the proof of (2.8).

By putting $f(t) = t$ in equation (1.7), we get

$$(A_n^*t)(x) = n\left(1+\frac{1}{n}\right)^2\sum_{k=0}^{\infty}\binom{n+k-1}{k}\frac{x^k}{(1+x)^{n+k}}$$

$$\text{….}\int_0^{\frac{n}{n+1}}\left(\frac{n+1}{n}\right)^n\binom{n}{k}t^k\left(\frac{n}{n+1}-t\right)^{n-k}tdt$$

$$= n\left(1+\frac{1}{n}\right)^2 \sum_{k=0}^{\infty} \binom{n+k-1}{k} \frac{x^k}{(1+x)^{n+k}} \cdots$$

$$\cdots \frac{(k+1)}{n(n+2)} \left(\frac{n}{n+1}\right)^3$$

$$= \frac{n}{(n+1)(n+2)} \left\{ \sum_{k=0}^{\infty} \binom{n+k-1}{k}\frac{x^k}{(1+x)^{n+k}}\, k + \sum_{k=0}^{\infty}\binom{n+k-1}{k}\frac{x^k}{(1+x)^{n+k}}\, 1\right\}$$

$$= \frac{n}{(n+1)(n+2)}\,[nx+1]$$

$$= \frac{n^2x+n}{(n+1)(n+2)}$$

$(A_n^* t)(x) \to x \quad as \quad n \to \infty.$

This completes the proof of (2.9).

By putting $f(t) = t^2$ in equation (1.7), we get

$$(A_n^* t^2)(x) = n\left(1+\frac{1}{n}\right)^2 \sum_{k=0}^{\infty} \binom{n+k-1}{k} \frac{x^k}{(1+x)^{n+k}}$$

$$\cdots \int_0^{\frac{n}{n+1}} \left(\frac{n+1}{n}\right)^n \binom{n}{k} t^k \left(\frac{n}{n+1}-t\right)^{n-k} t^2\, dt$$

$$= n\left(1+\frac{1}{n}\right)^2 \sum_{k=0}^{\infty} \binom{n+k-1}{k} \frac{x^k}{(1+x)^{n+k}} \cdots$$

$$\cdots \frac{(k+1)(k+2)}{n(n+2)(n+3)} \left(\frac{n}{n+1}\right)^4$$

$$= \frac{n^2}{(n+1)^2(n+2)(n+3)}\left\{\sum_{k=0}^{\infty}\binom{n+k-1}{k}\frac{x^k}{(1+x)^{n-k}}\, k^2 + \sum_{k=0}^{\infty}\binom{n+k-1}{k}\frac{x^k}{(1+x)^{n+k}}\, 3k + \sum_{k=0}^{\infty}\binom{n+k-1}{k}\frac{x^k}{(1+x)^{n+k}}\, 2\right\}$$

$$= \frac{n^2[n^2x^2+nx(4+x)+2]}{(n+1)^2(n+2)(n+3)}$$

$(A_n^* t^2)(x) \to x^2 \quad as \quad n \to \infty.$

This completes the proof of (2.10).

In the same way by taking $f(t) = t^3$ & $f(t) = t^4$ respectively in (1.7) and after little calculations we get required results (2.11) to (2.16).

VI. MAIN RESULTS

In this section we shall give our main results.

LEMMA: Let $f \epsilon\, C\left[0\ ,\ \frac{n}{n+1}\right]$ then the sequence of positive linear operator defined by $\{A_n^* f\}$ is converges uniformly to $f \quad as \quad n \to \infty.$

Proof : Since from basic results (2.8), (2.9) and (2.10), we get

$$(A_n^* 1)(x) \xrightarrow{uniformly} 1 \quad as \quad n \to \infty$$

$$(A_n^* t)(x) \xrightarrow{uniformly} x \quad as \quad n \to \infty$$

$$(A_n^* t^2)(x) \xrightarrow{uniformly} x^2 \quad as \quad n \to \infty$$

Then using Korokin theorem we can conclude that

$$(A_n^* f)(x) \xrightarrow{uniformly} f \quad as \quad n \to \infty.$$

This completes the proof.

Theorem : Let f be the integrable and bounded in the interval $\left[0\ ,\ \frac{n}{n+1}\right]$ and let if f'' exists at a point x in $\left[0\ ,\ \frac{n}{n+1}\right]$, then one gets that

$$\lim_{n\to\infty} n[(A_n^* f)(x) - f(x)] = (1-3x)f'(x) + x\, f''(x).$$

where $\{A_n^* f\}$ are defined in (1.7).

Proof : Since f'' exists at a point x in $\left[0\ ,\ \frac{n}{n+1}\right]$, then by using Taylor's expansion, we write

$$f(t) = f(x) + (t-x)f'(x) + \frac{(t-x)^2}{2} f''(x) + (t-x)^2\lambda(t-x) \qquad \ldots\ldots\ldots\ldots(3.1)$$

where $\lambda(t-x) \to 0 \quad as \quad t \to x.$

Now for each $\varepsilon > 0$, there corresponds $\delta > 0$ such that

$|\lambda(t-x)| \le \varepsilon \qquad wherever \qquad |t-x| \le \delta.$

Again for $|t-x| > \delta,$ then there exist a positive number M such that

$$|\lambda(t-x)| \le M \le M\frac{(t-x)^2}{\delta^2} \quad .$$

Thus for all t and $x \epsilon \left[0\ ,\ \frac{n}{n+1}\right]$, we get

$$|\lambda(t-x)| \le \varepsilon + M\frac{(t-x)^2}{\delta^2} \qquad \ldots\ldots\ldots\ldots\ldots\ldots(3.2)$$

Applying $\{A_n^*\}$ on (3.1), we get

$$(A_n^* f)(x) = f(x)(A_n^* 1)(x) + f'(x)\left(A_n^*(t-x)\right)(x) + \frac{f''(x)}{2}(A_n^*(t-x)^2)(x) + \left(A_n^*(t-x)^2\lambda(t-x)\right)(x)$$

$$= f(x) + f'(x)\left[\frac{n[1-3x]-2x}{(n+1)\ \ (n+2)}\right] + \frac{f''(x)}{2}\left[\frac{2xn^3+n^2[11x^2-8x+2]+n[17x^2-6x]+6x^2}{(n+1)^2\ \ (n+2)\ \ (n+3)}\right]$$

$$+\, n\left(1+\frac{1}{n}\right)^2\sum_{k=0}^{\infty}\binom{n+k-1}{k}\frac{x^k}{(1+x)^{n+k}} \cdots$$

$$\cdots \int_0^{\frac{n}{n+1}} p_{n,k}(t)\,(t-x)^2\lambda(t-x)\, dt$$

Multiplying n both side, we get

$$n[(A_n^* f)(x) - f(x)] = f'(x)\left[\frac{n[1-3x]-2x}{(n+1)\ \ (n+2)}\right]n + \frac{f''(x)}{2}\left[\frac{2xn^3+n^2[11x^2-8x+2]+n[17x^2-6x]+6x^2}{(n+1)^2\ \ (n-2)\ \ (n+3)}\right]n + nR_n(t,x)$$

$(say) \qquad \ldots\ldots\ldots(3.3)$

Here we write,

$$nR_n(t,x) = nn\left(1+\frac{1}{n}\right)^2\sum_{k=0}^{\infty}\binom{n+k-1}{k}\frac{x^k}{(1+x)^{n+k}} \cdots$$

$$\cdots \int_0^{\frac{n}{n+1}} p_{n,k}(t)\,(t-x)^2\lambda(t-x)\, dt$$

$$|nR_n(t,x)| = \left| nn\left(1+\frac{1}{n}\right)^2\sum_{k=0}^{\infty}\binom{n+k-1}{k}\frac{x^k}{(1+x)^{n+k}} \cdots\cdots \int_0^{\frac{n}{n+1}} p_{n,k}(t)(t-x)^2\lambda(t-x)\, dt \right|$$

$$\leq nn\left(1+\frac{1}{n}\right)^2 \sum_{k=0}^{\infty} \binom{n+k-1}{k} \frac{x^k}{(1+x)^{n+k}} \dots$$
$$\int_0^{\frac{n}{n+1}} p_{n,k}(t) \| (t-x)^2 \lambda(t-x) | \, dt \dots (3.4)$$

Using (3.2) in equation (3.4), we get

$$|nR_n(t,x)| \leq n\varepsilon (A_n^*(t-x)^2)(x) + \frac{nM}{\delta^2}(A_n^*(y-t)^4)(x)$$
$$\leq n\varepsilon\, o\left(\frac{1}{n}\right) + \frac{nM}{\delta^2}\, o\left(\frac{1}{n^2}\right)$$
$$\leq \varepsilon + \frac{M}{\delta^2}\, o\left(\frac{1}{n}\right)$$

By choosing $\delta = n^{-1/4}$, we get that

$$|nR_n(t,x)| \leq \varepsilon + \frac{M}{n^{-1/2}}\, o\left(\frac{1}{n}\right)$$
$$\leq \varepsilon + M\, o\left(\frac{1}{\sqrt{n}}\right).$$

Since ε is arbitrary and small, we get

$$|nR_n(t,x)| \to 0 \quad as \quad n \to \infty. \qquad \dots\dots\dots..(3.5)$$

Using (3.5) in equation (3.3), we get

$$\lim_{n\to\infty} n[(A_n^* f)(x) - f(x)] = (1-3x)f'(x) + x f''(x).$$

This completes the proof.

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