ISSN 2250-3153

On the Zeros of a Polynomial

M.H.Gulzar

Department of Mathematics,
University of Kashmir, Srinagar, India

Email: gulzarmh@gmail.com

Abstract: In this paper we consider the problem of finding the number of zeros of a polynomial in a prescribed region by subjecting the real and imaginary parts of its coefficients to certain restrictions.
(AMS) Mathematics Subject Classification (2000): 30C10, 30C15

Index Terms: Zeros, Polynomial, Prescribed region

1. Introduction and Statement of Results

Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n. Regarding the number of zeros of P(z) in $|z|\le\frac{1}{2}$, Q. G. Mohammad [3] proved the following result:

Theorem A: Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n such that

$$a_n \ge a_{n-1} \ge \dots \ge a_1 \ge a_0 > 0.$$

Then the number of zeros of P(z) in $|z|\le\frac{1}{2}$ does not exceed

$$1+\frac{1}{\log 2}\log\frac{a_n}{a_0}.$$

K. K Dewan [1] generalized Theorem A to polynomials with complex coefficients and proved the following result:

Theorem B: Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with complex coefficients .If $\operatorname{Re} a_j=\alpha_j, \operatorname{Im} a_j=\beta_j,\ j=0,1,2,....,n$ and

$$\alpha_n \ge \alpha_{n-1} \ge \dots \ge \alpha_1 \ge \alpha_0 > 0,$$

then the number of zeros of P(z) in $|z|\le\frac{1}{2}$ does not exceed

$$1+\frac{1}{\log 2}\log\frac{\alpha_n+\sum_{j=0}^{n}|\beta_j|}{|a_0|} .$$

C. M. Upadhye [5] gave a generalization of Theorem B for the region $|z|\le\delta, 0<\delta<1$.In fact, she proved the following result:

TheoremC: Let $P(z)=\sum_{j=0}^{n}a_j z^j$ be a polynomial of degree n with complex coefficients. If $\operatorname{Re}a_j=\alpha_j, \operatorname{Im}a_j=\beta_j, j=0,1,2,....,n$ and

$$k\alpha_n \ge \alpha_{n-1} \ge ... \ge \alpha_1 \ge \alpha_0 > 0, k \ge 1,$$

then the number of zeros of P(z) in $\frac{|a_0|}{M_1} \le |z| \le \delta, 0<\delta<1$ does not exceed

$$\frac{1}{\log\frac{1}{\delta}}\log\frac{k(|\alpha_n|+\alpha_n)+|\alpha_0|-\alpha_0+2\sum_{j=0}^{n}|\beta_j|}{|a_0|},$$

where

$$M_1=k(|\alpha_n|+\alpha_n)+|\beta_0|-\alpha_0+2\sum_{j=1}^{n}|\beta_j|$$

Gulzar [2] proved the following generalization of Theorem C:

Theorem D: Let $P(z)=\sum_{j=0}^{n}a_j z^j$ be a polynomial of degree n with complex coefficients.If $\operatorname{Re}a_j=\alpha_j, \operatorname{Im}a_j=\beta_j, j=0,1,2,....,n$ and

$$k\alpha_n \ge \alpha_{n-1} \ge ... \ge \alpha_1 \ge \tau\alpha_0 ,$$

for $k \ge 1, 0<\tau \le 1$,then the number of zeros of P(z) in $\frac{|a_0|}{M_2} \le |z| \le \delta, 0<\delta<1$ does not exceed

$$\frac{1}{\log\frac{1}{\delta}}\log\frac{k(|\alpha_n|+\alpha_n)+2|\alpha_0|-\tau(|\alpha_0|+\alpha_0)+2\sum_{j=0}^{n}|\beta_j|}{|a_0|},$$

where

$$M_2=k(|\alpha_n|+\alpha_n)+|\alpha_0|+|\beta_0|-\tau(|\alpha_0|+\alpha_0)+2\sum_{j=1}^{n}|\beta_j| .$$

In this paper we give a generalization of Theorem D and hence of Theorems A,B and C as well. In fact we prove the following results:

Theorem 1: Let $P(z)=\sum_{j=0}^{n}a_j z^j$ be a polynomial of degree n with complex coefficients. If $\operatorname{Re}a_j=\alpha_j, \operatorname{Im}a_j=\beta_j, j=0,1,2,....,n$ and

$$\alpha_n \ge \alpha_{n-1} \ge ... \ge \alpha_\lambda, k\alpha_\lambda \ge \alpha_{\lambda-1} \ge ...... \ge \alpha_1 \ge \tau\alpha_0 ,$$

for $k \ge 1, 0<\tau \le 1, 0 \le \lambda \le n,$ then the number of zeros of P(z) in $\frac{|a_0|}{M_3} \le |z| \le \delta, 0<\delta<1$ does not exceed

$$\frac{1}{\log\frac{1}{\delta}}\log\frac{|\alpha_n|+\alpha_n+(k-1)(|\alpha_\lambda|+\alpha_\lambda)+2|\alpha_0|-\tau(|\alpha_0|+\alpha_0)+2\sum_{j=0}^{n}|\beta_j|}{|a_0|},$$

where

$$M_3=|\alpha_n|+\alpha_n+(k-1)(|\alpha_\lambda|+\alpha_\lambda)+|\alpha_0|+|\beta_0|-\tau(|\alpha_0|+\alpha_0)+2\sum_{j=1}^{n}|\beta_j| .$$

Remark 1: Taking $\lambda=n$ in Theorem 1, we get Theorem D. For $\lambda=n$, $\tau=1$,Theorem 1 reduces to Theorem C ;for $\lambda=n$, $\tau=1, k=1$,it reduces to Theorem B and for $\lambda=n$, $\tau=1, k=1$, $\alpha_0>0$, $\beta_j=0, \forall j=0,1,...,n,$ it reduces to Theorem A.

Theorem 2: Let $P(z)=\sum_{j=0}^{n}a_jz^j$ be a polynomial of degree n with complex coefficients. If $\operatorname{Re}a_j=\alpha_j, \operatorname{Im}a_j=\beta_j, j=0,1,2,....,n$ and

$$k\alpha_n\le\alpha_{n-1}\le......\le\alpha_\lambda\ge\alpha_{\lambda-1}\ge......\ge\alpha_1\ge\tau\alpha_0,$$

for $0\le\lambda\le n, 0<\tau\le1, k\ge1,$ then the number of zeros of P(z) in $\frac{|a_0|}{M_4}\le|z|\le\delta, 0<\delta<1$ does not exceed

$$\frac{1}{\log\frac{1}{\delta}}\log\frac{2\alpha_\lambda+k(|\alpha_n|-\alpha_n)+2|\alpha_0|-\tau(|\alpha_0|+\alpha_0)+2\sum_{j=0}^{n}|\beta_j|}{|a_0|},$$

where

$$M_4=2\alpha_\lambda+k(|\alpha_n|-\alpha_n)+|\alpha_0|+|\beta_0|-\tau(|\alpha_0|+\alpha_0)+2\sum_{j=1}^{n}|\beta_j| .$$

Remark 2: Taking $\lambda=n$ in Theorem 2, we get Theorem D by noting that $k\ge1$ .For $\lambda=n$, $\tau=1$,Theorem 2 reduces to Theorem C ;for $\lambda=n$, $\tau=1, k=1$,it reduces to Theorem B and for $\lambda=n$, $\tau=1$, $k=1$, $\alpha_0>0$, $\beta_j=0, \forall j=0,1,...,n,$ it reduces to Theorem A .

2.Lemmas

For the proofs of the above theorems, we need the following lemma:

Lemma 1: If f(z) is regular , f(0)$\neq 0$ and $|f(z)|\le M$ in $|z|\le1$, then the number of zeros of f(z) in $|z|\le\delta, 0<\delta<1$, does not exceed

$$\frac{1}{\log\frac{1}{\delta}}\log\frac{M}{|f(0)|}$$ (see [4]).

3.Proofs of Theorems

Proof of Theorem 1: Consider the polynomial

$$F(z)=(1-z)P(z)$$

$$
\begin{aligned}
&= (1-z)(a_n z^n + a_{n-1} z^{n-1} + ... + a_1 z + a_0) \\
&= -a_n z^{n+1} + (a_n - a_{n-1}) z^n + ... + (a_{\lambda+1} - a_\lambda) z^{\lambda+1} \\
&\quad + [\ (k a_\lambda - a_{\lambda-1}) - (k a_\lambda - a_\lambda)\ ] z^\lambda + (a_{\lambda-1} - a_{\lambda-2}) z^{\lambda-1} + ... \\
&\quad + [(a_1 - \tau a_0) + (\tau a_0 - a_0)] z + a_0 \\
&= -(\alpha_n + i\beta_n) z^{n+1} + [(\alpha_n - \alpha_{n-1}) + i(\beta_n - \beta_{n-1})] z^n \\
&\quad + ... + [(\alpha_{\lambda+1} - \alpha_\lambda) + i(\beta_{\lambda+1} - \beta_\lambda)] z^{\lambda+1} \\
&\quad + [(k\alpha_\lambda - \alpha_{\lambda-1}) - (k\alpha_\lambda - \alpha_\lambda) + i(\beta_\lambda - \beta_{\lambda-1})] z^\lambda \\
&\quad + [(\alpha_{\lambda-1} - \alpha_{\lambda-2}) + i(\beta_{\lambda-1} - \beta_{\lambda-2})] z^{\lambda-1} + ... \\
&\quad + [(\alpha_1 - \tau\alpha_0) + (\tau\alpha_0 - \alpha_0) + i(\beta_1 - \beta_0)] z + (\alpha_0 + i\beta_0)\ .
\end{aligned}
$$

For $|z| \le 1$,we have, by using the hypothesis,

$$
\begin{aligned}
|F(z)| &\le |\alpha_n| + |\beta_n| + \alpha_n - \alpha_{n-1} + |\beta_n| + |\beta_{n-1}| + \alpha_{n-1} - \alpha_{n-2} + |\beta_{n-1}| + |\beta_{n-2}| \\
&\quad + ... + \alpha_{\lambda+1} - \alpha_\lambda + |\beta_{\lambda+1}| + |\beta_\lambda| + k\alpha_\lambda - \alpha_{\lambda-1} + (k-1)|\alpha_\lambda| + |\beta_\lambda| + |\beta_{\lambda-1}| + ... \\
&\quad + \alpha_1 - \tau\alpha_0 + (1-\tau)|\alpha_0| + |\beta_1| + |\beta_0| + |\alpha_0| + |\beta_0| \\
&= |\alpha_n| + \alpha_n + (k-1)(|\alpha_\lambda| + \alpha_\lambda) + 2|\alpha_0| - \tau(|\alpha_0| + \alpha_0) + 2\sum_{j=0}^{n} |\beta_j|\ .
\end{aligned}
$$

Therefore, by using Lemma 1,we conclude that the number of zeros of F(z) and hence P(z) in $|z| \le \delta, 0 < \delta < 1$ does not exceed

$$
\frac{1}{\log\frac{1}{\delta}} \log \frac{|\alpha_n| + \alpha_n + (k-1)(|\alpha_\lambda| + \alpha_\lambda) + 2|\alpha_0| - \tau(|\alpha_0| + \alpha_0) + 2\sum_{j=0}^{n} |\beta_j|}{|a_0|}\ .
$$

To prove that P(z) has no zero in $\frac{|a_0|}{M_3} < |z|$, we consider

$$
\begin{aligned}
F(z) &= (1-z)P(z) \\
&= (1-z)(a_n z^n + a_{n-1} z^{n-1} + ... + a_1 z + a_0) \\
&= -a_n z^{n+1} + (a_n - a_{n-1}) z^n + ... + (a_{\lambda+1} - a_\lambda) z^{\lambda+1} \\
&\quad + [\ (k a_\lambda - a_{\lambda-1}) - (k a_\lambda - a_\lambda)\ ] z^\lambda + (a_{\lambda-1} - a_{\lambda-2}) z^{\lambda-1} + ... \\
&\quad + [(a_1 - \tau a_0) + (\tau a_0 - a_0)] z + a_0 \\
&= -(\alpha_n + i\beta_n) z^{n+1} + [(\alpha_n - \alpha_{n-1}) + i(\beta_n - \beta_{n-1})] z^n \\
&\quad + ... + [(\alpha_{\lambda+1} - \alpha_\lambda) + i(\beta_{\lambda+1} - \beta_\lambda)] z^{\lambda+1} \\
&\quad + [(k\alpha_\lambda - \alpha_{\lambda-1}) - (k\alpha_\lambda - \alpha_\lambda) + i(\beta_\lambda - \beta_{\lambda-1})] z^\lambda \\
&\quad + [(\alpha_{\lambda-1} - \alpha_{\lambda-2}) + i(\beta_{\lambda-1} - \beta_{\lambda-2})] z^{\lambda-1} + ... \\
&\quad + [(\alpha_1 - \tau\alpha_0) + (\tau\alpha_0 - \alpha_0) + i(\beta_1 - \beta_0)] z + a_0 \\
&= a_0 + Q(z)\ ,
\end{aligned}
$$

where

$$\begin{aligned}Q(z) &= -(\alpha_n + i\beta_n)z^{n+1} + \left[(\alpha_n - \alpha_{n-1}) + i(\beta_n - \beta_{n-1})\right]z^n \\ &+ ...... + \left[(\alpha_{\lambda+1} - \alpha_\lambda) + i(\beta_{\lambda+1} - \beta_\lambda)\right]z^{\lambda+1} \\ &+ \left[(k\alpha_\lambda - \alpha_{\lambda-1}) - (k\alpha_\lambda - \alpha_\lambda) + i(\beta_\lambda - \beta_{\lambda-1})\right]z^\lambda \\ &+ \left[(\alpha_{\lambda-1} - \alpha_{\lambda-2}) + i(\beta_{\lambda-1} - \beta_{\lambda-2})\right]z^{\lambda-1} + ... \\ &+ \left[(\alpha_1 - \tau\alpha_0) + (\tau\alpha_0 - \alpha_0) + i(\beta_1 - \beta_0)\right]z\end{aligned}$$

For $|z| \le 1$,we have

$$\begin{aligned}\max_{|z|=1}|Q(z)| &\le |\alpha_n| + |\beta_n| + \alpha_n - \alpha_{n-1} + |\beta_n| + |\beta_{n-1}| + \alpha_{n-1} - \alpha_{n-2} + |\beta_{n-1}| + |\beta_{n-2}| \\ &+ ...... + \alpha_{\lambda+1} - \alpha_\lambda + |\beta_{\lambda+1}| + |\beta_\lambda| + k\alpha_\lambda - \alpha_{\lambda-1} + (k-1)|\alpha_\lambda| + |\beta_\lambda| + |\beta_{\lambda-1}| \\ &+ ..... + \alpha_1 - \tau\alpha_0 + (1-\tau)|\alpha_0| + |\beta_1| + |\beta_0| + |\alpha_0| + |\beta_0| \\ &= |\alpha_n| + \alpha_n + (k-1)(|\alpha_\lambda| + \alpha_\lambda) + 2|\alpha_0| - \tau(|\alpha_0| + \alpha_0) + 2\sum_{j=0}^{n}|\beta_j| = M_3 .\end{aligned}$$

Hence

$$\begin{aligned}|F(z)| &= |a_0 + Q(z)| \\ &\ge |a_0| - |Q(z)| \\ &\ge |a_0| - |z|\max_{|z|=1}|Q(z)| \\ &\ge |a_0| - |z|M_3 \\ &> 0 \text{ if } |z| < \frac{|a_0|}{M_3} .\end{aligned}$$

This shows that F(z) and hence P(z) has no zero in $|z| < \dfrac{|a_0|}{M_3}$ and the proof of Theorem 1 is complete.

Proof of Theorem 2: Consider the polynomial

$$\begin{aligned}F(z) &= (1-z)P(z) \\ &= (1-z)(a_n z^n + a_{n-1}z^{n-1} + ... + a_1 z + a_0) \\ &= -a_n z^{n+1} + \left[\ (ka_n - a_{n-1}) - (ka_n - a_n)\ \right]z^n + ... + (a_{\lambda+1} - a_\lambda)z^{\lambda+1} \\ &+ (a_\lambda - a_{\lambda-1})z^\lambda + ... + \left[(a_1 - \tau a_0) + (\tau a_0 - a_0)\right]z + a_0 \\ &= -(\alpha_n + i\beta_n)z^{n+1} + \left[(k\alpha_n - \alpha_{n-1}) - (k\alpha_n - \alpha_n) + i(\beta_n - \beta_{n-1})\right]z^n \\ &+ ...... + \left[(\alpha_{\lambda+1} - \alpha_\lambda) + i(\beta_{\lambda+1} - \beta_\lambda)\right]z^{\lambda+1} \\ &+ \left[(\alpha_\lambda - \alpha_{\lambda-1}) + i(\beta_\lambda - \beta_{\lambda-1})\right]z^\lambda \\ &+ \left[(\alpha_{\lambda-1} - \alpha_{\lambda-2}) + i(\beta_{\lambda-1} - \beta_{\lambda-2})\right]z^{\lambda-1} + ... \\ &+ \left[(\alpha_1 - \tau\alpha_0) + (\tau\alpha_0 - \alpha_0) + i(\beta_1 - \beta_0)\right]z + (\alpha_0 + i\beta_0) .\end{aligned}$$

For $|z| \le 1$,we have, by using the hypothesis,

$$\begin{aligned}|F(z)| &\le |\alpha_n| + |\beta_n| + \alpha_{n-1} - k\alpha_n + (k-1)|\alpha_n| + |\beta_n| + |\beta_{n-1}| + \alpha_{n-2} - \alpha_{n-1} + |\beta_{n-1}| + |\beta_{n-2}| \\ &+ ...... + \alpha_\lambda - \alpha_{\lambda+1} + |\beta_{\lambda+1}| + |\beta_\lambda| + \alpha_\lambda - \alpha_{\lambda-1} + |\beta_\lambda| + |\beta_{\lambda-1}| + ...\end{aligned}$$

$$+\alpha_1-\tau\alpha_0+(1-\tau)|\alpha_0|+|\beta_1|+|\beta_0|+|\alpha_0|+|\beta_0|$$
$$=2\alpha_\lambda+k(|\alpha_n|-\alpha_n)+2|\alpha_0|-\tau(|\alpha_0|+\alpha_0)+2\sum_{j=0}^{n}|\beta_j| .$$

Therefore, by using Lemma 1,we conclude that the number of zeros of F(z) and hence P(z) in $|z|\le\delta, 0<\delta<1$ does not exceed

$$\frac{1}{\log\frac{1}{\delta}}\log\frac{2\alpha_\lambda+k(|\alpha_n|-\alpha_n)+2|\alpha_0|-\tau(|\alpha_0|+\alpha_0)+2\sum_{j=0}^{n}|\beta_j|}{|a_0|} .$$

To prove that P(z) has no zero in $\frac{|a_0|}{M_4}<|z|$, we consider

$$\begin{aligned}F(z)&=(1-z)P(z)\\&=(1-z)(a_nz^n+a_{n-1}z^{n-1}+......+a_1z+a_0)\\&=-a_nz^{n+1}+[\ (ka_n-a_{n-1})-(ka_n-a_n)\ ]z^n+......+(a_{\lambda+1}-a_\lambda)z^{\lambda+1}\\&+(a_\lambda-a_{\lambda-1})z^\lambda+......+[(a_1-\tau a_0)+(\tau a_0-a_0)]z+a_0\\&=-(\alpha_n+i\beta_n)z^{n+1}+[(k\alpha_n-\alpha_{n-1})-(k\alpha_n-\alpha_n)+i(\beta_n-\beta_{n-1})]z^n\\&+......+[(\alpha_{\lambda+1}-\alpha_\lambda)+i(\beta_{\lambda+1}-\beta_\lambda)]z^{\lambda+1}\\&+[(\alpha_\lambda-\alpha_{\lambda-1})+i(\beta_\lambda-\beta_{\lambda-1})]z^\lambda\\&+[(\alpha_{\lambda-1}-\alpha_{\lambda-2})+i(\beta_{\lambda-1}-\beta_{\lambda-2})]z^{\lambda-1}+......\\&+[(\alpha_1-\tau\alpha_0)+(\tau\alpha_0-\alpha_0)+i(\beta_1-\beta_0)]z+a_0\\&=a_0+Q(z)\ ,\end{aligned}$$

where

$$\begin{aligned}Q(z)\ &=-(\alpha_n+i\beta_n)z^{n+1}+[(k\alpha_n-\alpha_{n-1})-(k\alpha_n-\alpha_n)+i(\beta_n-\beta_{n-1})]z^n\\&+......+[(\alpha_{\lambda+1}-\alpha_\lambda)+i(\beta_{\lambda+1}-\beta_\lambda)]z^{\lambda+1}\\&+[(\alpha_\lambda-\alpha_{\lambda-1})+i(\beta_\lambda-\beta_{\lambda-1})]z^\lambda\\&+[(\alpha_{\lambda-1}-\alpha_{\lambda-2})+i(\beta_{\lambda-1}-\beta_{\lambda-2})]z^{\lambda-1}+......\\&+[(\alpha_1-\tau\alpha_0)+(\tau\alpha_0-\alpha_0)+i(\beta_1-\beta_0)]z\end{aligned}$$

For $|z|\le1$, we have

$$\begin{aligned}\max_{|z|=1}|Q(z)|\le\ &|\alpha_n|+|\beta_n|+\alpha_{n-1}-k\alpha_n+(k-1)|\alpha_n|+|\beta_n|+|\beta_{n-1}|+\alpha_{n-2}-\alpha_{n-1}\\&+......+\alpha_\lambda-\alpha_{\lambda+1}+|\beta_{\lambda+1}|+|\beta_\lambda|+\alpha_\lambda-\alpha_{\lambda-1}+|\beta_\lambda|+|\beta_{\lambda-1}|\\&+.....+\alpha_1-\tau\alpha_0+(1-\tau)|\alpha_0|+|\beta_1|+|\beta_0|\\&=2\alpha_\lambda+k(|\alpha_n|-\alpha_n)+|\alpha_0|+|\beta_0|-\tau(|\alpha_0|+\alpha_0)+2\sum_{j=1}^{n}|\beta_j| .\end{aligned}$$

Hence

$$\begin{aligned}|F(z)| &= |a_0 + Q(z)| \\ &\geq |a_0| - |Q(z)| \\ &\geq |a_0| - |z| \max_{|z|=1} |Q(z)| \\ &\geq |a_0| - |z| M_4 \\ &> 0 \text{ if } |z| < \frac{|a_0|}{M_4} .\end{aligned}$$

This shows that F(z) and hence P(z) has no zero in $|z| < \frac{|a_0|}{M_4}$ and the proof of Theorem 2 is complete .

REFERENCES

[1] K.K.Dewan, Extremal Properties and Coefficient estimates for polynomials with restricted zeros and on location of zeros of polynomials, Ph. D Thesis, IIT Delhi, 1980.
[2] M.H.Gulzar, On the Number of Zeros of a Polynomial in a prescribed region, Research Journal of Pure Algebra, vol.2(2),2012,35-46.
[3] Q.G. Mohammad, On the zeros of polynomials, Amer. Math. Monthly 72 (1965), 631-633.
[4] E.c.Titchnarsh, Theory of Functions, 2nd edition, Oxford University Press, London, 1939.
[5] C.M.Upadhye, On the zeros of a polynomial, Theory of Polynomials and Applications, Deep and Deep Publications, 2007, 197-202.