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Further Results on Generalized Incomplete Extended Beta Function

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Abstract: Recently an extension of beta function is defined by introducing an extra parameter is proved to be useful earlier (Aslam Chaudhry 1997 [8] and A. R. Miller 1998 [1]). In this short research note, we further obtained results for generalized incomplete beta function and obtained the various integral representations and properties.

Index Terms: Beta Function, Incomplete Beta Function, Extended Beta Function, Extended Incomplete Beta Function.

I. INTRODUCTION

The classical incomplete beta function is defined by [3, 5, 9]

$$B_x(\nu,\mu) = \int_0^z t^{\nu-1}(1-t)^{\mu-1}dt \quad (\nu > 0,\ \mu > 0 \text{ and } 0 \le x < 1) \tag{1.1}$$

It is expressed in terms of the Gauss function to give ([6], [7])

$$B_x(\nu,\mu) = \frac{x^\nu}{\nu}(1-x)^\mu \; {}_2F_1(1, \nu+\mu; 1+\nu; x), \tag{1.2}$$

$$B_x(\nu,\mu) = \frac{x^\nu}{\nu} \; {}_2F_1(\nu, 1-\mu; 1+\nu; x), \tag{1.3}$$

The adjective 'incomplete' reflects the fact that the upper limit of Euler's integral of the first kind is less than the value of unity that is required to complete the integral. This incompleteness prevents the interchangeability of the ν and μ parameters. The formula

$$B_x(\mu,\nu) = B(\nu,\mu) - B_{1-x}(\nu,\mu) \tag{1.4}$$

shows the effect of interchanging the parameters and provides an argument reflection formula.

The recurrence formulas

$$B_x(\nu+1,\mu) = \frac{\nu}{\mu}B_x(\nu,\mu+1) - \frac{x^\nu(1-x)^\mu}{\mu} \tag{1.5}$$

and

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$$B_x(\nu,\mu+1) = \frac{\mu}{\nu} B_x(\nu+1,\mu) + \frac{x^\nu (1-x)^\mu}{\nu} \tag{1.6}$$

link two incomplete beta functions. The interrelationship

$$B_x(\nu,\mu) = B_x(\nu+1,\mu) + B_x(\nu,\mu+1) \tag{1.7}$$

connects three incomplete beta functions.

It is to be noted that several integral functions involving powers of trigonometric or hyperbolic functions are special cases of the incomplete beta function. Some of these representations are as follows [7]

$$B_x(\nu,\mu) = 2\int_0^T \sin^{2\nu-1}(t)\cos^{2\mu-1}(t)dt$$

$$\left(0 \le T = \arcsin(\sqrt{x}) < \frac{\pi}{2}\right) \tag{1.8}$$

$$B_x(\nu,\mu) = \int_0^T \frac{t^{\nu-1}dt}{(1+t)^{\nu+\mu}}$$

$$\left(0 \le T = \frac{x}{1-x} < \infty\right) \tag{1.9}$$

$$B_x(\nu,\mu) = 2\int_0^T \tanh^{2\nu-1}(t)\,\mathrm{sech}^{2\mu}(t)dt$$

$$\left(0 \le T = \mathrm{arctanh}\ (\sqrt{x}) < \infty\right) \tag{1.10}$$

In recent years several extensions of well known special functions have been considered by several authors [2, 4, 8, 9]. Especially, Chaudhry et al. [8] gave an extension of Euler's beta function. Namely, they defined the following extended beta function and extended incomplete beta function as

$$B(p,q;p) = \int_0^1 t^{p-1}(1-t)^{q-1}\exp\left[-\frac{b}{t(1-t)}\right]dt$$

(Re (b) > 0; For b = 0, Re (p) > 0, Re (q) > 0) (1.11)

and

$$B_x(p,q;b) = \int_0^x t^{p-1}(1-t)^{q-1}\exp\left[-\frac{b}{t(1-t)}\right]dt$$

(Re (b) > 0; For b = 0, p > 0, q > 0 and 0 < x < 1) (1.12)

respectively.

Clearly, B (p, q; 0) = B (p, q) and B_x (p, q; 0) = B_x (p, q)

The extension will be seen to be useful in that most properties of the beta function and incomplete beta function carry over naturally and simply for it. They also obtained integral representation, various properties, Mellin transform, beta distribution and express in terms of special function as Macdonald, Whittaker and error function.

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Afterward, Kim et al. [2] considered the following generalizations of extended beta function and extended incomplete beta function as

$$B(p,q;b;m)=\int_0^1 t^{p-1}(1-t)^{q-1}\exp\left[-\frac{b}{t^m(1-t)^m}\right]dt$$

(Re (b) > 0, Re (m) > 0; For b = 0, Re (p) > 0, Re (q) > 0) (1.13)

and

$$B_x(p,q;b;m)=\int_0^x t^{p-1}(1-t)^{q-1}\exp\left[-\frac{b}{t^m(1-t)^m}\right]dt$$

(Re (b) > 0, Re (m) > 0; For b = 0, p > 0, q > 0 and 0 < x < 1) (1.14)

respectively.

Clearly, B (p, q; b; 1) = B (p, q; b), B (p, q; 0; m) = B (p, q),

B_x (p, q; b; 1) = B_x (p, q; b) and B_x (p, q; 0; m) = B_x (p, q)

In this paper we further obtained integral representation and some properties of generalized incomplete beta function (1.14) defined earlier by Kim et al. [2]. The plan of the paper is as follows:

This present paper is divided into three sections. In section 2, various integral representations of generalized incomplete beta function are obtained. In section 3, some properties of generalized incomplete beta function are obtained.

II. INTEGRAL REPRESENTATIONS

Theorem 2.1. If Re (b) > 0, Re(m) > 0; For b = 0, p > 0, q > 0 and 0 < x < 1, we have following integral representations:

$$B_x(p,q;b;m)=2\int_0^T \sin^{2p-1}\theta\cos^{2q-1}\theta\exp[-b\sec^{2m}\theta\,\mathrm{cosec}^{2m}\theta]d\theta$$

$$\left(0<T=\sin^{-1}(\sqrt{x})\le\frac{\pi}{2}\right) \tag{2.1}$$

$$B_x(p,q;b;m)=\int_0^T \frac{u^{p-1}}{(1+u)^{p+q}}\exp\left[-b(2+u+\frac{1}{u})^m\right]du$$

$$\left(0<T=\frac{x}{1-x}<\infty\right) \tag{2.2}$$

$$B_x(p,q;b;m)=2\int_0^T \tanh^{2p-1}\theta\,\mathrm{sech}^{2q}\theta\exp\left[-b(2+\sinh^2\theta+\mathrm{cosech}^2\theta)^m\right]d\theta$$

$$\left(0\le T=\sinh^{-1}\left(\sqrt{\frac{x}{1-x}}\right)<\infty\right) \tag{2.3}$$

Proof. Letting $t = \sin^2 \theta$ in (1.14), we get

$$B_x(p,q;b;m) = \int_0^x t^{p-1}(1-t)^{q-1} \exp\left[-\frac{b}{t^m(1-t)^m}\right]dt$$

$$= 2\int_0^T \sin^{2p-1}\theta \cos^{2q-1}\theta \exp\left[-b\sec^{2m}\theta \operatorname{cosec}^{2m}\theta\right]d\theta$$

On the other hand, letting $t = \frac{u}{1+u}$ in (1.14), we get

$$B_x(p,q;b;m) = \int_0^T \frac{u^{p-1}}{(1+u)^{p+q}} \exp\left[-b(2+u+\frac{1}{u})^m\right]du$$

Finally, substituting $u = \sinh^2 \theta$ in (2.2), we get

$$B_x(p,q;b;m) = 2\int_0^T \tanh^{2p-1}\theta \operatorname{sech}^{2q}\theta \exp\left[-b(2+\sinh^2\theta + \operatorname{cosech}^2\theta)^m\right]d\theta$$

This completes the proof.

III. PROPERTIES AND SUMMATION FORMULA

Theorem 3.1. For the generalized incomplete beta function, we have the following argument reflection formula:

$$B_x(p,q;b;m) = B(p,q;b;m) - B_{1-x}(q,p;b;m) \tag{3.1}$$

Proof. The Right-hand side of (3.1) yields

$$\int_{1-x}^1 u^{q-1}(1-u)^{p-1} \exp\left[-\frac{b}{u^m(1-u)^m}\right]du$$

Putting u = 1- t, we have

$$\int_0^x t^{p-1}(1-t)^{q-1} \exp\left[-\frac{b}{t^m(1-t)^m}\right]dt$$

which is the left-hand side of (3.1).

Remark 3.1. It is to be noted that (3.1) shows the effect of interchanging the parameters and provides an argument reflection formula. In particular, by letting b = 0, we obtain the argument reflection formula [9] for the classical incomplete beta function.

Theorem 3.2. For the generalized incomplete beta function, we have following relation between three generalized incomplete beta function:

$$B_x(p,q;b;m) = B_x(p+1,q;b;m) + B_x(p,q+1;b;m) \tag{3.2}$$

Proof. The right-hand side of (3.2) yields

$$\int_0^x \left\{t^p(1-t)^{q-1} + t^{p-1}(1-t)^q\right\}\exp\left[-\frac{b}{t^m(1-t)^m}\right]dt$$

Which, after simple algebraic manipulations, yields

$$\int_0^x t^{p-1}(1-t)^{q-1}\exp\left[-\frac{b}{t^m(1-t)^m}\right]dt$$

Which is equal to the left-hand side of (3.2).

Remark 3.2. From the argument reflection formula (3.1), we find that

$$B_x(p,q;b;m) + B_{1-x}(q,p;b;m) = B(p,q;b;m)$$

Setting m = 1, q = p and x = $\frac{1}{2}$ in, we find that

$$B_{\frac{1}{2}}(p,p;b) = \frac{1}{2}B(p,p;b),$$

Which can be further written in terms of the Whittaker function to give

$$B_{\frac{1}{2}}(p,p;b) = \sqrt{\pi}\quad 2^{-p-1}\quad b^{(p-1)/2}\quad e^{-2b}\quad W_{-\frac{p}{2},\frac{p}{2}}(4b)$$

(Re (b) > 0)

In particular, when p = $\frac{1}{2}$, we find

$$B_{\frac{1}{2}}\left(\frac{1}{2},\frac{1}{2};b\right) = \frac{\pi}{2}\quad \mathrm{erfc}(2\sqrt{b})$$

Theorem 3.3. For the generalized incomplete beta function, we have the following summation formula:

$$B_x(\alpha,-\alpha-n;b;m) = \sum_{k=0}^{n}\binom{n}{k}B_x(\alpha+k,-\alpha-k;b;m)$$

(Re (b) > 0, Re (m) > 0, n = 0, 1, 2, 3, …) (3.3)

Proof. On setting p = α and q = - α - n in (3.2), we get

$$B_x(\alpha,-\alpha-n;b;m) = B_x(\alpha,-\alpha-n+1;b;m) + B_x(\alpha+1,-\alpha-n;b;m)$$

We can write this formula recursively, starting with n = 1, to obtain

$$B_x(\alpha,-\alpha-1;b;m) = B_x(\alpha,-\alpha;b;m) + B_x(\alpha+1,-\alpha-1;b;m)$$

$$B_x(\alpha,-\alpha-2;b;m) = B_x(\alpha,-\alpha;b;m) + 2B_x(\alpha+1,-\alpha-1;b;m) + B_x(\alpha+2,-\alpha-2;b;m)$$

$$B_x(\alpha,-\alpha-3;b;m) = B_x(\alpha,-\alpha;b;m) + 3B_x(\alpha+1,-\alpha-1;b;m)$$
$$+3B_x(\alpha+2,-\alpha-2;b;m) + B_x(\alpha+3,-\alpha-3;b;m)$$

and so on. The series arises exactly as the binomial series does and so we can guess that

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$$B_x(\alpha, -\alpha - n; b; m) = \sum_{k=0}^{n} \binom{n}{k} B_x(\alpha + k, -\alpha - k; b; m)$$

This result can be simply proved by induction, assuming it to be true for some n and writing $B_x(\alpha, -\alpha - n; b; m)$ as above. It immediately follows that (3.3) holds for (n + 1). Thus, the result (3.3) is true for all n = 0, 1, 2, 3, …

It is to be noted that the problem of expressing $B_x(p, q; b; m)$ in terms of other special functions remains open.

Presumably, this distribution should be useful in extending the statistical results for strictly positive variables to deal with variables that can take arbitrarily large negative values as well.

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