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Minimum Shannon Entropy for two specified Moments

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Abstract- Entropy optimization includes maximization and minimization. Maximization of entropy is easy and it can be done by using Lagrange's method since entropy is concave function. Due to the concavity minimization of entropy is not so simple. But calculation of minimum entropy probability distribution is necessary because knowledge of both maximum and minimum entropy probability distribution gives complete information about probability distribution. In this paper we obtain analytical expressions for minimum Shannon entropy for given r^{th} & s^{th} order moments.

Index Terms- Switching point, consistent, minimum entropy.

I. Introduction

Maximum entropy probability distribution has various applications in Science and Engineering. Various measures have been used to obtain maximum entropy probability distribution. It has been studied in detail [3, 4, 5, 7, 8]. But these measures have not been used to obtain minimum entropy probability distribution so much.

Maximum entropy probability distribution is most unbiased, most uniform and most random while minimum entropy probability distribution is most biased, least uniform and least random. Entropy is concave function so minimization of entropy is complicated than maximization.

When information is least we get maximum entropy probability distribution. As we add information consistent with initial information in the form of moments, entropy decreases. When we reach $S_{min,}$ we need not search for additional constraints because we have completed information. Need to obtain minimum entropy lies in the fact that true distribution lies in between maximum & minimum entropy probability distribution and to know the information contained in moments, we need both S_{max} and S_{min} . Knowing the S_{min} we can recognize the pattern.

A good progress to obtain S_{min} was done by Kapur [6]. Anju Rani [2] continued the work and obtained minimum Shannon entropy as well as Havrda – Charvat entropy when one moment is prescribed.

In the present paper, we obtain minimum Shannon entropy when two moments are prescribed. We shall obtain analytical expressions for minimum Shannon entropy S_{min} when r^{th} and s^{th} order moments are prescribed.

II. Analytical Expressions for Minimum Shannon Entropy when r^{th} and s^{th} Order Moments are Prescribed

Let x be a discrete variate which takes all values from 1 to n with probabilities $p_1, p_2, \ldots, p_n$. The r^{th} and s^{th} order moments of this probability distribution are prescribed as $\left(\mu_r'\right)^{1/r}$ and $\left(\mu_s'\right)^{1/s}$. There will be many distributions having these particular values of r^{th} order and s^{th} order moments and each of these distributions will have a particular value of entropy. Out of these entropies our aim is to find minimum value of entropy say S_{min}. Mathematically we have to minimize

$$S = -\sum_{i=1}^{n} p_i \ln p_i \qquad \text{--(1)}$$

subject to

$$\sum_{i=1}^{n} p_i = 1,\ \sum_{i=1}^{n} p_i i^r = \mu_r',\ \sum_{i=1}^{n} p_i i^s = \mu_s' \qquad \text{--(2)}$$

Since there are three linear constraints, the minimum entropy probability distribution will have at most three non zero components. Let these be p_h, p_k and p_l. Then

$$p_h + p_k + p_l = 1,\ h^r p_h + k^r p_k + l^r p_l = \mu_r',$$
$$h^s p_h + k^s p_k + l^s p_l = \mu_s' \qquad \text{-(3)}$$

simplifying these equations, we get

$$p_h = \frac{\mu_s'(l^r - k^r) - \mu_r'(l^s - k^s) + l^s k^r - l^r k^s}{(l^r - h^r)(l^s - k^s) - (l^s - h^s)(l^r - k^r)} \qquad \text{--(4)}$$

$$p_k = \frac{\mu_r'(l^s - h^s) - \mu_s'(l^r - h^r) + h^s l^r - l^s h^r}{(l^r - h^r)(l^s - k^s) - (l^s - h^s)(l^r - k^r)} \qquad \text{--(5)}$$

$$p_l = \frac{\mu_s'(k^r - h^r) - \mu_r'(k^s - h^s) + h^r k^s - h^s k^r}{(l^r - h^r)(l^s - k^s) - (l^s - h^s)(l^r - k^r)} \qquad \text{--(6)}$$

To calculate p_h , p_k & p_l we take set of consistent values of r^{th} & s^{th} order moments. Here s^{th} order moment is greater than r^{th} order moment but it does not mean that all values of s^{th} order moment are possible for given value of r^{th} order moment. In fact there is a range for feasible values of s^{th} order moment for given r^{th} order moment (Anju Rani [1]). These values are given by

(i) If $\left(\mu_r'\right)^{1/r}$ is an integer , then

$$\left(\mu_s'\right)_{min} = \left(\mu_r'\right)^{s/r} \qquad \text{--(7)}$$

If $(\mu'_r)^{1/r}$ is not an integer, let $(\mu'_r)^{1/r} = [(\mu'_r)^{1/r}]+L$, $0 < L < 1$, where $[(\mu'_r)^{1/r}]$ represents integral part of $(\mu'_r)^{1/r}$. Then

$$(\mu'_s)_{min} = \frac{[(\mu'_r)^{1/r}+1]^r - \mu'_r}{[(\mu'_r)^{1/r}+1]^r - [(\mu'_r)^{1/r}]^r}[(\mu'_r)^{1/r}]^s + \frac{\mu'_r - [(\mu'_r)^{1/r}]^r}{[(\mu'_r)^{1/r}+1]^r - [(\mu'_r)^{1/r}]^r}[(\mu'_r)^{\frac{1}{r}}+1]^s \quad \text{- -(8)}$$

(ii) The expression for maximum value of s^{th} order moment is given as

$$(\mu'_s)_{max} = \frac{\mu'_r\{n^s-1\}+n^r-n^s}{\{n^r-1\}} \quad \text{- -(9)}$$

For the given values of r^{th} order moment and $(\mu'_s)^{1/s}_{min}$ probability p_h is zero at point (1, A, A+1) or p_l is zero at point (A, A+1, n) & for the given values of r^{th} order moment and $(\mu'_s)^{1/s}_{max}$, probability $p_k = 0$ at point (1, n-1, n).Probability $p_h = 0$ for $\{1 \le h < k < (\mu'_r)^{1/r} \le l \le n\}$ or $\{1 \le h < k \le (\mu'_r)^{1/r} < l \le n\}$ and $p_l = 0$ for $\{1 \le h \le (\mu'_r)^{1/r} < k < l \le n\}$ or $\{1 \le h < (\mu'_r)^{1/r} \le k < l \le n\}$. As we go on increasing the values of p_h & p_l probability p_k tends to zero. For the given values of r^{th} order moment and $(\mu'_s)^{1/s}_{min}$, the values of entropies are same at all existing points and similarly for the given values of r^{th} order moment and $(\mu'_s)^{1/s}_{max}$, the values of entropies are the same at all existing points.

Every interval is divided into many subintervals such that at common values of s^{th} order moment, the values of minimum entropy for any two subintervals are same. These values of s^{th} order moment are called switching points. At these values, we switch over entropy from one set of values of (*h*,*k*,*l*) to another set of values of (*h*,*k*,*l*).

Let $(\mu'_r)^{1/r} \in$ (A, A+1] , $1 \le A < n$, where A is an integer. *h* can take values 1,2,….,A; *k* can take values *h*+1,….,n-1; *l* can take values A+1,……..,n. We calculate probability distributions in each possible interval for different values of r^{th} & s^{th} order moments.

For calculating minimum entropy, we consider four types of points. These points are:

(1) (A+α, A+β, n)
(2) (1, A+ α, A+γ)
(3) (A+ α, A+δ, A+ξ)
(4) (1, A+ α, n)

Before considering these points, we calculate minimum and maximum values of s^{th} order moment from equations (7), (8) and (9).

For some value of s^{th} order moment which is slightly greater than minimum value of s^{th} order moment, we calculate entropies at points (A+ α, A+β, n) and (1, A+ α, A+γ). By doing this we can obtain that point at which entropy is minimum for given r^{th} order moment . Initiallly α = 0, β = α+1.

Now , we are considering points as prescribed above-

(1) WHEN MINIMUM ENTROPY OCCURS AT POINT (A+α, A+β, n): While calculating minimum entropy, we observe that minimum entropy shifts from point (A+α, A+β, n) to one of these points (A+α, A+λ, A+β+1), (1, A+α, A+γ), (A+α, A+μ, n) or (1, A+α, n). Here β >α, A+γ < n, α < λ < β+1, μ≠β. To observe shifting of minimum entropy from point (A+α, A+β, n) to any of above points , first we equate entropy at point (A+α, A+β, n) with entropies at points (1, A+λ, A+β+1) and (1, A+α. A+γ). By equating entropies we obtain some values of s^{th} order moment . Then we check the existence of s^{th} order moment and the minimum value out of these two s^{th} order moments is called switching point. If both calculated values of s^{th} order moments do not lie in the feasible subinterval , we equate entropies at point (A+α, A+β, n) with points (A+α, A+μ, n) and (1, A+α, n). When we get any switching point by equating entropies at points (A+α, A+β, n) & (A+α, A+μ, n), minimum entropy shifts from point (A+α, A+β, n) to point (A+α, A+μ, n). Further minimum entropy shifts from point (A+α, A+μ, n) to point (1, A+α, n) and when we do not get any switching point by equating entropies at points (A+α, A+β, n) & (A+α, A+μ, n) then minimum entropy shifts from point (A+α, A+β, n) to point (1, A+α, n).

Hence, mathematically we equate entropies as following:

S(A+α, A+β, n) = S(A+α, A+λ, A+β+1)

from equations (1),(4),(5),(6)

$$p_1 \ln p_1 + q_1 \ln q_1 + r_1 \ln r_1 = p_2 \ln p_2 + q_2 \ln q_2 + r_2 \ln r_2 \quad \text{- -(10)}$$

We get values of p_1, q_1, r_1 for *h*=A+α, *k*=A+β, *l*=n and values of p_2, q_2, r_2 for *h*=A+α, *k*=A+λ, *l*=A+β+1 from equations (4), (5), (6) respectively.

By solving equation (10), we obtain the value of μ'_s say $(\mu'_s)_A$.

The value of $(\mu'_s)_A$ can be obtained numerically.

Again

S(A+α, A+β, n) = S(1, A+α, A+γ)

from equations (1),(4),(5),(6)

$$p_1 \ln p_1 + q_1 \ln q_1 + r_1 \ln r_1 = p_3 \ln p_3 + q_3 \ln q_3 + r_3 \ln r_3 \quad \text{- -(11)}$$

equation (11) can be solved for value of μ'_s say $(\mu'_s)_B$

We check whether $(\mu'_s)_A$ and $(\mu'_s)_B$ lie in the feasible region or not. When both values lie in the feasible region, then minimum value out of $(\mu'_s)_A$ and $(\mu'_s)_B$ is considered as switching point and if only one value out of these lies in the feasible region, then that value is considered as switching point.

When $(\mu_s')_A$ lies in the feasible region and is minimum, then minimum entropy shifts from point (A+α, A+β, n) to point (A+α, A+λ, A+β+1), hence

$$S_{min} = -p_1 \ln p_1 - q_1 \ln q_1 - r_1 \ln r_1, \text{ for } \mu_s' \leq (\mu_s')_A$$

$$S_{min} = -p_2 \ln p_2 - q_2 \ln q_2 - r_2 \ln r_2, \text{ for } (\mu_s')_A \leq \mu_s'$$

When $(\mu_s')_B$ lies in the feasible region and is minimum. Then minimum entropy shifts from point (A+α, A+β, n) to point (1, A+α, A+γ), so

$$S_{min} = -p_1 \ln p_1 - q_1 \ln q_1 - r_1 \ln r_1, \text{ for } \mu_s' \leq (\mu_s')_B$$

$$S_{min} = -p_3 \ln p_3 - q_3 \ln q_3 - r_3 \ln r_3, \text{ for } (\mu_s')_B \leq \mu_s'$$

If both values of s[th] order moments do not lie in the feasible region then we equate entropies as:

S(A+α, A+β, n) = S(A+α, A+μ, n)

$$p_1 \ln p_1 + q_1 \ln q_1 + r_1 \ln r_1 = p_4 \ln p_4 + q_4 \ln q_4 + r_4 \ln r_4 \quad \text{- -(12)}$$

by solving equation (12) , we can get value $(\mu_s')_C$ (say)

Again, we are equating entropies as:

S(A+α, A+β, n) = S(1, A+α, n)

$$p_1 \ln p_1 + q_1 \ln q_1 + r_1 \ln r_1 = p_5 \ln p_5 + q_5 \ln q_5 + r_5 \ln r_5 \quad \text{- -(13)}$$

by solving equation (13), we get value $(\mu_s')_D$ (say). This value is given as:

$$(\mu_s')_D = \frac{\mu_r'\{n^s - (A+\alpha)^s\} - n^s (A+\alpha)^r + n^r (A+\alpha)^s}{\{n^r - (A+\alpha)^r\}} \quad \text{- -(14)}$$

This value can also be obtained by equating $q_1 = 0$.

When condition $(\mu_s')_P < (\mu_s')_C < (\mu_s')_D$ holds, then minimum entropy shifts from point (A+α, A+β, n) to point (A+α, A+μ, n), where $(\mu_s')_P$ is previous switching point and further from point (A+α, A+μ, n) minimum entropy shifts to point (1, A+α, n). If above condition is not satisfied then minimum entropy shifts from point (A+α, A+β, n) to point (1, A+α, n). Since for $(\mu_s')_C > (\mu_s')_D$ i.e. $(\mu_s')_C$ does not lie in the feasible region, it can not be considered as switching point.

When minimum entropy shifts from point (A+α, A+β, n) to point (A+α, A+μ, n). Then,

$$S_{min} = -p_1 \ln p_1 - q_1 \ln q_1 - r_1 \ln r_1, \text{ for } \mu_s' \leq (\mu_s')_C$$

$$S_{min} = -p_4 \ln p_4 - q_4 \ln q_4 - r_4 \ln r_4, \text{ for } (\mu_s')_C \leq \mu_s'$$

Now, minimum entropy shifts from point (A+α, A+μ, n) to point (1, A+α, n), so

$$S_{min} = -p_4 \ln p_4 - q_4 \ln q_4 - r_4 \ln r_4, \text{ for } (\mu_s')_C \leq \mu_s' \leq (\mu_s')_D$$

$$S_{min} = -p_5 \ln p_5 - q_5 \ln q_5 - r_5 \ln r_5, \text{ for } (\mu_s')_D \leq \mu_s'$$

When minimum entropy shifts from point (A+α, A+β, n) to point (1, A+α, n) and in this situation,

$$S_{min} = -p_1 \ln p_1 - q_1 \ln q_1 - r_1 \ln r_1, \text{ for } \mu_s' \leq (\mu_s')_D$$

$$S_{min} = -p_5 \ln p_5 - q_5 \ln q_5 - r_5 \ln r_5, \text{ for } (\mu_s')_D \leq \mu_s'$$

(2) WHEN MINIMUM ENTROPY OCCURS AT POINT (1, A+α, A+γ): From calculation we observe minimum entropy shifts from point (1, A+α, A+γ) to one of these points (A+α, A+δ, A+ξ), (A+α-1, A+ν, A+γ), (1, A+α, A+Φ), (1, A+Ψ, A+γ), (1, A+γ, n). Here A+γ < n, A+ξ < n, A+ Φ < n, α < δ < ξ , α-1 < ν < γ, α≠ Ψ First we equate entropies at point (1, A+α, A+γ) with points (A+α, A+δ, A+ξ), (A+α-1, A+ ν, A+γ) and (1, A+α, A+Φ). By doing this three values of s[th] order moment are obtained. It is not necessary that three values lie in the feasible region. When all values lie in the feasible region then minimum value out of these is considered as switching point and minimum entropy shifts to the corresponding point. When any one value of s[th] order moment lies in the feasible region then that value is considered as switching point. So, here we equate entropies as:

S(1, A+α, A+γ) = S(A+α, A+δ, A+ξ)

$$p_3 \ln p_3 + q_3 \ln q_3 + r_3 \ln r_3 = p_6 \ln p_6 + q_6 \ln q_6 + r_6 \ln r_6 \quad \text{- -(15)}$$

by solving equation (15), we get value of μ_s' say $(\mu_s')_E$.

Again, we equate entropies as:

S(1, A+α, A+γ) = S(A+α-1, A+ν, A+γ)

$$p_3 \ln p_3 + q_3 \ln q_3 + r_3 \ln r_3 = p_7 \ln p_7 + q_7 \ln q_7 + r_7 \ln r_7 \quad \text{- -(16)}$$

by solving equation (16) we get value of μ_s' say $(\mu_s')_F$.

Again, we equate entropies as:

S(1, A+α, A+γ) = S(1, A+α, A+Φ)

$$p_3 \ln p_3 + q_3 \ln q_3 + r_3 \ln r_3 = p_8 \ln p_8 + q_8 \ln q_8 + r_8 \ln r_8 \quad \text{- -(17)}$$

by solving equation (17), we get value $(\mu_s')_G$(say)

Now, we check the existence of $(\mu_s')_E$, $(\mu_s')_F$ and $(\mu_s')_G$ in the feasible region. If three values lie in the feasible region then minimum value out of these is considered as switching point.

If $(\mu_s')_E$ lies in the feasible region and is minimum then minimum entropy shifts from point (1, A+α, A+γ) to point (A+α, A+δ, A+ξ). So, in this case

$$S_{min} = -p_3 \ln p_3 - q_3 \ln q_3 - r_3 \ln r_3, \text{ for } \mu_s' \leq (\mu_s')_E$$

$$S_{min} = -p_6 \ln p_6 - q_6 \ln q_6 - r_6 \ln r_6, \text{ for } (\mu_s')_E \leq \mu_s'$$

Again, if $(\mu_s')_F$ lies in the feasible region and is minimum then minimum entropy shifts from point (1, A+α, A+γ) to point (A+α-1, A+ν, A+γ). Then,

$$S_{min} = -p_3 \ln p_3 - q_3 \ln q_3 - r_3 \ln r_3, \text{ for } \mu_s' \leq (\mu_s')_F$$

$$S_{min} = -p_7 \ln p_7 - q_7 \ln q_7 - r_7 \ln r_7, \text{ for } (\mu_s')_F \leq \mu_s'$$

And, if $(\mu_s')_G$ lies in the feasible region and is minimum then minimum entropy shifts from point (1, A+α, A+γ) to point (1, A+α, A+Φ). Then,

$S_{min} = -p_3\ln p_3 - q_3\ln q_3 - r_3\ln r_3$, for $\mu_s' \le (\mu_s')_G$

$S_{min} = -p_8\ln p_8 - q_8\ln q_8 - r_8\ln r_8$, for $(\mu_s')_G \le \mu_s'$

When all values $(\mu_s')_E$, $(\mu_s')_F$ and $(\mu_s')_G$ do not lie in the feasible region, then we equate entropies at point (1, A+α, A+γ) with points (1, A+Ψ, A+γ) & (1, A+γ, n) and obtain values of switching points. So, entropies are equated as follow:

S(1, A+α, A+γ) = S(1, A+ Ψ, A+γ)

$$p_3\ln p_3 + q_3\ln q_3 + r_3\ln r_3 = p_9\ln p_9 + q_9\ln q_9 + r_9\ln r_9 \quad \text{- -(18)}$$

by solving equation (18), we get value $(\mu_s')_H$ (say)

Again, entropies are equated as follow :

S(1, A+α, A+γ) = S(1, A+γ, n)

$$p_3\ln p_3 + q_3\ln q_3 + r_3\ln r_3 = p_{10}\ln p_{10} + q_{10}\ln q_{10} + r_{10}\ln r_{10} \quad \text{- -(19)}$$

by solving equation (19), we get value $(\mu_s')_I$ (say). This value can be given as

$$(\mu_s')_I = \frac{\mu_r'\{(A+\gamma)^s - 1\} + (A+\gamma)^r - (A+\gamma)^s}{\{(A+\gamma)^r - 1\}} \quad \text{- -(20)}$$

above expression can also be obtained by equating $q_3 = 0$.

When condition $(\mu_s')_P < (\mu_s')_H < (\mu_s')_I$ holds, minimum entropy shifts from point (1, A+α, A+γ) to point (1, A+ Ψ, A+γ) and further from point (1, A+Ψ, A+γ) to point (1, A+γ, n). So,

$S_{min} = -p_3\ln p_3 - q_3\ln q_3 - r_3\ln r_3$, for $\mu_s' \le (\mu_s')_H$

$S_{min} = -p_8\ln p_8 - q_8\ln q_8 - r_8\ln r_8$, for $(\mu_s')_H \le \mu_s' \le (\mu_s')_I$

$S_{min} = -p_9\ln p_9 - q_9\ln q_9 - r_9\ln r_9$, for $(\mu_s')_I \le \mu_s'$

If condition $(\mu_s')_P < (\mu_s')_H < (\mu_s')_I$ does not hold, then minimum entropy shifts from point (1, A+α, A+γ) to point (1, A+γ, n). In this situation,

$S_{min} = -p_3\ln p_3 - q_3\ln q_3 - r_3\ln r_3$, for $\mu_s' \le (\mu_s')_I$

$S_{min} = -p_{10}\ln p_{10} - q_{10}\ln q_{10} - r_{10}\ln r_{10}$, for $(\mu_s')_I \le \mu_s'$

(3) WHEN MINIMUM ENTROPY OCCURS AT POINT (A+α, A+δ, A+ξ): We observe that minimum entropy shifts from point (A+α, A+δ, A+ξ) to one of these points (A+α, A+σ, A+ξ), (A+α, A+ξ, n) and (1, A+α, A+ξ). Here δ ≠ σ. So, we equate entropies at point (A+α, A+δ, A+ξ) with points (A+α, A+ σ, A+ξ), (A+α, A+ξ, n) and (1, A+α, A+ξ). By doing this we get two values of s[th] order moment. Since the values of s[th] order moment are same by equating entropies at point (A+α, A+δ, A+ξ) with points (A+α, A+ξ, n) and (1, A+α, A+ξ). Minimum value out of these two values of s[th] order moment is considered as switching point. When any one value lies in the feasible region then that value is considered as switching point. Further we are equating entropies as follow:

S(A+α, A+δ, A+ξ) = S(A+α, A+σ, A+ξ)

$$p_6\ln p_6 + q_6\ln q_6 + r_6\ln r_6 = p_{11}\ln p_{11} + q_{11}\ln q_{11} + r_{11}\ln r_{11} \quad \text{- -(21)}$$

by solving equation (21), we get value $(\mu_s')_J$ (say)

Again, entropies are equated as:

S(A+α, A+δ, A+ξ) = S(A+α, A+ξ, n)

$$p_6\ln p_6 + q_6\ln q_6 + r_6\ln r_6 = p_{12}\ln p_{12} + q_{12}\ln q_{12} + r_{12}\ln r_{12} \quad \text{- -(22)}$$

And, entropies are equated as:

S(A+α, A+δ, A+ξ) = S(1, A+α, A+ξ)

$$p_6\ln p_6 + q_6\ln q_6 + r_6\ln r_6 = p_{13}\ln p_{13} + q_{13}\ln q_{13} + r_{13}\ln r_{13} \quad \text{- -(23)}$$

equations (22) and (23) can be solved to get value $(\mu_s')_K$ (say). This value can be given as-

$$(\mu_s')_K = \frac{\mu_r'\{(A+\xi)^s - (A+\alpha)^s\} - (A+\alpha)^r(A+\xi)^s + (A+\alpha)^s(A+\xi)^r}{\{(A+\xi)^r - (A+\alpha)^r\}} \quad \text{- -(24)}$$

The expression for $(\mu_s')_K$ can be obtained by equating $q_6 = 0$.

When condition $(\mu_s')_P < (\mu_s')_J < (\mu_s')_K$ holds, minimum entropy shifts from point (A+α, A+δ, A+ξ) to point (A+α, A+σ, A+ξ) and from this point it shifts to point (1, A+α, A+ξ). Here $(\mu_s')_P$ is previous switching point. In this situation,

$S_{min} = -p_6\ln p_6 - q_6\ln q_6 - r_6\ln r_6$, for $\mu_s' \le (\mu_s')_J$

$S_{min} = -p_{11}\ln p_{11} - q_{11}\ln q_{11} - r_{11}\ln r_{11}$, for $(\mu_s')_J \le \mu_s' \le (\mu_s')_K$

$S_{min} = -p_{12}\ln p_{12} - q_{12}\ln q_{12} - r_{12}\ln r_{12}$ or

$S_{min.} = -p_{13}\ln p_{13} - q_{13}\ln q_{13} - r_{13}\ln r_{13}$, for $(\mu_s')_K \le \mu_s'$

If condition $(\mu_s')_P < (\mu_s')_J < (\mu_s')_K$ does not hold, then minimum entropy shifts from point (A+α, A+δ, A+ξ) to point (A+α, A+ξ, n) or point (1, A+α, A+ξ). In this situation,

$S_{min} = -p_{11}\ln p_{11} - q_{11}\ln q_{11} - r_{11}\ln r_{11}$, for $\mu_s' \le (\mu_s')_K$

$S_{min} = -p_{12}\ln p_{12} - q_{12}\ln q_{12} - r_{12}\ln r_{12}$ or

$S_{min} = -p_{13}\ln p_{13} - q_{13}\ln q_{13} - r_{13}\ln r_{13}$, for $(\mu_s')_K \le \mu_s'$

Now, we calculate entropies at point (A+α, A+ξ, n) & (1, A+α, A+ξ) for $\mu_s' \gtrsim (\mu_s')_K$. If entropy is minimum at point (A+α, A+ξ, n), then entropy shifts from point (A+α, A+δ, A+ξ) to point (A+α, A+ξ, n) and if entropy is minimum at point (1, A+α, A+ξ) then minimum entropy shifts from point (A+α, A+δ, A+ξ) to point (1, A+α, A+ξ).

(4) WHEN MINIMUM ENTROPY OCCURS AT POINT (1, A+α, n):

There are two cases:-

(a) A < n-1 (b) A= n-1, α = 0

(a) When A < n-1

Here, first we equate entropies at point (1, A+α, n) with point (A+α-1, A+α, A+α+1). By equating these entropies we may get a value of s[th] order moment. In this situation, minimum entropy shifts from point (1, A+α, n) to point (A+α -1, A+α, A+α +1). If we do not get any switching point then we equate entropies at point (1,A+α, n) with points (A+α-1, A+Є, A+ α+2), (A+α-1, A+α+1, n), (1, A+α-1, A+α+1) and (A+α-2, A+ω, A+α+1). Here α-1 < Є < α+2, α-2 < ω < α+1. By equating entropies we get four values of s^{th} order moment. We check the existence of these values in the feasible region. Then we search minimum value out of these four values, this value is considered as switching point. If only one value lies in the feasible region then that value is considered as switching point. Now, we are equating entropies as follow:

S(1, A+α, n) = S(A+α-1, A+α, A+α+1)

$$p_5 \ln p_5 + q_5 \ln q_5 + r_5 \ln r_5 = p_{14} \ln p_{14} + q_{14} \ln q_{14} + r_{14} \ln r_{14} \quad \text{- -(25)}$$

equation (25) can be solved for value $(\mu_s')_L$. Now, we check whether this value lies in the feasible region or not. If this value lies in the feasible region, then minimum entropy shifts from point (1, A+α, n) to point (A+α-1, A+α, A+α+1). So,

$$S_{min} = -p_5 \ln p_5 - q_5 \ln q_5 - r_5 \ln r_5, \text{ for } \mu_s' \le (\mu_s')_L$$

$$S_{min} = -p_{14} \ln p_{14} - q_{14} \ln q_{14} - r_{14} \ln r_{14}, \text{ for } (\mu_s')_L \le \mu_s'$$

If $(\mu_s')_L$ does not lie in the feasible region then we equate entropies as follow:

S(1, A+α, n) = S(A+α-1, A+Є, A+α+2)

$$p_5 \ln p_5 + q_5 \ln q_5 + r_5 \ln r_5 = p_{15} \ln p_{15} + q_{15} \ln q_{15} + r_{15} \ln r_{15} \quad \text{- -(26)}$$

this equation can be solved for value $(\mu_s')_M$ (say)

Entropies are equated as:

S(1, A+α, n) = S(A+α-1, A+α+1, n)

$$p_5 \ln p_5 + q_5 \ln q_5 + r_5 \ln r_5 = p_{16} \ln p_{16} + q_{16} \ln q_{16} + r_{16} \ln r_{16} \quad \text{- -(27)}$$

this equation can be solved for value $(\mu_s')_N$ (say).

Entropies are equated as:

S(1, A+α, n) = S(1, A+α-1, A+α+1)

$$p_5 \ln p_5 + q_5 \ln q_5 + r_5 \ln r_5 = p_{17} \ln p_{17} + q_{17} \ln q_{17} + r_{17} \ln r_{17} \quad \text{- -(28)}$$

this equation can be solved for value $(\mu_s')_O$ (say).

Entropies are equated as:

S(1, A+α, n) = S(A+α-2, A+ω, A+α+1)

$$p_5 \ln p_5 + q_5 \ln q_5 + r_5 \ln r_5 = p_{18} \ln p_{18} + q_{18} \ln q_{18} + r_{18} \ln r_{18} \quad \text{- -(29)}$$

this equation can be solved for value $(\mu_s')_Q$ (say)

If $(\mu_s')_M$ lies in the feasible region and is minimum then minimum entropy shifts from point (1, A+α, n) to point (A+α-1, A+Є, A+α +2). So, in this case

$$S_{min} = -p_5 \ln p_5 - q_5 \ln q_5 - r_5 \ln r_5, \text{ for } \mu_s' \le (\mu_s')_M$$

$$S_{min} = -p_{15} \ln p_{15} - q_{15} \ln q_{15} - r_{15} \ln r_{15}, \text{for } (\mu_s')_M \le \mu_s'$$

If $(\mu_s')_N$ lies in the feasible region and is minimum then minimum entropy shifts from point (1, A+α, n) to point (A+α-1, A+α+1, n). So, in this case

$$S_{min} = -p_5 \ln p_5 - q_5 \ln q_5 - r_5 \ln r_5, \text{ for } \mu_s' \le (\mu_s')_N$$

$$S_{min} = -p_{16} \ln p_{16} - q_{16} \ln q_{16} - r_{16} \ln r_{16}, \text{for } (\mu_s')_N \le \mu_s'$$

If $(\mu_s')_Q$ lies in the feasible region and is minimum then minimum entropy shifts from point (1, A+α, n) to point (A+α-2, A+ω, A+α+1). So, in this case

$$S_{min} = -p_5 \ln p_5 - q_5 \ln q_5 - r_5 \ln r_5, \text{ for } \mu_s' \le (\mu_s')_Q$$

$$S_{min} = -p_{18} \ln p_{18} - q_{18} \ln q_{18} - r_{18} \ln r_{18}, \text{for } (\mu_s')_Q \le \mu_s'$$

If values $(\mu_s')_M$, $(\mu_s')_N$, $(\mu_s')_O$ and $(\mu_s')_Q$ do not lie in the feasible region, then we consider shifting of minimum entropy for points (A+α-1, A+Є, A+α+2) and (A+α-2, A+ω, A+α+1), as we have discussed above for point (A+α-1, A+α, A+α+1).

(b) When A = n-1, α = 0: While calculating we observe that minimum entropy shifts from point (1, A+α, n) to point (A+α-1, A+α, n) then from point (A+α-1, A+α, n) to point (1, A+α-1, n). Hence, we obtain switching points. We proceed similarly upto point (1, 2, n) and obtain switching points. Now, we are equating entropies as:

S(1, A+α, n) = S(A+α-1, A+α, n)

$$p_5 \ln p_5 + q_5 \ln q_5 + r_5 \ln r_5 = p_{19} \ln p_{19} + q_{19} \ln q_{19} + r_{19} \ln r_{19} \quad \text{- -(30)}$$

by solving equation (30), we get value of μ_s' say $(\mu_s')_R$.

Further, minimum entropy shifts from point (A+α-1, A+α, n) to point (1, A+α-1, n), then entropies are equated as:

S(A+α-1, A+α, n) = S(1, A+α-1, n)

$$p_{19} \ln p_{19} + q_{19} \ln q_{19} + r_{19} \ln r_{19} = p_{20} \ln p_{20} + q_{20} \ln q_{20} + r_{20} \ln r_{20} \quad \text{- -(31)}$$

by solving equation (31) we get value of μ_s' say $(\mu_s')_S$. This value can also be obtained by equating $q_{19} = 0$.

$$(\mu_s')_S = \frac{\mu_r'\{n^s - (A+\alpha-1)^s\} + n^r (A+\alpha-1)^s - n^s (A+\alpha-1)^r}{\{n^r - (A+\alpha-1)^r\}} \quad \text{- -(32)}$$

Further shifting of minimum entropy will be proceed as with point (1, A+α, n) upto point (1,2,n).

In this way we have observed shifting of minimum entropy. But in the particular range entropies exist only at points (1, A+τ, n), where (A+τ) vary from 2 to n-1. We can obtain this range from equation (5) by $p_k = 0$ at points (2, 3, n) & (1, n-2, n-1) and from equation (9) i.e. $(\mu_s')_{max}$. We get following expressions.

$$\frac{\mu_r'\{n^s - 2^s\} + 2^s n^r - n^s 2^r}{\{n^r - 2^r\}} < \mu_s' < \frac{\mu_r'\{n^s - 1\} + n^r - n^s}{\{n^r - 1\}} \quad \text{- -(33)}$$

and

$$\frac{\mu_r'\{(n-1)^s - 1\} + (n-1)^r - (n-1)^s}{\{(n-1)^r - 1\}} < \mu_s' < \frac{\mu_r'\{n^s - 1\} + n^r - n^s}{\{n^r - 1\}} \quad \text{- -(34)}$$

If these two conditions hold then minimum entropy shifts from point (1, A+α, n) to point (1, A+τ, n), where α ≠ τ.

S(1, A+α, n) = S(1, A+τ, n)

$$p_5 \ln p_5 + q_5 \ln q_5 + r_5 \ln r_5 = p_{21} \ln p_{21} + q_{21} \ln q_{21} + r_{21} \ln r_{21} \quad \text{- -(35)}$$

by solving equation (35) , we get value of μ_s' say $(\mu_s')_T$.

$$S_{min} = -p_5 \ln p_5 - q_5 \ln q_5 - r_5 \ln r_5, \text{ for } \mu_s' \le (\mu_s')_T$$

$$S_{min}= -p_{21}\ln p_{21} - q_{21}\ln q_{21} - r_{21}\ln r_{21}, \text{for } \left(\mu_s'\right)_T \leq \mu_s'$$

We get expressions for p_i, q_i, r_i from equations (4), (5), (6).

So, in this way we can observe the shifting of minimum entropy and obtain the expressions for minimum entropy and switching points. Minimum entropy can be obtained by giving different values to r, s & t for special cases.

III. CONCLUDING REMARKS

We have obtained the expressions of minimum Shannon entropy for the given values of r^{th} and s^{th} order moments. So, we observe that

(1) For given values of $\left(\mu_r'\right)^{1/r}$ & $\left(\mu_s'\right)^{1/s}_{min}$, entropies are same for all existing points and similarly for given values of $\left(\mu_r'\right)^{1/r}$ & $\left(\mu_s'\right)^{1/s}_{max}$, entropies are same for all existing points.

(2) When both moments take discrete and equal values, S_{min} is zero.

(3) Number of switching points are small when $\left(\mu_r'\right)^{1/r}$ is near to 1 & n and number of switching point are large when $\left(\mu_r'\right)^{1/r}$ is far away from 1 & n.

(4) S_{min} is a piecewise concave function.

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