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On Rolf Nevanlinna Prize Winners Collaboration Graph-II

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Abstract- The problem of determining the collaboration graph of co-authors of Paul Erdos is a challenging task. Here we take up this problem for the case of Rolf Nevalinna Prize Winners. Even though the number of these prizewinners as on date is 7, the collaboration graphs has 20 vertices and 41 edges and possess several interesting properties. In this paper we have obtained this graph and determined standard graph parameters for both this graph and its complement besides probing its structural properties. Several new results were obtained.

Index Terms- Collaboration Graph, Erdos, Rolf Nevanlinna Prize Winner Domination, Global Domination, Total Domination, Connected Domination, Strong Domination, Spectrum 2000 Subject Classification: 05C07, 05C12, 05C15, 05C35, 05C38, 05C40, 05C45, 05C90

I. Introduction

In the past decade, graph theory has gone through a remarkable shift and a profound transformation. The change is in large part due to humongous amount of information that are confronted with. A main way to sort through massive data sets is to build and examine the network formed by interrelations. For example, Google's successful web search algorithms are based on the www graph, which contain all WebPages as vertices and hyper links as edges. These are sorts of information networks, such as biological networks built from biological databases and social networks formed by email, phone calls, instant messaging and various other types of physical networks. Of particular interest to mathematicians is the collaboration graph, which is based on the data from Mathematical Reviews. In the collaboration graph, every mathematician is a vertex, and two mathematicians who wrote a joint paper are connected by an edge.

The graph considered in his paper is finite, simple and undirected. For any undefined terms see [1] and [2]. For any graph G, we denote by $V(G)$ and $E(G)$ the vertex set and the edge set of G respectively. The collaboration graph G has as vertices all researchers (dead or alive) from all academic disciplines with an edge joining vertices u and v if u and v have jointly published a paper or book. The distance between two vertices u and v denoted d(u,v), is the number of edges in the shortest path between u and v in case if such a path exists and ∞ otherwise.Clearly d(u,u) = 0. We now consider the collaboration subgraph centered at Paul Erdos (1913-1996). For a researcher v, the number d(Erdos,v) is called the Erdos number of v. That is, Paul Erdos himself has Erdos number 0, and his coauthors have Erdos number 1. People not having Erdos number 0 or 1 but who have published with some one with Erdos number 1 have Erdos number 2, and so on. Those who are not linked in this way to Paul Erdos have Erdos number ∞. The collection of all individuals with a finite Erdos number constitutes the Erdos component of G. 511 people have Erdos number 1, and over 5000 have Erdos number 2. In the history of scholarly publishing in Mathematics, no one has ever matched Paul Erdos number of collaborators or papers (about 1500, almost 70% of which were joint works). Many important people in academic areas other than mathematics proper-as diverse as physics, chemistry, crystallography, economics, finance, biology, medicine, biophysics, genetics, metrology, astronomy, geology, aeronautical engineering, electrical engineering, computer Science, linguistics, psychology and philosophy do indeed have finite Erdos numbers. Also see [4] for more details.

Problem: For the sake of brevity we denote the Rolf Nevanlinna Prize Winners Collaboration Graph by G^*. In this paper we have indicated the method of obtaining this graph. Further we have computed for this graph several interesting parameters like domination number, total domination number, global domination number, total global domination number, k-domination number, connected domination number, strong domination number and spectrum.

1.1 About Rolf Nevanlinna Prize (RNP)

Front side view

Rear side view

The RNP is awarded once in every four years at the International Congress of Mathematicians for outstanding contribution in mathematical aspects of information science that includes:1)computer science areas like complexity theory, logic of programming languages, analysis of algorithms, cryptography, computer vision, pattern recognition, information processing and modeling of intelligence;

computing and numerical analysis; 3)computational aspects of optimization and control theory. The RNP committee is chosen by the executive committee of the International Mathematical Union. The name of the Chair of the committee is made public, but the names of other members of the committee remain anonymous until the award of the prize at the Congress. A candidate's 40th birthday must not occur before January 1st of the year of the congress at which the Prize is awarded. If a former student (Ph.D. thesis only) of a committee member is seriously considered, such a member shall not continue to serve on the committee for its final decision.

1.2 History of the Rolf Nevanlinna Prize (RNP)

The RNP in mathematical aspects of information science was established by the Executive Committee of the International Mathematical Union (IMU) in April 1981. It was decided that the prize should consist of a gold medal and a cash prize similar to the ones associated with the Fields Medal and that one prize should be given at each International Congress of Mathematicians. One year later, in April 1982, the IMU accepted the offer by the University of Helsinki to finance the prize. The prize was named the RNP in honor of Rolf Nevanlinna (1895-1980), who had been Rector of the University of Helsinki and President of the IMU and who in the 1950s had taken the initiative to the computer organization at Finnish universities. On its obverse side, the medal represents Nevanlinna and bears the text "RNP". In addition, there is in very small characters "RH 83", RH refers to the Finnish sculptor Raimo Heino (1932-95) who designed the medal, and 83 to the year 1983 when the first medal was minted. On the reverse side, the two figures are related to the University of Helsinki. On the University's seal in the lower right, the text "Universitas Helsingiensis" is readable. The seal is from the 17th century, except for the Cross of Liberty, which was added to it in 1940. In the upper left part, the word "Helsinki" is in coded form. The name of the prizewinner is engraved on the rim of the medal.

2. Construction of G^*

G^* is constructed as follows: G^* has twenty vertices and forty one edges. $V(G^*) = \{u_1, u_2, \ldots, u_{20}\}$ here u_1 = Paul Erdos, u_2 = Maria Margarat Klawe, u_3 = Siemion Fajtlowicz, u_4 = Robert Robinson, u_5 = George Gunthar Lorentz, u_6 = Endre Szemeredi, u_7 = Laszlo Lovasz, u_8 = Nathan Linial, u_9 = Alon Noga, u_{10} = Boris Aronov, u_{11} = Andrej Ehrenfeucht, u_{12} = Mark Jerrum, u_{13} = Alok Aggarwal, u_{14} = Robert Endre Tarjan, u_{15} = Leslie Valiant, u_{16} = A.A.Razborov, u_{17} = Avi Wigderson, u_{18} = Peter W.Shor, u_{19} = Madhu Sudan, u_{20} = Jon Kleinberg. Note that the chronological orders of prize winners are defined in order by u_j, j =14 to 20, $E(G^*) = \{e_1, e_2, \ldots e_{41}\}$ where $e_1=(u_1,u_2)$, $e_2=(u_1,u_3)$, $e_3=(u_1,u_4)$, $e_4=(u_1,u_5)$, $e_5=(u_1,u_6)$, $e_6=(u_1,u_7)$, $e_7=(u_1,u_8)$, $e_8=(u_1,u_9)$, $e_9=(u_1,u_{10})$, $e_{10}=(u_2,u_8)$, $e_{11}=(u_2,u_{13})$, $e_{12}=(u_2,u_{14})$, $e_{13}=(u_2,u_{17})$, $e_{14}=(u_2,u_{18})$, $e_{15}=(u_3,u_{11})$, $e_{16}=(u_4,u_{12})$, $e_{17}=(u_5,u_{16})$, $e_{18}=(u_6,u_9)$, $e_{19}=(u_6,u_{16})$, $e_{20}=(u_6,u_{17})$, $e_{21}=(u_7,u_8)$, $e_{22}=(u_7,u_9)$, $e_{23}=(u_7,u_{17})$, $e_{24}=(u_7,u_{18})$, $e_{25}=(u_8,u_9)$, $e_{26}=(u_8,u_{13})$, $e_{27}=(u_8,u_{17})$, $e_{28}=(u_8,u_{18})$, $e_{29}=(u_9,u_{10})$, $e_{30}=(u_9,u_{17})$, $e_{31}=(u_9,u_{19})$, $e_{32}=(u_{10},u_{13})$, $e_{33}=(u_{11},u_{15})$, $e_{34}=(u_{12},u_{15})$, $e_{35}=(u_{13},u_{17})$, $e_{36}=(u_{13},u_{18})$, $e_{37}=(u_{13},u_{19})$, $e_{38}=(u_{13},u_{20})$, $e_{39}=(u_{16},u_{17})$, $e_{40}=(u_{17},u_{19})$, $e_{41}=(u_{19},u_{20})$. None of the seven RNPW'S have Erdos number 1. Out of the 511 direct co-authors of Paul Erdos, with Erdos Number 1, only Nine members are connected by a path of length 1 or 2 with the RNPW'S. Out of the seven RNPW'S only five members namely $u_{14}, u_{16}, u_{17}, u_{18}, u_{19}$ have Erdos number 2, the remaining members namely u_{15}, u_{20} have Erdos number 3. G^* is shown in Figure 1.

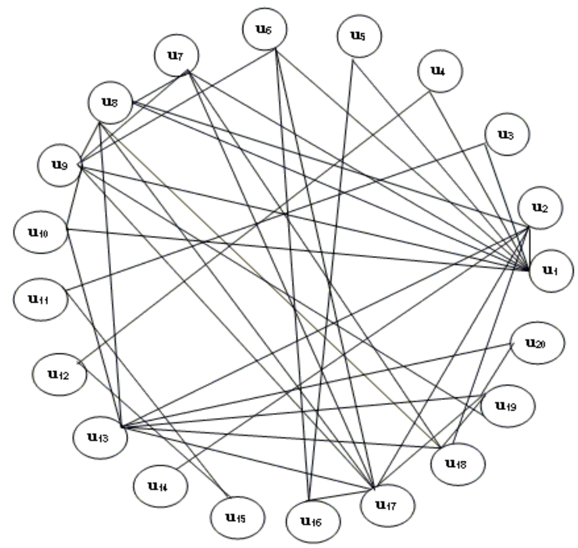
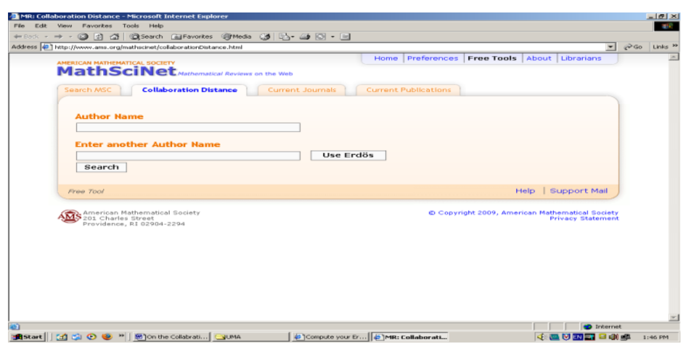


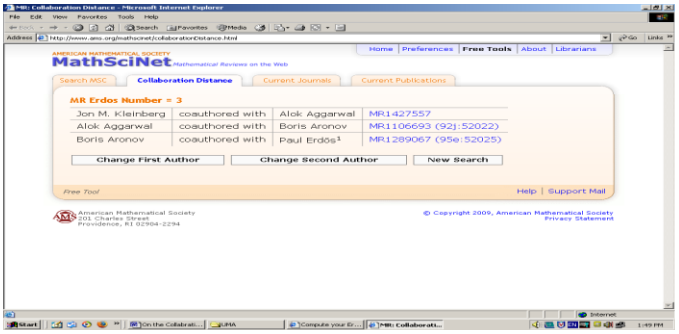
Figure 1: G*

The method of obtaining the G^* is described as follows:
Step1: Click on the link: http://www.ams.org/ mathscinet/ collaborationDistance.html

The result of step 1 is the following screen:



Step 2: Enter the Author name and Enter another author name or click on the use Erdos icon. For example, if the author name is: Jon.M. Kleinberg and another author name are: Paul Erdos then we obtain the following screen:



To know more details about the joint work of these authors, just click on the respective MR number. For example, if we click on MR1427557 then we derive the following screen:

Proceeding like this, one can obtain all the seven RNPW'S collaboration details one by one. Since the number of RNPW'S is a small number, the above procedure is recommended. It is vital to record a fact that, if there is no co author relationship at all between two persons say X and Y, then the result of our action of doing the Step 2 will be: "No path found". We have thoroughly checked all possible combinations. That is, first, we have checked the co author relationship between any of the RNPW'S with any of the 9 applicable co-authors at level 1 with Erdos number 1. This action leads to 7x9=63 combinations. Then we have looked for the same among 5 of the RNPW'S having Erdos Number 2. This leads to 5(5-1)/2=10 combinations. Next we repeated the same for 2 of the RNPW'S at level 3 with Erdos no 3. This leads to 2x5+2x3+1x1=17 combinations. Also we have ascertained the coauthor relationship of the non RNPW'S at level 2 having Erdos number 2 with any of the 5 RNPW'S at the same level having Erdos number 2 and also between the non RNPW'S. This leads to (3x5)+3(3-1)/2=18 combinations. A scrupulous implementation of the above said procedure has led to the graph **G*** in Figure1.One can see [9,10] for more about G*.

3. Domination and Total Domination

A set S of vertices in a graph G =(V,E) is a dominating set of G if every vertex in V-S is adjacent to some vertex in S. The domination number $\gamma(G)$ of G is the minimum cardinality of a dominating set. A total dominating set S of G is a dominating set such that the induced subgraph $\langle S \rangle$ has no isolates, or if any vertex v of G is adjacent to at least one vertex of the set S. The total domination number $\gamma_T(G)$ of G is the minimum cardinality of a total dominating set. The maximum degree of a graph *G*, denoted by $\Delta(G)$.

Theorem 1 $\gamma(G^*)=5$

Proof First we claim that no dominating set of G*can have cardinality 4. Suppose not, then it means there exists a dominating set S with cardinality exactly 4. The degree distribution of vertices of G* reveals the fact that S should contain u_1 as $\deg(u_1)=\Delta(G^*)=9$, either u_2 or u_{14} as $\deg(u_{14})=1$. Since $\deg(u_2)>\deg(u_{14})$, it is clear that $u_2 \in S$ .Now look at all the 2-degree vertices of G^*. They are $u_3,u_4,u_5,u_{11}, u_{12},u_{15},u_{20}$. We find no single vertex of G* adjacent with all the above given 2-degree vertices, to be included as, the third choice of element of S. However, as u_{11},u_{12} are both adjacent only to u_{15}, we allow u_{15} as the third preferred element of S. This is because, if we did not allow u_{15} as an element of S, then, in order to accommodate u_{11},u_{12}, we need to include u_3,u_4 in S, in which case, $S=\{u_1,u_2,u_3,u_4\}$ and there exists an element u_{16} which is not adjacent to any element of S. So $S=\{u_1,u_2,u_{15}\}$. Now the fourth element of S can be any one of the remaining seventeen vertices of G*. But it is clear that, none of the remaining 2-degree vertices can be an element of S. Therefore, we have the number of choices reduced from seventeen to eleven. But obviously u_{14} cannot be the fourth element of S. So it is enough to consider only the vertices $u_6,u_7,u_8,u_9,u_{10},u_{13},u_{16},u_{17},u_{18}$ and u_{19}. Now as $\deg(u_8)=\deg(u_9)=\deg(u_{13})=7$, they are more privileged to be the fourth element. However, as (u_8,u_{16}), (u_9,u_{16}), $(u_{13},u_{16}) \notin E(G^*)$ implies that $u_i \notin S$, for $i=8,9,13$. Again as (u_7,u_{16}), (u_{10},u_{16}), (u_{18},u_{16}), $(u_{19},u_{16}) \notin E(G^*)$ we have that $u_i \notin S$ for $i=7,10,18,19$. That is, we have now only two choices, viz., u_6 and u_{17} .Suppose $S=\{u_1,u_2,u_6,u_{15}\}$ then we find u_{19} not adjacent with any element of S. So, $u_6 \notin S$. If $S=\{u_1,u_2, u_{15},u_{17}\}$ then we find u_{20} not adjacent with any element of S. That is, $u_{17} \notin S$. Hence it follows that a four element subset cannot be a dominating set of G*. This means that a probable dominating set of G* must contain at least 5 elements.

Now let S be a subset of V(G) and $v \in S$ .Then the Private Neighbourhood of v with respect to the set S denoted by $\Pr[v,S]$ is defined as :

$\Pr[v,S] = \{w \in V(G) : N(w) \cap S = \{v\}\}$. Note that 1) if $w \in V(G)-S$ and w is adjacent to only $v \in S$, then $w \in \Pr[v,S]$, 2) if $w \in S$ and $w \neq v$, then $w \notin \Pr[v,S]$, 3)

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If $w = \mathrm{v}$ is not adjacent to any vertex of S, then $w \in \Pr[\mathrm{v}, \mathrm{S}]$ .Haynes, Hedetniemi and Slater[3] have showed that a dominating set in a graph G is a minimal dominal set if and only if if for every vertex $\mathrm{v} \in S, \Pr[\mathrm{v}, \mathrm{S}] \neq \phi$. Now consider the set $S_1 = \{u_1, u_{13}, u_{14}, u_{15}, u_{16}\}$, a subset of $V(G^*)$. As (u_i, u_1) for $i = 2,3,...,10$; (u_i, u_{15}), for $i = 11,12$; (u_i, u_{13}), for $i = 17,18,19,20$ are all edges of $G*$, we see that S_1 is a dominating set of G*.We claim that S_1 is a minimal dominating set. As $N(u_9) \cap S_1 = \{u_1\}$ we see that $u_9 \in \Pr[u_1, S_1]$ and hence $\Pr[u_1, S_1] \neq \phi$. Further, u_j is not adjacent to any vertex of S_1 for $j = u_1, u_{13}, u_{14}, u_{15}, u_{16}$. Therefore $\Pr[u_j, S_1] \neq \phi$ for $j = u_1, u_{13}, u_{14}, u_{15}, u_{16}$ and S_1 is a minimal dominating set. Next we claim that G* has no minimum dominating set. That is, we show that $S_2 = \{u_1, u_2, u_{13}, u_{15}, u_{17}\}$ is another minimal dominating set. S_2 is a dominating set, since , (u_i, u_1) for $i = 2,3,...,10$; (u_i, u_{15}) for $i = 11,12$; (u_{14}, u_2); (u_{16}, u_{17}); (u_i, u_{13}) for $i = 18,19,20$ are all edges of G*. Moreover, $u_5 \in \Pr[u_1, S_2]$; $u_{14} \in \Pr[u_2, S_2]$; $u_{20} \in \Pr[u_{13}, S_2]$; $u_{12} \in \Pr[u_{15}, S_2]$; $u_6 \in \Pr[u_{17}, S_2]$ and hence $\Pr[u_i, S_2] \neq \phi$ for all $i \in S_2$. Therefore, S_2 is a minimal dominating set.Hence $\gamma(G*.) = |\square| \ (\text{or} = |S_2|) = 5$.

Note1 It is interesting to note that even though $\deg(u_{14}) < \deg(u_2)$, u_{14} can also become an element of a dominating set in general and a minimal dominating set in particular.

Theorem 2 $\gamma_T(G*) = 6$

Proof By Theorem 1, we have seen that a dominating set of G* should have at least five elements. Now we claim that any total dominating set of G* must have at least six elements. To see this, let us first start with an arbitrary set $S \subseteq V(G*)$ with indispensable elements as dictated by the structure of G*. By Theorem 1, the compulsory elements of G* are $\{u_1, u_{13}, u_{14}, u_{15}\}$ (or) $\{u_1, u_2, u_{13}, u_{15}\}$.Suppose $S = \{u_1, u_2, u_{13}, u_{15}\}$ then the all possible dominating sets are $S_1 = \{u_1, u_2, u_5, u_{13}, u_{15}\}, S_2 = \{u_1, u_2, u_6, u_{13}, u_{15}\}$, $S_3 = \{u_1, u_2, u_{13}, u_{15}, u_{16}\}$ and $S_4 = \{u_1, u_2, u_{13}, u_{15}, u_{17}\}$. This is because $\{u_1, u_2, u_j, u_{13}, u_{15}\}$ is not a dominating set as $(u_j, u_{16}) \notin E(G*)$ for the following different possible combinations: $j = (u_1, u_2, u_3, u_{13}, u_{15})$ (or) $(u_1, u_2, u_4, u_{13}, u_{15})$ (or) $(u_1, u_2, u_7, u_{13}, u_{15})$ (or) $(u_1, u_2, u_8, u_{13}, u_{15})$ (or) $(u_1, u_2, u_9, u_{13}, u_{15})$ (or) $(u_1, u_2, u_{10}, u_{13}, u_{15})$ (or) $(u_1, u_2, u_{11}, u_{13}, u_{15})$ (or) $(u_1, u_2, u_{12}, u_{13}, u_{15})$ (or) $(u_1, u_2, u_{18}, u_{13}, u_{15})$ (or) $(u_1, u_2, u_{19}, u_{13}, u_{15})$ (or) $(u_1, u_2, u_{20}, u_{13}, u_{15})$. Again if $S = \{u_1, u_{13}, u_{14}, u_{15}\}$ then the all possible dominating sets are $S_5 = \{u_1, u_{13}, u_{14}, u_{15}, u_5\}, S_6 = \{u_1, u_{13}, u_{14}, u_{15}, u_6\}$, $S_7 = \{u_1, u_{13}, u_{14}, u_{15}, u_{16}\}$ and $S_8 = \{u_1, u_{13}, u_{14}, u_{15}, u_{17}\}$ for the same reason given above. In view of this, the total number of distinct dominating sets possible for G* is eight. It is easy to see that none of these eight dominating sets can be a total dominating set. This is because, the u_{15} is not adjacent with any of the elements of any of these eight dominating sets. Hence we infer that a total dominating set of G* must have at least six elements.

Now by the definition of a total dominating set, we infer that every element in the total dominating set S must be adjacent with at least one element of the S. The presence of u_{15} as an indispensable element in the construction of a total dominating set reveals that the fact that the sixth element must be either u_{11} or u_{12}. Hence the all possible total dominating sets of G* are $T_1 = \{u_1, u_2, u_{13}, u_{15}, u_5, u_{11}\}, T_2 = \{u_1, u_2, u_{13}, u_{15}, u_5, u_{12}\}, T_3 = \{u_1, u_2, u_{13}, u_{15}, u_6, u_{11}\}$, $T_4 = \{u_1, u_2, u_{13}, u_{15}, u_6, u_{12}\}, T_5 = \{u_1, u_2, u_{13}, u_{15}, u_{16}, u_{11}\}$, $T_6 = \{u_1, u_2, u_{13}, u_{15}, u_{16}, u_{12}\}$, $T_7 = \{u_1, u_2, u_{113}, u_{15}, u_{17}, u_{11}\}$ and $T_8 = \{u_1, u_2, u_{13}, u_{15}, u_{17}, u_{12}\}$. let G be a graph and S a subset of $V(G)$. For $\mathrm{v} \in S$,the total private neighbourhood of v with respect to S in G ,denoted $T\Pr[\mathrm{v}, S]$ is defined as: $T\Pr[\mathrm{v}, S] = \{w \in V(G): N[w] \cap S = \{\mathrm{v}\}\}$.Note that, 1)if $w \in V(G) - \{\mathrm{v}\}$ and w is adjacent to only $\mathrm{v} \in S$,then $w \in T\Pr[\mathrm{v}, S]$ .2)if $w = \mathrm{v} \in S$,then $w \notin T\Pr[\mathrm{v}, S]$ .we claim that a total dominating set S in G is a minimal total dominating set if and only if for every vertex $\mathrm{v} \in S, T\Pr[\mathrm{v}, S] \neq \phi$ .Let S be a minimal total dominating set in G and $\mathrm{v} \in S$ be any arbitrary vertex. So there exists a $w \in V(G)$ such that w is not adjacent to any vertex $S - \mathrm{v}$ .If $w = \mathrm{v}$,then w is not adjacent to any vertex of S and in which case, S will turn out to be a non total dominating set, a contradiction. Now let $w \neq \mathrm{v} \in V(G)$.As S is a total dominating set and w is adjacent to only v in S , we see that $w \in T\Pr[\mathrm{v}, S]$ .that is, $T\Pr[\mathrm{v}, S] \neq \phi$ for any $\mathrm{v} \in S$ .Conversely, suppose that S is a total dominating set in G for any vertex $\mathrm{v} \in S$, $T\Pr[\mathrm{v}, S] \neq \phi$. Let $S_1 = S - \{\mathrm{v}\}$ and

$w \in T\,\Pr[\mathrm{v}, S]$. Then $w \neq \mathrm{v}$ is adjacent to only v in S. Also w is not adjacent to any vertex of S_1. That is, S_1 is not a total dominating set in G. This is true, of course, for any vertex v of S. Hence S is a minimal total dominating set in G. Clearly $S = \{u_1, u_2, u_{11}, u_{13}, u_{15}, u_{17}\}$ is a minimum total dominating set. This is because (u_i, u_1) for $i = 3,\ldots,10$; (u_{12}, u_{15}); (u_{14}, u_2); (u_{16}, u_{17}); (u_i, u_{13}) for $i = 18,19,20$; (u_1, u_2); (u_2, u_{17}); (u_{11}, u_{15}); (u_{13}, u_{17}) are all edges of G^* or the subgraph induced by set S has no isolates. Further note that u_4 is adjacent to only $u_1 \in S$; u_{13} is adjacent to $u_2 \in S$; u_{15} is adjacent to only $u_{11} \in S$; u_{19} is adjacent to only $u_{13} \in S$; u_{12} is adjacent to only $u_{15} \in S$; u_{16} is adjacent to only $u_{17} \in S$. This shows that $T\,\Pr[u_i, S] \neq \phi$ for all $u_i \in S$. For the same reason we infer that $S_1 = \{u_1, u_2, u_5, u_{11}, u_{13}, u_{15}\}$ is a minimal total dominating set in G^*. Therefore we infer that G* has no minimum total dominating set. Hence□e conclude that $\gamma_T(G^*) = |S| = 6$.

Note 2 In Theorem 1, we have seen two dominating sets. In the course of the proof of Theorem 2, we have decisively found all the dominating sets of G*.

Note3 We have found in the course of the proof of Theorem 2 all the minimal total dominating sets of G*. They are $T_1 = \{u_1, u_2, u_{13}, u_{15}, u_5, u_{11}\}$, $T_2 = \{u_1, u_2, u_{13}, u_{15}, u_5, u_{12}\}$, $T_3 = \{u_1, u_2, u_{13}, u_{15}, u_6, u_{11}\}$, $T_4 = \{u_1, u_2, u_{13}, u_{15}, u_6, u_{12}\}$, $T_5 = \{u_1, u_2, u_{13}, u_{15}, u_{16}, u_{11}\}$, $T_6 = \{u_1, u_2, u_{13}, u_{15}, u_{16}, u_{12}\}$, $T_7 = \{u_1, u_2, u_{13}, u_{15}, u_{17}, u_{11}\}$ and $T_8 = \{u_1, u_2, u_{13}, u_{15}, u_{17}, u_{12}\}$.

Let us present here an algorithm to find a dominating set of any graph G

Algorithm

1.Pick a vertex $u \in V(G)$ and color it A.
2.Color all uncolored neighbours of all vertices with color A with the color B.
3.Color all uncolored neighbours of all vertices with color B with the color A.
4.If there are uncolored vertices,go to step 2.
5.Let $S_A = \{u \in V(G) : color(u) = A\}$ and $S_B = \{u \in V(G) : color(u) = B\}$.If $|S_A| > |S_B|$ then $S = S_B$; else $S = S_A$. S_A is the set of vertices with color A, S_B is the set of vertices with color B. The resulting dominating set is not necessarily minimal but $|S| \leq n/2$, where $n = V(G)$. We use the above algorithm and construct a dominating set of G^*. Pick u_1 and let $col(u_1) = A$ where $c : V(G^*) \to \{A, B\}$ is a color function. So initially $S_A = \{u_1\}$. Assign all the neighbours of u_1 with color B. As $N(u_1) = \{u_i : 2 \leq i \leq 10\}$, we get $S_B = \{u_2, u_3, u_4, u_5, u_6, u_7, u_8, u_9, u_{10}\}$. Now color all the uncolored neighbours of all the elements of S_B with the color A. Then the initial S_A gets revised to $S_A = \{u_1, u_{11}, u_{12}, u_{13}, u_{14}, u_{16}, u_{17}, u_{18}, u_{19}\}$. Now color all the uncolored neighbours of all the elements of S_A with the color B. This gives revised S_B with $S_B = \{u_2, u_3, u_4, u_5, u_6, u_7, u_8, u_9, u_{10}, u_{15}, u_{20}\}$. As $|S_B| > |S_A|$,we get $S = S_A$ and it is a dominating set. Also note that $|S_A| = 9 < 10(= 20/2)$.

We observe that the above algorithm is not an efficient one to produce a minimal dominating set. However, for networks of enormous size**,** this algorithm proves useful in producing an initial dominating set, which can be pruned later into a minimal one by employing other heuristic or greedy approaches.

4. Global Domination and Total Global Domination

A dominating set S of G is a global dominating set if S is also a dominating set of G^C.The global domination number $\gamma_g(G)$ of G is the minimum cardinality of a global dominating set. A global dominating set S of G is a total global dominating set, if S is also a total dominating set of G^C.The total global domination number $\gamma_{tg}(G)$ is the minimum cardinality of a total global dominating set. Note that $\gamma(G)$ and $\gamma_g(G)$ are defined for any graph G ,while $\gamma_t(G)$ is defined only for those G with $\delta(G) \geq 1$ and $\gamma_{tg}(G)$ is defined only for those G with $\delta(G) \geq 1$ and $\delta(G^C) \geq 1$.

Theorem 3 $\gamma_g(G^*) = 5$

Proof In the course of the proof of Theorem 2, we have found all the dominating sets of G*. They are $S_1 = \{u_1, u_2, u_5, u_{13}, u_{15}\}$, $S_2 = \{u_1, u_2, u_6, u_{13}, u_{15}\}$, $S_3 = \{u_1, u_2, u_{13}, u_{15}, u_{16}\}$, $S_4 = \{u_1, u_2, u_{13}, u_{15}, u_{17}\}$, $S_5 = \{u_1, u_{13}, u_{14}, u_{15}, u_5\}$, $S_6 = \{u_1, u_{13}, u_{14}, u_{15}, u_6\}$, $S_7 = \{u_1, u_{13}, u_{14}, u_{15}, u_{16}\}$, $S_8 = \{u_1, u_{13}, u_{14}, u_{15}, u_{17}\}$. As $(u_{13}, u_j) \in E(G^*)^C$ for $j = 3,4,6,7,9,11$; $(u_{14}, u_j) \in E(G^*)^C$ for $j = 8,10,13,16,17,18,19,20$; we find that S_1 is a global dominating set. Moreover, we find S_j is a global dominating set

□

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for j=1 to 8. In view of this, we infer that there is no minimum global dominating set. Hence $\gamma_g(G^*) = 5$.

Theorem 4 $\gamma_{tg}(G^*) = 6$

Proof In the course of the proof of Theorem 2, we have found all the total dominating sets of G*.They are $T_1 = \{u_1, u_2, u_{13}, u_{15}, u_5, u_{11}\}, T_2 = \{u_1, u_2, u_{13}, u_{15}, u_5, u_{12}\}, T_3 = \{u_1, u_2, u_{13}, u_{15}, u_6, u_{11}\}$, $T_4 = \{u_1, u_2, u_{13}, \{u_{15}, u_6, u_{12}\}$, $T_5 = \{u_1, u_2, u_{13}, u_{15}, u_{16}, u_{11}\}$, $T_6 = \{u_1, u_2, u_{13}, u_{15}, u_{16}, u_{12}\}, T_7 = \{u_1, u_2, u_{13}, u_{15}, u_{17}, u_{11}\}$ and $T_8 = \{u_1, u_2, u_{13}, u_{15}, u_{17}, u_{12}\}$.Consider the total dominating set $T_7 = \{u_1, u_2, u_{11}, u_{13}, u_{15}, u_{17}\}$ of G^*. We claim that T_7 is also a total global dominating set of G*.As (u_1, u_i) for $i = 11,13,14,17,18$; (u_2, u_i) for $i = 3,4, 5,6,7,9,10,12,15,16,19,20$; (u_8, u_{11}); are all edges of $(G^*)^C$, the claim follows. By Theorem 2, $\gamma_t(G^*) = 6$ with T_7 as the minimum total dominating set. Moreover, we find T_j is a global total dominating set forj=1 to 8. In view of this, we infer that there is no minimum total global dominating set and $\gamma_{tg}(G^*) = 6$.□

Theorem 5 The global dominating set and total global dominating set of G* are distinct with different cardinality.

Proof In [1], we proved that $diam(\overline{G^*}) = 3$.Kulli and Janakiram [5]have showed that if G and G^C both have no isolated vertices and $diam(G) \geq 5$ then a set $S \subseteq V(G)$ with $\delta(\langle S \rangle) \geq 1$is a global dominating set if and only if S is a total global dominating set. Here both $G^*$and $(G^*)^C$ have no isolated vertices and $diam(G^*) = 3$. This implies that G^* has global dominating set distinct from its total global dominating set. This result was further confirmed by Theorem 3 and Theorem 4.

Theorem 6 It is not necessary that diameter of a graph should be at least 5 to have the same total dominating set and total global dominating set.

Proof Kulli and Janakiram [5] have showed that for a graph G with $diam(G) \geq 5$,a set $S \subseteq V(G)$ is a total dominating set iff S is a total global dominating set. But we have proved in Theorem.4,that $\gamma_{tg}(G^*) = 6$ and $\gamma_t(G^*) = 6$ with the same set $S = \{u_1, u_2, u_{11}, u_{13}, u_{15}, u_{17}\}$serving as a total global dominating set and a total dominating set. Also in [1],we have proved that $diam(G^*) = 3$.Hence the result follows. □

Observation 1 Kulli and Janakiram [5] have proved: A total dominating set S of G is a total global dominating set iff for each $v \in V$ there exists a vertex $u \in S$ such that v is not adjacent to u .The graph G^*satisfies this result. This can be seen by noticing that for u_i, $i = 3,4,5,6,7,11,12,15,16$ there exists a vertex $u_2 \in S = \{u_1, u_2, u_{11}, u_{13}, u_{15}, u_{17}\}$ and for u_i, $i = 1,2,8,9,10,13,14,17,18,19,20$ there exists a vertex $u_{11} \in S$ such that u_i are not adjacent to u_2, u_{11} for respective $i's$. Also we have established by Theorem 2 and 4 above that S is the same for both total domination and total global domination.

Observation 2 Kulli and Janakiram [5] have proved : If G is a graph such that neither itself nor its complement has an isolated vertex, then $\gamma_{tg}(G) = |V(G)|$ iff G is isomorphic to one of P_4 , $mK_2, (mK_2)^C, m \geq 2$ .As G^* is different from these graphs we conclude that $\gamma_{tg}(G^*) \neq |V(G^*)| = 20$ .Moreover we have proved in Theorem 4 that $\gamma_{tg}(G^*) = 6$.

5. Connected Domination

Sampath Kumar and Walikar[6]defined a connected dominating set S to be a dominating set S whose induced subgraph is connected. Since a dominating set of a graph G must contain at least one vertex from each component of G , it follows that only connected graphs have connected dominating set. A connected dominating set S is said to be a minimal connected dominating set of G if and only if for any $v \in S$, $S - v$ is not a connected dominating set.

Theorem 7 $\gamma_C(G^*) = 7$

Proof Let S be a connected dominating set in G and $v \in S$ .Then note that 1) if $w \in V(G) - S$ and w is adjacent to only $v \in S$, then $w \in \Pr[v, S]$ 2)if $w \in S$ and S contains at least two vertices, then $w \notin \Pr[v, S]$,3)if $S = \{v\}$,then $v \in \Pr[v, S]$.

First we claim that a connected dominating set S in G is a minimal connected dominating set if and only if at least one of the following conditions are satisfied by every vertex $v \in S$.(i) $\Pr[v, S] \neq \phi$,(ii) v is a cut vertex of S .Let S be a minimal connected dominating set in G and $v \in S$ .Then, $S - v$ is not a connected dominating set. So, either the subgraph induced by $S - v$ is not connected or $S - v$ is not a dominating set.

Case-1 The subgraph induced by $S - v$ is connected. In this case, v is a cut-vertex of S .

Case-2 $S - v$ is not a dominating set in G .In this case, there exists a vertex $w \in V(G) - \{S - v\}$ such that w is not adjacent to any vertex of $S-v$.That is., w is adjacent to only one vertex of S.If $w=v$,then w is not adjacent to every vertex of S and therefore the subgraph induced by S is not connected, a contradiction. So, $w \neq v$ and w is adjacent to only $v \in S$.This means $w \in \Pr[v,S]$.That is., $\Pr[v,S] \neq \phi$.Conversely, suppose that S is a connected dominating set in G and at least one of the following conditions are satisfied by every vertex $v \in S$,viz.,(i) $\Pr[v,S] \neq \phi$,(ii) v is a cut vertex of S.Now two cases arise.

Case-1 v is not a cut vertex of S.Then the sub graph induced by $S-v$ is connected. That is, S contains at least two vertices. Let $w \in \Pr[v,S]$.Then $w \notin S$ and w is adjacent to only v in S. Let $S_1 = S - v$.Then S_1 is not a dominating set in G.Thus, if a vertex $v \in S$ is not a cut-vertex of S, then $S-v$ is not a connected dominating set.

Case-2 v is not a cut vertex of S and the sub graph induced by $S-v$ is not connected. In this case, $S-v$ is not a connected dominating set. To sum up, in all the above cases, we have that for any $v \in V(G)$, $S-v$ is not a connected dominating set. Hence S is a minimal connected dominating set in G.

By Theorem 1 and Theorem 2 it follows that the dominating sets of G* are $S_1 = \{u_1, u_2, u_5, u_{13}, u_{15}\}$, $S_2 = \{u_1, u_2, u_6, u_{13}, u_{15}\}$, $S_3 = \{u_1, u_2, u_{13}, u_{15}, u_{16}\}$, $S_4 = \{u_1, u_2, u_{13}, u_{15}, u_{17}\}$ $S_5 = \{u_1, u_{13}, u_{14}, u_{15}, u_5\}$, $S_6 = \{u_1, u_{13}, u_{14}, u_{15}, u_6\}$, $S_7 = \{u_1, u_{13}, u_{14}, u_{15}, u_{16}\}$ and $S_8 = \{u_1, u_{13}, u_{14}, u_{15}, u_{17}\}$. First we establish a connected dominating set must contain at least 7 elements. Now as $(u_{15}, u_i) \in E(G^*)$ only for i=11,12, the sixth element of S_j, j=1 to 8 can be either u_{11}, u_{12}.Suppose that u_{11} is the sixth element of S_j, j=1 to 8, We see that u_{11} and u_{15} constitutes a separate component, K_2, in G* and hence S_j does not induce a connected sub graph of G* for j=1 to 8. For a similar reason, u_{12} also cannot be a sixth element of S_j for j=1 to 8.This analysis therefore reveals that a connected dominating set of G* must contain at least 7 elements. As a possible connected dominating set should contain 7 elements, we enumerate the number of choices for the sixth and seventh positions. Out of 15 elements, two elements can be chosen in $\binom{15}{2}$ ways. But u_{11} or u_{12} must be an element of a connected dominating set by virtue of the fact that u_{15} is adjacent only with u_{11} or u_{12}. In view of this the possible choices for a dominating set to be a connected dominating set are: $T_j = S_j \cup \{u_{11}\}$ or $Q_j = S_j \cup \{u_{12}\}$ for j=1 to 8. Next we determine the possible choices for the seventh position of both T_j and Q_j:As u_{11} and u_{15} together contributes a component K_2, u_l cannot be the seventh element of T_j, for l=4,6 to 10,12,14,16 to 20. That is, we have only one choice for the seventh position, namely, the element u_3.This implies the existence of a connected dominating set, $T_j' = T_j \cup \{u_3\}$ for j=1,2,4. For the rest of the T_j' for j=3,5,6,7,8, we observe that either (u_{11}, u_{15}) constitutes a separate component or we find an isolated vertex. Specifically, for T_3' either u_{16} is isolated or (u_{11}, u_{15}) a separate component; for T_m' either (u_{11}, u_{15}) is a separate component or u_{14} is an isolated vertex for m=5 to 8 .Similarly, as u_{12} and u_{15} together constitutes a component K_2, u_l cannot be the seventh element of Q_j, for l=3,6 to 10,12,14,16 to 20.That is, we have u_{14} as the only choice for the seventh position. This yields a another set of connected dominating sets, $Q_j' = Q_j \cup \{u_4\}$ for j=1,2,4.For the rest of the Q_j' for j=3,5,6,7,8, we observe that either (u_{12}, u_{15}) constitutes a separate component; for Q_m' either (u_{12}, u_{15}) is a separate component or u_{14} is an isolated vertex for m=5 to 8. Therefore the connected components are exactly six and they are: $T_1' = \{u_1, u_2, u_5, u_{13}, u_{15}, u_{11}, u_3\}$, $T_2' = \{u_1, u_2, u_6, u_{13}, u_{15}, u_{11}, u_3\}$, $T_4' = \{u_1, u_2,, u_{13}, u_{15}, u_{17}, u_{11}, u_3\}$, $Q_1' = \{u_1, u_2, u_5, u_{13}, u_{15}, u_{12}, u_4\}$, $Q_2' = \{u_1, u_2, u_6, u_{13}, u_{15}, u_{12}, u_4\}$, $Q_4' = \{u_1, u_2, u_5, u_{13}, u_{15}, u_{17}, u_{12}, u_4\}$. Finally we determine by using the above criteria for a minimal connected dominating set, which of the T_j' and Q_j' are minimal connected dominating sets for j=1,2,4. Consider T_j' for j=1,2.Here u_1, u_2, u_3, u_{11} are cut vertices and as $u_{16} \in \Pr[u_5, T_1']$, $u_{16} \in \Pr[u_6, T_2']$, $u_{20} \in \Pr[u_{13}, T_j']$ and $u_{12} \in \Pr[u_{15}, T_j']$ we infer that T_j' is a minimal connected dominating set for j=1,2. For T_4', u_2, u_3 and u_{11} are cut vertices and $u_4 \in \Pr[u_1, T_4']$, $u_{20} \in \Pr[u_{13}, T_4']$, $u_{12} \in \Pr[u_{15}, T_4']$, $u_{16} \in \Pr[u_{17}, T_4']$ and hence T_4' is also a minimal connected dominating set. For Q_4', u_2, u_4, u_{12}, u_{13} are cut vertices and as $u_5 \in \Pr[u_1, Q_4']$, $u_{11} \in \Pr[u_{15}, Q_4']$, $u_{16} \in \Pr[u_{17}, Q_4']$ and hence Q_4' is also a minimal connected dominating sets hence it turns out that all connected dominating sets are minimal

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connected dominating sets. Clearly there is no minimum connected dominating set. Hence $\gamma_C(G^*) = 7$. □

6. k -Domination

The concept of k-Domination is stronger than the concept of Domination. There are dominating sets which are not k-dominating for $k \geq 2$.Let G be a graph and k be a positive integer. A subset S of $V(G)$ is said to be a k-dominating set in the graph G if every vertex $v \in V(G) - S$ is adjacent to at least k vertices of S.A k-dominating set S in G is said to be minimal k-dominating set if for any $v \in S$, $S - v$ is not a k-dominating set. A k-dominating set in G with minimum cardinality is called a minimum k-dominating set in G.The minimum cardinality of a k-dominating set, denoted $\gamma_k(G)$ is called the k-dominating number. If S is a k-dominating set in G, $\gamma_k(G) \leq |S|$.If $k = 1$, then $\gamma_1(G) = \gamma(G)$.If S is a k-dominating set in G then it is also j-dominating set for $1 \leq j \leq k$, and $\gamma_j(G) = \gamma_k(G)$.

Theorem 8 $\gamma_2(G^*) = 10$

Proof Let G be any graph and S a subset of $V(G)$.Let $v \in S$ and $k \geq 1$. The Private k-neighbourhood of v with respect to S, denoted $PR_k[v, S]$ is defined as:

$$PR_k[v,S] = \{w \in V(G) - S : w \text{ is adjacent to exactly } k \text{ vertices of } S \text{ including } v\} \cup \{v : v \text{ is adjacent to at most } k-1 \text{ vertices of } S\}$$

.First we claim that a k-dominating set S in a graph G is a minimal k-dominating set if and only if $PR_k[v,S] \neq \phi, \forall v \in S$.Let S be a minimal k-dominating set in G. Let $v \in S$ be any arbitrary vertex. Then $S - v$ is not a k-dominating set in G. So there exists a vertex $w \in V(G) - (S - v)$ which is adjacent to at most $(k-1)$ vertices of $S - v$. If $w = v$ and is adjacent to at most $(k-1)$ vertices of S. Then $v \in PR_k[v,S]$ and $PR_k[v,S] \neq \phi$.If $w \neq v \in V(G) - (S - v)$ then as S is k-dominating and w is adjacent to at most $(k-1)$ vertices of $S - v$, w must be adjacent to v. This means, w is adjacent to exactly k vertices of S and so $w \in PR_k[v,S]$ and and $PR_k[v,S] \neq \phi$. Conversely, suppose that $PR_k[v,S] \neq \phi$ for every $v \in S$. Let $v \in S$ be any arbitrary vertex and $w \in PR_k[v,S]$.If $w = v$ and w is adjacent to at most $(k-1)$ vertices of S. Then w is adjacent to at most $(k-1)$ vertices of $S - v$. That is., $S - v$ is not a k-dominating set in G. If $w \neq v$ then w is adjacent to exactly k vertices of S including v.That is., w is adjacent to exactly $(k-1)$ vertices of $S - v$ and $S - v$ is not k-dominating set in G. This means S is a minimal k-dominating set in G.

Now let us construct a minimal 2-connected dominating set S. First we allow all vertices of degree one as the element of S for obvious reasons. Therefore $S_1 = \{u_{14}\}$.Next let us include all vertices of degree two in S. Then S_1 gets refined to $S_2 = \{u_{14}, u_3, u_4, u_5, u_{11}, u_{12}, u_{15}, u_{20}\}$.Clearly S_2 is not a 2-connected dominating set as u_6 is not adjacent to any element of S_2.Now as u_{11} is adjacent to u_3, u_{15} and u_{12} is adjacent to u_4, u_{15}, we can conveniently drop u_{11} and u_{12} from S_2 to include other vital elements to produce a 2-connected dominating set. So S_2 gets refined to $S_3 = \{u_{14}, u_3, u_4, u_5, u_{15}, u_{20}\}$. We find that the inclusion of u_2 is mandatory, else, $\Pr_2(u_{14}, S)$ will become empty by the definition of a minimal 2-connected dominating set. So S_3 gets modified into $S_4 = \{u_{14}, u_3, u_4, u_5, u_{15}, u_{20}, u_2\}$. Now as u_6 to u_{10} and u_{21} are adjacent to u_1, the inclusion of u_1 will ensure at least one vertex of adjacency to these vertices in the proposed minimal 2-connected dominating set. So S_4 gets refined to $S_5 = \{u_1, u_2, u_3, u_4, u_5, u_{14}, u_{15}, u_{20}\}$. Similarly as u_{17} is adjacent to u_6 to u_9, u_{13}, u_{16} and u_{19}.For the same reason as above, we allow u_{17} into our proposed set. Hence S_5 gets modified into $S_6 = \{u_1, u_2, u_3, u_4, u_5, u_{14}, u_{15}, u_{17}, u_{20}\}$. But S_6 is still not a 2-connected dominating set. This is because, u_{10} is adjacent to only one vertex in S_6.Again as u_{13} is adjacent to u_{10}, u_{18} and u_{20}, we allow u_{13} into the proposed set. So S_6 gets modified into $S = \{u_1, u_2, u_3, u_4, u_5, u_{13}, u_{14}, u_{15}, u_{17}, u_{20}\}$. Now as u_6 to u_9 are adjacent to u_1 and u_{17}, u_{10} is adjacent to u_1 and u_{13}, u_{11} is adjacent to u_3 and u_{15}, u_{12} is adjacent to u_4 and u_{15}, u_{16} is adjacent to u_5 and u_{17}, u_{18} is adjacent to u_2 and u_{13}, u_{19} is adjacent to u_{13} and u_{17} we see that S is a 2-connected dominating set.

Moreover we find that $u_{10} \in \Pr_2[u_1, S]$, $u_{18} \in \Pr_2[u_2, S]$, $u_{11} \in \Pr_2[u_3, S]$, $u_{12} \in \Pr_2[u_4, S]$, $u_{16} \in \Pr_2[u_5, S]$, $u_{10} \in \Pr_2[u_{13}, S]$, $u_{11} \in \Pr_2[u_{15}, S]$, $u_7 \in \Pr_2[u_{17}, S]$ we see that $\Pr_2[u_j, S] \neq \phi$ for all j=1 to 5,13 to 15, 17,20 .Hence we deduce that S is a minimal 2-connected dominating set.

Interestingly we find another minimal 2-connected dominating set S' by just adding u_{16} to S and dropping u_5 from S. That is $S' = \{u_1, u_2, u_3, u_4, u_{13}, u_{14}, u_{15}, u_{16}, u_{17}, u_{20}\}$ is a minimal

2-connected dominating set as $u_5 \in \Pr_2\left[u_{16}, S^{'}\right]$.In view of this we conclude that there exists no minimum or a unique 2-connected dominating set with twelve elements. Hence $\gamma_2(\mathrm{G}^*) = 10$.

Note 4: It is easy to see from the structure of G^* that there exists no k -dominating set for $k \geq 3$ as there are a number of vertices (7 to be exact) with maximum degree equal to 2.

Note 5: In [9] we have found the vertex independence number β_o of G*. $\beta_o = 9$ and $I = \{u_2, u_3, u_4, u_5, u_6, u_7, u_{10}, u_{15}, u_{19}\}$ is an independent set. It is interesting to observe that an independent set need not be a 2-dominating set. That is., I is not 2-dominating as u_{14} is adjacent to only u_2 and out of the two adjacent elements u_{13}, u_{19} of u_{20} in G* only $u_{19} \in I$.

7. Strong domination

For a graph $G = (V, E)$,a set $S \subseteq V$ is a strong dominating set if every vertex $v \in V - S$ has a neighbour u in S such that the degree of u is not smaller than the degree of v.The minimum cardinality of a strong dominating set of G is the strong domination number, $\gamma_{strong}(G)$.

Theorem 9 $\gamma_{strong}(\mathrm{G}^*) = 5$

Proof We know from Theorem 1,2 that a dominating set of G*must have atleast five elements and the possible dominating sets are $S_1 = \{u_1, u_2, u_5, u_{13}, u_{15}\}$, $S_2 = \{u_1, u_2, u_6, u_{13}, u_{15}\}$, $S_3 = \{u_1, u_2, u_{13}, u_{15}, u_{16}\}$, $S_4 = \{u_1, u_2, u_{13}, u_{15}, u_{17}\}$, $S_5 = \{u_1, u_{13}, u_{14}, u_{15}, u_5\}$, $S_6 = \{u_1, u_{13}, u_{14}, u_{15}, u_6\}$, $S_7 = \{u_1, u_{13}, u_{14}, u_{15}, u_{16}\}$, $S_8 = \{u_1, u_{13}, u_{14}, u_{15}, u_{17}\}$. Let us now determine how many of these are strong dominating sets. It turns out that S_4 and S_8 are strong dominating sets .This is because, for $S_4 : u_j$ is adjacent to u_1 for $j = 3$ to 10 with $\deg(u_1) > \deg(u_j)$; u_j is adjacent to u_{15} for j=11,12 ; with $\deg(u_{15}) > \deg(u_j)$; u_4 is adjacent to u_2 with $\deg(u_2) > \deg(u_{14})$; u_{16} is adjacent to u_{17} with $\deg(u_{17}) > \deg(u_{16})$; u_j is adjacent to u_{13} for j=18 to 20 with $\deg(u_{13}) > \deg(u_j)$; u_j is adjacent to u_{15} for j=11,12 with $\deg(u_{15}) \geq \deg(u_j)$; u_{16} is adjacent to u_{17} with $\deg(u_{17}) > \deg(u_{16})$; u_j is adjacent to u_{13} for j=18 to 20 with $\deg(u_{13}) \geq \deg(u_j)$. For S_8, the same reason exactly as in S_4 holds good expect for one minor difference: instead of u_j is adjacent to u_1 for j=3 to 10 with $\deg(u_1) > \deg(u_j)$ read u_j is adjacent to u_1 for j=2 to 10 with $\deg(u_1) > \deg(u_j)$. Now S_j is not a strong dominating set for j=1,2,3,5,6,7 because□he degree of u_{17} an element of S_j has degree more than all the elements of S_j. Further as there exists more than one strong dominating set, we conclude that G* has no minimum strong dominating set. Hence $\gamma_{strong}(\mathrm{G}^*) = 5$.

Note 6 Harinarayanan et.al[3] have proved the following: Let G be a graph of order n and size m.Let the strong domination number of G be t.Then every t-subset of $V(G)$ is strong dominating iff G is either K_n or $K_n^{\ C}$ or $\left(\binom{n}{2}K_2\right)^C$. As our G^* is different from K_{20} or $K_{20}^{\ C}$ or $(10K_2)^C$,it is easy to see from the above that, not every 5-element subset of $V(\mathrm{G}^*)$ is strong dominating. For instance, $\{u_1, u_2, u_5, u_{13}, u_{15}\}$, $\{u_1, u_2, u_6, u_{13}, u_{15}\}$, $\{u_1, u_2, u_5, u_{13}, u_{15}\}$ are some 5-element subsets of G^* that are not strong dominating sets(even though they are dominating).

8. Spectrum of G^*

The eigenvalues of a matrix A are the numbers λ such that $Ax = \lambda x$ has a nonzero solution vector; each such solution is an eigen vector associated with λ.The eigenvalues of a graph are the eigenvalues of its adjacency matrix A.There are the roots $\lambda_{1, .., }\lambda_n$ of the characteristic polynomial $\phi(G : \lambda) = \det(\lambda I - A) = \prod_{i=1}^{n}(\lambda - \lambda_i)$.The spectrum is the list of distinct eigenvalues with their multiplicities $m_{1, .., }m_t$;write

$$\mathrm{Spec(G)} = \begin{pmatrix} \lambda_{1...}\ \lambda_i \\ m_{1...}\ m_i \end{pmatrix}.$$

It would be interesting to compute the spectrum of G^*. We made use of the computer software Sage notebook and found that the characteristic polynomial of G^* is:

$$\phi(\mathrm{G}^* : x) = \mathrm{x}^{20} - 41\mathrm{x}^{18} - 46\mathrm{x}^{17} + 511\mathrm{x}^{16} + 792\mathrm{x}^{15} - 2875\mathrm{x}^{14} - 5278\mathrm{x}^{13} + 8158\mathrm{x}^{12} + 17584\mathrm{x}^{11}$$
$$-11467\mathrm{x}^{10} - 31520\mathrm{x}^{9} + 6132\mathrm{x}^{8} + 30372\mathrm{x}^{7} + 1743\mathrm{x}^{6} - 14644\mathrm{x}^{5} - 2734\mathrm{x}^{4} + 2916\mathrm{x}^{3} + 588\mathrm{x}^{2} - 176\mathrm{x} - 16$$

Now solving $\phi(G^*: x) = 0$ we get the eigenvalues $\lambda_i, i = 1,...,20$ as follows:

1)1 2)-1 3)-3.30889 4)-2.59562 5)-2.16512
6)-1.72186 7)-1.65466 8) -1.30711 9)-0.98670 10)-0.69808
11)-0.49168 12)-0.07864 13)0.25157 14)0.51006 15)1.07442 16)1.32138 17)1.6

Observe that only two out of 20 are integers and the others are numbers with non-terminating decimal digits (given here as corrected to 5 decimal digits). Out of these 20 eigenvalues, 7 are positive and 7 are negative. As all the eigenvalues are distinct, the minimal polynomial of the adjacency matrix A□f G^* is $\psi(A) = \prod_{i=1}^{20}(\lambda - \lambda_i)$, where $\{\lambda_1,...,\lambda_{20}\}$are the distinct eigenvalues of A.Note that the adjacency matrix A of G^* given below is symmetric and hence they have real eigenvalues by spectral theorem and 20 orthonormal eigenvectors as given below.

Corresponding to 1:
(0, 0, 1, -1, 0, 0, 0, 0, 0, 0, 1, -1, 0, 0, 0, 0, 0, 0, 0, 0);
Corresponding to -1 :
(0, 0, 1, -1, 0, 0, 0, 0, 0, 0, -1, 1, 0, 0, 0, 0, 0, 0, 0, 0);
Corresponding to -3.30889 :
(1, -0.59718, -0.34024,-0.34024, -0.33251, -0.55915,-0.53435,-0.36986, 0.18998, -0.42533, 0.12580,0.12581, 0.21739, 0.18048, -0.07604, 0.10025,0.55994, 0.38805, -0.29987, 0.02495);
Corresponding to -2.59562 :
(1, -15.74683, -0.48835,-0.48835, 7.18851, 6.98239, -17.98015,22.08629,-36.32039, 32.17126, 0.26758,0.26758, -48.18393, 6.06670, -0.20618,-19.65862, 36.85539, 23.04830, 13.15869,13.49398); Corresponding to -2.16512 :
(1, -0.14376,-0.73553, -0.73553, -0.76080, 0.300070.56919, 0.58883, -1.35634, 0.10875,0.59251, 0.59251, 0.12088, 0.06640,-0.54733, 0.64721, -0.94056, -0.52428,1.31035, -0.66104);
Corresponding to -1.72186:
(1,-74.79231, 15.91217, 15.91217,24.47158, 39.80277, 10.51548,21.23822, -36.39862, -18.38331,-28.39846, -28.39846, 67.05203,43.43704, 32.98588, -43.13652,10.00051, -13.94624, -1.500499,-38.07029);
Corresponding to -1.65466:
(1, -0.38617,1.70155, 1.70155, -3.20688, -2.28203,-1.23140, 1.68897, 0.10626,0.25348, -3.81550, -3.81550,-1.52569, 0.23338, 4.61181, 4.30631,-1.63658, 0.87890, 2.03174,-0.30583),
Corresponding to -1.30711:
(1, 1.32671,-0.17266, -0.17266, -0.13435,0.03594, 1.62533, -2.71663,-1.39857, 0.29976, -0.77432,-0.77432, 0.00675, -1.01500,1.18478, -0.82439, 1.17598,-0.18528, 0.40771, -0.31708);
Corresponding to -0.98670:
(1, -2.72873, -0.51334,-0.51334, -8.67837, -8.12056,12.82638, 10.35370, -9.88692,6.27449, -0.49348, -0.49348,2.69585, 2.76551, 1.00026,7.56298, 9.33652, -23.45913,-21.91507, 19.47822) ;
Corresponding to -0.69808:
(1, 1.10210, -0.86239, -0.86239,0.46764, 0.73902, -1.55588,1.48011, 0.09124, -1.29752,-0.39798, -0.39798, -0.18547,-1.57876, 1.14022, -1.32645,-0.28069, -1.20453, -0.14873,0.47875);
Corresponding to -0.49168:
(1,-0.93030, -1.29648, -1.29648,-2.04419, 6.0183, 1.44346,-1.78841?, 0.01246?, -0.61002?,-0.36255, -0.36255, -0.71252,1.89211, 1.47473, 0.00508,-3.97661, 4.04284, -2.09284,5.70568);
Corresponding to -0.07865:
(1,-1.14661, -8.46777, -8.46776,15.65384 -11.124608, 8.97250,-0.22027, 6.45984, -1.73780,-0.33402, -0.33402, -7.32316,14.57885, 8.49404, -2.23115,-4.35375, -3.59149, 6.95587,4.67001);
Corresponding to 0.25157 :
(1,0.24555, 2.62147, 2.62147,-0.63595, -0.27389, 1.25620,0.04835, -0.52694, -5.10469,-0.34051, -0.34051, -1.75724,0.97605, -2.70714, -1.15999,0.61802, -0.82342, 2.32344,2.25067);
Corresponding to 0.51006:
(1,-13.43408, 1.24500, 1.24499,19.42510, -9.19602, 10.29204,4.92133, -8.91280, -5.07549,-0.36498, -0.36498, 5.32402,-26.33846, -1.43116, 8.90788,-5.68556, 13.92654, -0.80232,8.86512);
Corresponding to 1.07442:
(1, 0.78285,0.42644, 0.42644, 1.30307,-0.11604, -1.34670, -0.58757,-0.76750, 0.95344, -0.54182,-0.54182, 0.79190, 0.72863,-1.00859, 0.40004, -0.75721,-0.33463, 0.02948, 0.76448);
Corresponding to 1.32138:
(1, -1.47321, 0.15327,0.15327, -1.19917, -0.88567,0.70072, 0.06045, 1.74457,2.06714, -0.79747, -0.79747,-0.01311, -1.11490, -1.20703,-2.58455, -1.33033, -0.54878,0.69293, 0.51448);
Corresponding to 1.69002:
(1,2.45644, -3.52210, -3.52210, 1.07705,0.64529, 3.91898, 3.22667,-0.39357, -2.19663,-6.95242,-6.95242, -4.31879, 1.4535,-8.22762, 0.82023, -0.33612,3.12618, -6.92331, -6.65205);
Corresponding to 2.00038:
(1, 1.39966, 0.99906,0.99906, -0.32998, -1.55880, 0.50743,0.99358, -1.02610, 0.01645, 0.99850,0.99850, 0.05900, 0.69970, 0.99831,-1.66008, -1.43200, 1.47956, -1.57924,-0.75998);
Corresponding to 2.47911: (1, -0.48386,0.53159, 0.53159, 0.58748, 0.70959,0.17957, -0.24155, 0.46828, 0.19644,0.31786, 0.31786, -0.98128, 0.19518,0.25643, 0.45642, -0.16555, -0.61599,-0.51758, -0.60460);
Corresponding to 5.68144:
(1,0.88363, 0.18202, 0.18202, 0.23751, 0.61837, 0.87713,1.14651, 1.03482, 0.51941, 0.03416, 0.03416, 0.91619,0.15553, 0.01202, 0.34938, 1.12908, 0.67297, 0.58875,0.26489)

Let $w_i, i = 1 \; to \; 20$ denote these eigenvectors. Using this, we can write $x^T Ax = x^T S \wedge S^T x$, where $\wedge$ is the diagonal matrix of eigenvalues $\lambda_1 \geq ... \geq \lambda_{20}$ and S

has columns $\omega_1,...,\omega_{20}$.If S has 7 positive and 7 negative eigenvalues, $x^T Ax$ becomes, $x^T Ax = \sum_{i=1}^{7}(y^i \cdot x)^2 - \sum_{i=14}^{20}(z^i \cdot x)^2$,where y^i or z^i is $|\lambda_i|^{\left(\frac{1}{2}\right)}\omega_i$.

Proposition10 The diameter of G^* is less than the number of distinct eigenvalues of G^*.

Proof In [1] ,we have already computed the diameter of G^* as $diam(G^*) = 3$.As the number of distinct eigenvalues of G^* are 20, the result follows. □

As Spectral Theorem guarantees real eigenvalues, arrange the eigenvalues of G^* in the descending order as $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_{20}$.

Proposition11 If G' is an induced subgraph of G^* then $\lambda_{\min}(G^*) = \lambda_{20} \leq \lambda_{\min}(G') \leq \lambda_{\max}(G') \leq \lambda_{\max}(G^*) = \lambda_1$.

Observation 3 Wilf[8] have proved that for any graph G, $\chi(G) = 1 + \lambda_{\max}(G)$.In [1] we have computed the chromatic number of G^* as 4. As $\lambda_{\max}(G^*) = 5.681$, the inequality is very well intact.

Observation 4 We notice that for G^* the number of bicliques needed to decompose $G^*$is at least 7,which is equal to the maximum of the number of positive and number of negative eigenvalues of the adjacency matrix $A(G^*)$. For instance,

$$41 = |E(G^*)| > |E(G_1)| + |E(G_2)| + |E(G_3)| + |E(G_4)| + |E(G_5)| + |E(G_6)| + |E(G_7)| + |E(G_8)|$$

where $G_i \cong K_{1,2}$ if $i = 1$ *to* 4; $G_i \cong K_{1,3}$ if $i = 5,6$; $G_7 \cong K_{1,4}$; $G_8 \cong K_{1,7}$ where G_i 's $i = 1,2,...,8$ are shown below. The vertex set of these G_i 's are given below. In all the graphs, the first vertex denotes the central vertex and the others are pendent vertices, connected to the central vertex and thereby forming stars $K_{1,3}, K_{1,3}, K_{1,3}, K_{1,3}, K_{1,4}, K_{1,4}, K_{1,5}, K_{1,8}$ respectively in order. $V(G_1) = \{u_8, u_1, u_{18}\}$;

$V(G_2) = \{u_{15}, u_{11}, u_{12}\}$; $V(G_3) = \{u_{16}, u_5, u_6\}$;

$V(G_4) = \{u_2, u_{13}, u_{14}\}$; $V(G_5) = \{u_{13}, u_8, u_{10}, u_{20}\}$;

$V(G_6) = \{u_{18}, u_2, u_{17}, u_{19}\}$;

$V(G_7) = \{u_{17}, u_2, u_6, u_7, u_{16}\}$;

$V(G_8) = \{u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_{10}\}$

Observation 5 Notice that $\Delta(G^*) = 9$ and G^* has only one vertex namely u_1 with $\deg(u_1) = 9$.Hence there is no 9-regular subcomponent in G^*.This means the eigenvalue of G^* with largest absolute value namely $\lambda = 5.681$ is less than $\Delta(G^*) = 9$, a fact which is in support of the truth that: The necessary and sufficient condition for the eigenvalue of a graph G with largest absolute value is $\Delta(G)$ is that it should have some $\Delta(G)$-regular subcomponent.

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Appendix

Rolf Nevanlinna Prize Winners

Name	Photo	Year	Country of Origin	Erdos Number	Affiliation
ROBERT ENDRE TARJAN		1982	USA	2	Dept of Computer Science, Princeton University, 35,Olden Street, Room 324,Princeton. NJ 08544-2087
LESLIE G. VALIANT		1986	UK	3	Dept of Computer Science and Applied Mathematics, Harvard University; School of Engg.&Applied Science, 351,Maxwell Dworkin,33,Oxford street,Cambridge,MA02138
A.A. RAZBOROV		1990	Russia	2	Dept of Computer Science,Eotvos University,H-1088,Budapest Hungary, Dept of Computer Science, Princeton University,NJ 08544,USA
AVI WIGDERSON		1994	Israel	2	School of Mathematics, Institute for Advanced Study, Princeton
PETER W. SHOR		1998	USA	2	AT&T Labs Florham Park,New Jersey, USA
MADHU SUDAN		2002	INDIA	2	Massachusetts Institute of Technology, USA.
JON M.KLEINBERG		2006	USA	3	Dept of Computer Science, Cornell University,Inthana,NY14853

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