Bayes' Estimators of an Exponentially Distributed Random Variables Using Al-Bayyati's Loss Function

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Abstract — This research work discusses on the estimation of an unknown rate parameter of an Exponential distribution using Bayesian methodology under the Al-Bayyati's loss function with different prior distributions. The rate parameter of an Exponential distribution is assumed to follow non-informative prior distribution (such as extension of Jeffrey's prior distribution) and informative prior distribution (such as Gamma prior distribution, Gamma-Chi-square prior distribution, Gamma – Exponential prior distribution and Chi-square-Exponential prior distribution). The posterior distributions for the unknown rate of an Exponential distribution were derived using Bayes' theorem and the estimates under Al-Bayatti's loss function was obtained for the different prior distributions. A simulation study was considered to investigate the performance of the estimators under different prior distribution and various sample sizes. The estimators are then compared in terms of mean square error (MSE) which is computed using R programming Language. It was observed that the estimates of the unknown parameter under different priors are very close to the true parameter and that the mean square errors (MSE)of the estimates of the rate parameter increases as the increase of the rate parameter vale with all sample size. It was observed that the Bayesian rate estimates under informative prior distributions proves to be better than the estimates under the non-informative prior distributions proves to be efficient with minimum mean square error.

Keywords: *MSE, Al-Bayyati's loss function, non-informative prior distribution, Informative prior distribution, Posterior distribution*

1. INTRODUCTION

In Bayesian analysis, the unknown parameter is regarded as being the value of a random variable from a given probability distribution, with the knowledge of some information about the value of parameter prior to observing the data $x_1, x_2 \dots x_n$. The exponential distribution was the first widely used lifetime distribution model in areas such as the lifetimes of manufactured items [5], [7], [8], survival or remission times in chronic diseases [10] and in agricultural experiment [8]. The widely use of an Exponential distribution was its ability to model life time dataset due to its availability of simple statistical methods for it [7] and also because of its suitability for representing the lifetimes of many things such as various types of manufactured items [5]. In real-world scenarios, the assumption of a constant rate (or probability per unit time) is rarely satisfied. For example, the rate of incoming phone calls differs according to the time of day. In queuing theory, the service times of agents in a system (e.g. how long it takes for a bank teller etc. to serve a customer) are often modeled as exponentially distributed variables.

Maximum likelihood estimation has been the widely used method to estimate the parameter of a probability distributions. However, Bayes method has begun to get the attention of researchers in the estimation processes. The only aspect of statistics that can combine both modelling inherit uncertainty and statistical theory is Bayesian statistics. This method provides a way to learn from the dataset.

[6] presented some thought-provoking insights on the relationship between Bayesian and classical estimation using the exponential distribution. He proved how the classical estimators can be obtained from various choices made within Bayesian framework. [12], and [14] focused on the estimation of Exponential and Gamma distribution using Bayesian methods and adopted the use of Normal and Laplace approximations to estimate the parameters.

Exponential distribution is a special case of Gamma distribution

$$G(a,\gamma) = \frac{\gamma^a}{\Gamma(a)} e^{-\gamma x} x^{a-1} \quad a > 0, \gamma > 0 \qquad (1)$$

When a = 1, it becomes an exponential distribution with probability mass function of the form

$$f(x,\gamma) = \gamma e^{-\gamma x}\ ; \ x = 0,1,2,3 \dots , \gamma > 0 \qquad (2)$$

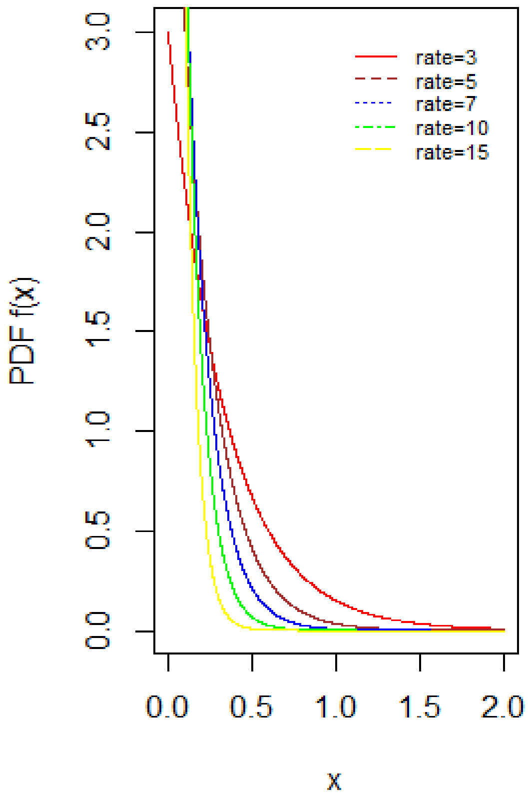

Figure 1: PDF curve of an Exponential Distribution at Different Values of Rate Parameter

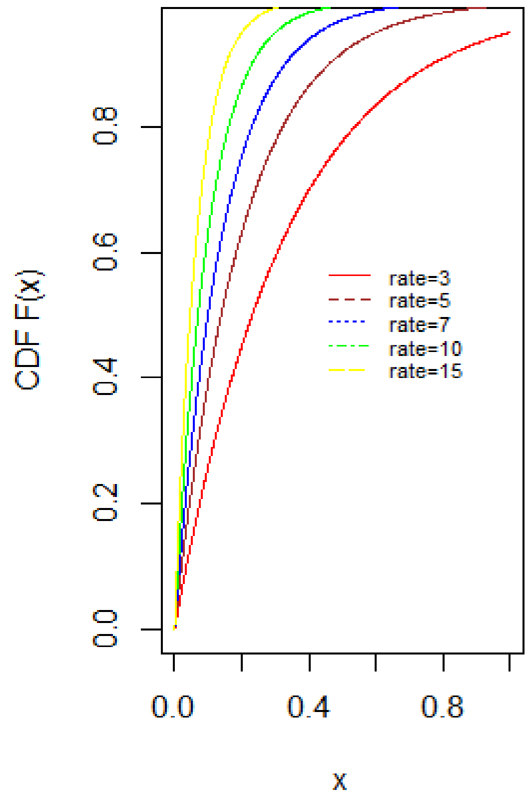

Figure 2: CDF curve of an Exponential Distribution at Different Values of Rate Parameter

2.0 MATERIALS AND METHODS

2.1 Prior Inormation

In Bayesian statistics, a prior probability information which is also known as prior distribution of an uncertain quantity is the probability distribution that would express one's beliefs about this quantity before evidence is considered. The most important part of a Bayesian estimation is the specification of a prior distribution.

Although, at times priors are chosen according to one's subjective knowledge and belief that is why Bayesian approach is sometimes called a subjective approach [4]. In this research work two types of prior information will be adopted namely; non-informative prior distribution (such as extension of Jeffrey's prior distribution) and informative prior distribution (such as Gamma prior distribution, Gamma-Chi-square prior distribution, Gamma – Exponential prior distribution and Chi-square-Exponential prior distribution).

2.1.1 Extension of the Jefferey's Prior:

The prior proposed by [3], [11] and [13], known as extension of Jeffrey's prior is given by

$$g(\gamma) \propto [I(\gamma)]^k;\ k \in \mathcal{R}^+ \quad (3)$$

Where $I(\gamma) = -nE\left[\frac{\partial^2}{\partial\gamma^2} logf(x;\gamma)\right]$ is known as Fisher's information matrix. Therefore $I(\gamma)$ for Exponential distribution will be

$$I(\gamma) = -n\left[\frac{\partial^2}{\partial\gamma^2}\{\ln\gamma - \gamma x\}\right] \quad (4)$$

Thus, the resulting extension of Jeffrey's prior will be

$$g_1(\gamma) \propto \left(\frac{1}{\gamma^2}\right)^k \quad (5)$$

Remark 1: if $k = \frac{1}{2}$, we will have Jeffrey's prior

$$g_{11}(\gamma) \propto \frac{1}{\gamma} \quad (6)$$

Remark 2: if $k = \frac{3}{2}$, we will have Hartigan's prior

$$g_{12}(\gamma) \propto \frac{1}{\gamma^3} \quad (7)$$

Remark 3: if $k = 0$, we will have U (0, k=$\frac{1}{p}$) prior

$$g_{13}(\gamma) \propto 1 \quad (8)$$

2.1.2 The Gamma Prior

The single prior distribution of γ is a Gamma distribution with hyper parameters α_1 and β_1 is

$$g_2(\gamma) = \frac{{\beta_1}^{\alpha_1}}{\Gamma(\alpha_1)}\gamma^{\alpha_1-1}e^{-\beta_1\gamma};\alpha_1 > 0\ ,\beta_1 > 0 \quad (9)$$

2.1.3 Gamma-Chi-Square Prior

Assume that the prior distribution of γ is a Gamma distribution with hyper parameters α_2 and β_2 as

$$g_{31}(\gamma) = \frac{\beta_2^{\alpha_2}}{\Gamma(\alpha_2)}\gamma^{\alpha_2-1}e^{-\beta_2\gamma}; \alpha_2 > 0 , \beta_2 > 0 \qquad (10)$$

The second prior assumed is a Chi-square distribution with hyper parameter ψ_1 given by

$$g_{32}(\gamma) = \frac{1}{2^{\frac{\psi_1}{2}}\Gamma\left(\frac{\psi_1}{2}\right)}\gamma^{\frac{\psi_1}{2}-1}e^{-\frac{\gamma}{2}}; \psi_1 > 0, \gamma > 0 \qquad (11)$$

[1] defined the double prior for γ by combining the two priors in (10) and (11) as:

$$g_3(\gamma) \propto g_{31}(\gamma)g_{32}(\gamma)$$

$$g_3(\gamma) \propto \gamma^{\alpha_2+\frac{\psi_1}{2}-2}e^{-\gamma\left(\frac{1}{2}+\beta_2\right)} \qquad (12)$$

2.1.4 Gamma - Exponential distribution

The double prior for γ is defined to be a Gamma distribution with hyper parameters (α_3, β_3) and an exponentialdistribution with hyper parameter $\delta_1$as

$$g_{41}(\gamma) = \frac{\beta_3^{\alpha_3}}{\Gamma(\alpha_3)}\gamma^{\alpha_3-1}e^{-\beta_3\gamma}; \alpha_3 > 0 , \beta_3 > 0 \qquad (13)$$

$$g_{42}(\gamma) = \delta_1 e^{\delta_1\gamma};\ \delta_1 > 0 \qquad (14)$$

[1] defined the double prior for γ by combining the two priors in (13) and (14) as:

$$g_4 \propto p_3(\gamma)p_4(\gamma)$$

$$g_4 \propto \gamma^{\alpha_3-1}e^{-\gamma(\delta_1+\beta_3)} \qquad (15)$$

2.1.5 Chi-Square-Exponential Prior

The double prior for γ is defined to be a Gammadistribution with hyper parameters ψ_2 and an exponentialdistribution with hyper parameter $\delta_2$as

$$g_{51}(\gamma) = \frac{1}{2^{\frac{\psi_2}{2}}\Gamma\left(\frac{\psi_2}{2}\right)}\gamma^{\frac{\psi_2}{2}-1}e^{-\frac{\gamma}{2}}; \psi_2 > 0, \gamma > 0 \qquad (16)$$

$$g_{52}(\gamma) = \delta_2 e^{\delta_2\gamma};\ \delta_2 > 0 \qquad (17)$$

[1] defined a double prior for γ by combining the two priors in (16) and (17) as:

$$g_5 \propto \gamma^{\frac{\psi_2}{2}-1}e^{-\gamma\left(\frac{1}{2}+\delta_2\right)} \qquad (18)$$

2.1 Loss Function

The concept of loss function is as old as Laplace and wad reintroduced in statistics by Abraham Wald in the middle of 20th Century. Sound statistical practice requires selecting an estimator consistent with the actual acceptable variation experienced in the context of a particular applied problem. In this paper we considered the use of Al-Bayyati's loss function for better comparison of Bayes' estimators.

2.1.1 Al-Bayyati's Loss Function

The Al-Bayyati's loss function was stated by [7] as

$$R_{AL}(\hat{\gamma}, \gamma) = \gamma^{c_2}(\hat{\gamma} - \gamma)^2;\ c_2 \in \mathcal{R}^+ \qquad (19)$$

Where $\hat{\gamma}$ is the estimated value of parameter γ.

3. ESTIMATION

3.1 Maximum Likelihood Estimation

Let$x_1, x_2, \cdots, x_n$ be a random sample size n from Exponential distribution. The maximum likelihood estimator γ of the parameter γ which maximizes the likelihood function will be as follows [15] and [16]

$$L(x;\gamma) = \prod_{i=1}^{n} f(x;\gamma)$$

$$L(x;\gamma) = \gamma^n e^{-\gamma\sum x}$$

The log-likelihood function for an Exponential distribution can be expressed as

$$\log L(\gamma|x) = n\log\gamma - \gamma\sum x \qquad (20)$$

Now solving $\frac{\partial}{\partial\gamma}log\{L(\gamma|x)\} = 0$

$$\hat{\gamma} = \left(\frac{\sum x}{n}\right)^{-1} \qquad (21)$$

3.2 Bayesian Estimation

To obtain Bayes estimators, we assume thatγ is a real valued random variable with probability density function $g(\gamma)$. The posterior distribution of γ i. e, $h(\gamma|x)$ is theconditional probability density function ofγ given the data. In this section weconsider Bayes estimation of the unknown parameter γ based on the above-mentioned priors and loss function.

3.2.1 Posterior Distribution of Unknown Parameter γ Using extension of the Jeffrey's Prior

The posterior distribution for γ using extension of the Jeffrey's prior

$$h_1(\gamma|x) = \frac{L(x|\gamma)g_1(\gamma)}{\int_0^\infty L(x|\gamma)g_1(\gamma)\,\partial\gamma}$$

$$h_1(\gamma|x) \propto \frac{\gamma^n e^{-\gamma\sum x}\left(\frac{1}{\gamma^2}\right)^k}{\int_0^\infty \gamma^n e^{-\gamma\sum x}\left(\frac{1}{\gamma^2}\right)^k\partial\gamma}$$

$$p(\gamma|y) \propto \gamma^{n-2k}e^{-\gamma\sum x}$$

which is the density function of a Gamma distribution of

$$h_1(\gamma|x) \propto \frac{(\sum x)^{n-2k+1}}{\Gamma(n-2k+1)}\gamma^{n-1}e^{-\gamma\sum x} \qquad (22)$$

with parameters (n-2k+1 , $\sum x$).

$$Mean = E(X) = \frac{n-2k+1}{\sum x}$$

$$Var(X) = \frac{n-2k+1}{\sum x^2}$$

3.2.2 Posterior Distribution of Unknown Parameter γ Using Gamma Prior

The posterior distribution for γ using Gamma prior

$$h_2(\gamma|x) = \frac{L(x|\gamma)g_2(\gamma)}{\int_0^\infty L(x|\gamma)g_2(\gamma)\,\partial\gamma}$$

$$h_2(\gamma|x) \propto \frac{\gamma^n e^{-\gamma\sum x}\gamma^{\alpha_1-1}e^{-\beta_1\gamma}}{\int_0^\infty \gamma^n e^{-\gamma\sum x}\gamma^{\alpha_1-1}e^{-\beta_1\gamma}\,\partial\gamma}$$

$$h_2(\gamma|x) \propto \gamma^{n+\alpha_1-1}e^{-\gamma(\beta_1+\sum x)}$$

which is the density function of a Gamma distribution of

$$h_2(\gamma|x) = \frac{(\beta_1+\sum x)^{n+\alpha_1}}{\Gamma(n+\alpha_1)}\gamma^{n+\alpha_1-1}e^{-\gamma(\sum x+\beta_1)} \qquad (23)$$

With parameters $(n + \alpha_1, \sum x + \beta_1)$

$$Mean = E(x) = \frac{n+\alpha_1}{\beta_1+\sum x}$$

$$Var(x) = \frac{n+\alpha_1}{(\beta_1+\sum x)^2}$$

3.2.3 Posterior Distribution of Unknown Parameter γ Using Gamma-Chi-Square Prior

The posterior distribution for γ using Gamma-Chi-Square Prior

$$h_3(\gamma|x) = \frac{L(x|\gamma)g_3(\gamma)}{\int_0^\infty L(x|\gamma)g_3(\gamma)\,\partial\gamma}$$

$$h_3(\gamma|x) \propto \frac{\gamma^n e^{-\gamma\sum x}\gamma^{\alpha_2+\frac{\psi_1}{2}-2}e^{-\gamma\left(\frac{1}{2}+\beta_2\right)}}{\int_0^\infty \gamma^n e^{-\gamma\sum x}\gamma^{\alpha_2+\frac{\psi_1}{2}-2}e^{-\gamma\left(\frac{1}{2}+\beta_2\right)}\,\partial\gamma}$$

$$h_3(\gamma|x) \propto \gamma^{\left(\alpha_2+\frac{\psi_1}{2}+n-1\right)-1}e^{-\gamma(\beta_1+\sum x+\frac{1}{2})}$$

which is the density function of a Gamma distribution of

$$h_3(\gamma|x) = \frac{\left(\beta_1+\sum x+\frac{1}{2}\right)^{\alpha_2+\frac{\psi_1}{2}+n-1}}{\Gamma(\alpha_2+\frac{\psi_1}{2}+n-1)}\gamma^{\left(\alpha_2+\frac{\psi_1}{2}+n-1\right)-1}e^{-\gamma(\beta_1+\sum x+\frac{1}{2})} \quad (24)$$

With parameters $(\alpha_2+\frac{\psi_1}{2}+n-1, \beta_1+\sum x+\frac{1}{2})$

$$Mean = E(x) = \frac{n+\alpha_2+\frac{\psi_1}{2}-1}{\beta_2+\sum x+\frac{1}{2}}$$

$$Var(x) = \frac{\mathbf{n}+\boldsymbol{\alpha}_2+\frac{\boldsymbol{\psi}_1}{2}-\mathbf{1}}{\left(\boldsymbol{\beta}_2+\sum \mathbf{x}+\frac{1}{2}\right)^2}$$

3.2.4 Posterior Distribution of Unknown Parameter γ Using Gamma-Exponential Prior

The posterior distribution for γ using Gamma-Exponential Prior

$$h_4(\gamma|x) = \frac{L(x|\gamma)g_4(\gamma)}{\int_0^\infty L(x|\gamma)g_4(\gamma)\,\partial\gamma}$$

$$h_4(\gamma|x) \propto \frac{\gamma^n e^{-\gamma\sum x}\gamma^{\alpha_3-1}e^{-\gamma(\delta_1+\beta_3)}}{\int_0^\infty \gamma^n e^{-\gamma\sum x}\gamma^{\alpha_3-1}e^{-\gamma(\delta_1+\beta_3)}\,\partial\gamma}$$

$$p(\gamma|x) \propto \gamma^{n+\alpha_3-1}e^{-\gamma(\delta_1+\beta_3+\sum x)}$$

which is the density function of a Gamma distribution of

$$p(\gamma|X) = \frac{(\delta_1+\beta_3+\sum x)^{n+\alpha_3}}{\Gamma(n+\alpha_3)}\gamma^{n+\alpha_3-1}e^{-\gamma(\delta_1+\beta_3+\sum x)} \quad (25)$$

With parameters $(n+\alpha_3, \delta_1+\beta_3+\sum x)$

$$Mean = E(X) = \frac{n+\alpha_3}{\delta_1+\beta_3+\sum x}$$

$$Var(X) = \frac{n+\alpha_3}{(\delta_1+\beta_3+\sum x)^2}$$

3.2.5 Posterior Distribution of Unknown Parameter γ Using Chi-Square-Exponential Prior

$$h_5(\gamma|x) = \frac{L(x|\gamma)g_5(\gamma)}{\int_0^\infty L(x|\gamma)g_5(\gamma)\,\partial\gamma}$$

$$h_5(\gamma|x) \propto \frac{\gamma^n e^{-\gamma\sum x}\gamma^{\frac{\psi_2}{2}-1}e^{-\gamma\left(\frac{1}{2}+\delta_2\right)}}{\int_0^\infty \gamma^n e^{-\gamma\sum x}\gamma^{\frac{\psi_2}{2}-1}e^{-\gamma\left(\frac{1}{2}+\delta_2\right)}\,\partial\gamma}$$

$$h_5(\gamma|x) \propto \gamma^{\left(n+\frac{\psi_2}{2}-1\right)}e^{-\gamma(\delta_2+\sum x+\frac{1}{2})}$$

which is the density function of a Gamma distribution of

$$h_5(\gamma|x) = \frac{\left(\delta_2+\sum x+\frac{1}{2}\right)^{n+\frac{\psi_2}{2}}}{\Gamma(n+\frac{\psi_2}{2})}\gamma^{\left(\frac{\psi_2}{2}+n-1\right)}e^{-\gamma(\delta_2+\sum x+\frac{1}{2})} \quad (26)$$

With parameters $(n+\frac{\psi_2}{2}, \delta_2+\sum x+\frac{1}{2})$

$$Mean = E(x) = \frac{n+\frac{\psi_2}{2}}{\delta_2+\sum x+\frac{1}{2}}$$

$$Var(x) = \frac{n+\frac{\psi_2}{2}}{\left(\delta_2+\sum x+\frac{1}{2}\right)^2}$$

3.3 Bayes' Estimator Under Al-Bayyati's Loss Function

The risk function under Al-Bayyati's loss function can be define is

$$R_{(AL,EJ)}(\hat{\gamma}) = \int_0^\infty \gamma^{c_2}(\hat{\gamma}-\gamma)^2 h_i(\gamma|x)\partial\gamma \quad (27)$$

3.3.1 Extension of Jeffrey's Prior

The risk function under Al-Bayyati's loss function using extension of Jeffrey's prior is given by

$$R_{(AL,EJ)}(\hat{\gamma}) = \int_0^\infty \gamma^{c_2}(\hat{\gamma}-\gamma)^2 h_1(\gamma|x)\partial\gamma$$

$$R_{(AL,EJ)}(\hat{\gamma}) = \int_0^\infty \gamma^{c_2}(\hat{\gamma}-\gamma)^2 \frac{(\sum x)^{n-2k+1}}{\Gamma(n-2k+1)}\gamma^{n-1}e^{-\gamma\sum x}\partial\gamma \quad (28)$$

$$R_{(AL,EJ)}(\hat{\gamma}) = \frac{\hat{\gamma}^2\Gamma(n-2k+c_2+1)}{\Gamma(n-2k+1)(\sum x)^{c_2+1}} - \frac{2\hat{\gamma}\Gamma(n-2k+c_2+2)}{\Gamma(n-2k+1)(\sum x)^{c_2+1}} + \frac{\Gamma(n-2k+c_2+3)}{\Gamma(n-2k+1)(\sum x)^{c_2+2}}$$

Now, solving $\frac{R_{(AL,EJ)}(\hat{\gamma})}{\partial\hat{\gamma}} = 0$ for $\hat{\gamma}$ yield the required Bayes' estimator under the assumed combination of loss function and prior distribution in given by

$$\hat{\gamma}_{(AL,EJ)} = \frac{n-2k+c_2+1}{\sum x} \quad (29)$$

3.3.2 Gamma Prior Distribution

The risk function under Al-Bayyati's loss function using extension of Gamma prior is given by

$$R_{(AL,GP)}(\hat{\gamma}) = \int_0^\infty \gamma^{c_2}(\hat{\gamma}-\gamma)^2 h_2(\gamma|x)\partial\gamma$$

$$R_{(AL,GP)}(\hat{\gamma}) = \int_0^\infty \gamma^{c_2}(\hat{\gamma}-\gamma)^2 \frac{(\beta_1+\sum x)^{n+\alpha_1}}{\Gamma(n+\alpha_1)}\gamma^{n+\alpha_1-1}e^{-\gamma(\sum x+\beta_1)}\partial\gamma$$

(30)

$$R_{(AL,GP)}(\hat{\gamma}) = \frac{\hat{\gamma}^2\Gamma(n+\alpha_1+c_2)}{\Gamma(n+\alpha_1)(\sum x+\beta_1)^{c_2}} - \frac{2\hat{\gamma}\Gamma(n+\alpha_1+c_2+1)}{\Gamma(n+\alpha_1)(\sum x+\beta_1)^{c_2+1}} + \frac{\Gamma(n+\alpha_1+c_2+2)}{\Gamma(n+\alpha_1)(\sum x+\beta_1)^{c_2+2}}$$

Now, solving $\frac{R_{(AL,GP)}(\hat{\gamma})}{\partial\hat{\gamma}} = 0$ for $\hat{\gamma}$ yield the required bayes estimator under the assumed combination of loss function and prior distribution in given by

$$\hat{\gamma}_{(AL,GP)} = \frac{n+\alpha_1+c_2}{\sum x+\beta_1} \quad (31)$$

3.3.3 Gamma- Chi-Square Prior Distribution

The risk function under Al-Bayyati's loss function using extension of Gamma- Chi-square prior is given by

$$R_{(AL,GC)}(\hat{\gamma}) = \int_0^\infty \gamma^{c_2}(\hat{\gamma}-\gamma)^2 h_3(\gamma|x)\partial\gamma$$

$$R_{(AL,GC)}(\hat{\gamma}) = \int_0^\infty \gamma^{c_2}(\hat{\gamma}-\gamma)^2 \frac{\left(\beta_1+\sum x+\frac{1}{2}\right)^{\alpha_2+\frac{\psi_1}{2}+n-1}}{\Gamma(\alpha_2+\frac{\psi_1}{2}+n-1)}\gamma^{\left(\alpha_2+\frac{\psi_1}{2}+n-1\right)-1} e^{-\gamma(\beta_1+\sum x+\frac{1}{2})}\partial\gamma \quad (32)$$

$$R_{(AL,GC)}(\hat{\gamma}) = \frac{\hat{\gamma}^2\Gamma\left(n+\alpha_2+\frac{\psi_1}{2}+c_2-1\right)}{\Gamma\left(n+\alpha_2+\frac{\psi_1}{2}-1\right)\left(\sum x+\frac{1}{2}+\beta_1\right)^{c_2}} - \frac{2\hat{\gamma}\Gamma\left(n+\alpha_2+\frac{\psi_1}{2}+c_2\right)}{\Gamma\left(n+\alpha_2+\frac{\psi_1}{2}-1\right)\left(\sum x+\frac{1}{2}+\beta_1\right)^{c_2+1}} + \frac{\Gamma\left(n+\alpha_2+\frac{\psi_1}{2}+c_2\right)}{\Gamma\left(n+\alpha_2+\frac{\psi_1}{2}-1\right)\left(\sum x+\frac{1}{2}+\beta_1\right)^{c_2+2}}$$

Now, solving $\frac{R_{(AL,GC)}(\hat{\gamma})}{\partial\hat{\gamma}} = 0$ for $\hat{\gamma}$ yield the required bayes estimator under the assumed combination of loss function and prior distribution in given by

$$\hat{\gamma}_{(AL,GC)} = \frac{n+\alpha_2+\frac{\psi_1}{2}+c_2-1}{\sum x+\frac{1}{2}+\beta_1} \quad (33)$$

3.3.4 Gamma- Exponential Prior Distribution

The risk function under Al-Bayyati's loss function using extension of Gamma- Exponential prior is given by

$$R_{(AL,GE)}(\hat{\gamma}) = \int_0^\infty \gamma^{c_2}(\hat{\gamma}-\gamma)^2 h_3(\gamma|x)\partial\gamma$$

$$R_{(AL,GE)}(\hat{\gamma}) = \int_0^\infty \gamma^{c_2}(\hat{\gamma}-\gamma)^2 \frac{(\delta_1+\beta_3+\sum x)^{n+\alpha_3}}{\Gamma(n+\alpha_3)}\gamma^{n+\alpha_3-1} e^{-\gamma(\delta_1+\beta_3+\sum x)}\partial\gamma \quad (34)$$

$$R_{(AL,GC)}(\hat{\gamma}) = \frac{\hat{\gamma}^2\Gamma(n+\alpha_3+c_2+2)}{\Gamma(n+\alpha_3+2)(\sum x+\beta_3+\delta_1)^{c_2}} - \frac{2\hat{\gamma}\Gamma(n+\alpha_3+c_2+3)}{\Gamma(n+\alpha_3+2)(\sum x+\beta_3+\delta_1)^{c_2+1}} + \frac{\Gamma(n+\alpha_3+c_2+4)}{\Gamma(n+\alpha_3+2)(\sum x+\beta_3+\delta_1)^{c_2+2}}$$

Now, solving $\frac{R_{(AL,GE)}(\hat{\gamma})}{\partial\hat{\gamma}} = 0$ for $\hat{\gamma}$ yield the required bayes estimator under the assumed combination of loss function and prior distribution in given by

$$\hat{\gamma}_{(AL,GE)} = \frac{n+\alpha_3+c_2+2}{\sum x+\beta_3+\delta_1} \quad (35)$$

3.3.5 Chi-Square-Exponential Prior Distribution

The risk function under Al-Bayyati's loss function using extension of Chi-square-Exponential prior is given by

$$R_{(AL,CE)}(\hat{\gamma}) = \int_0^\infty \gamma^{c_2}(\hat{\gamma}-\gamma)^2 h_3(\gamma|x)\partial\gamma$$

$$R_{(AL,CE)}(\hat{\gamma}) = \int_0^\infty \gamma^{c_2}(\hat{\gamma}-\gamma)^2 \frac{\left(\delta_2+\sum x+\frac{1}{2}\right)^{n+\frac{\psi_2}{2}}}{\Gamma(n+\frac{\psi_2}{2})}\gamma^{\left(\frac{\psi_2}{2}+n-1\right)} e^{-\gamma(\delta_2+\sum x+\frac{1}{2})}\partial\gamma \quad (36)$$

$$R_{(AL,CE)}(\hat{\gamma}) = \frac{\hat{\gamma}^2\Gamma\left(\frac{\psi_2}{2}+n+c_2\right)}{\Gamma\left(\frac{\psi_2}{2}+n\right)\left(\delta_2+\sum x+\frac{1}{2}\right)^{c_2}} - \frac{2\hat{\gamma}\Gamma\left(n\frac{\psi_2}{2}+n+c_2+1\right)}{\Gamma\left(\frac{\psi_2}{2}+n\right)\left(\delta_2+\sum x+\frac{1}{2}\right)^{c_2+1}} + \frac{\Gamma\left(\frac{\psi_2}{2}+n+c_2+2\right)}{\Gamma\left(\frac{\psi_2}{2}+n\right)\left(\delta_2+\sum x+\frac{1}{2}\right)^{c_2+2}}$$

Now, solving $\frac{R_{(AL,CE)}(\hat{\gamma})}{\partial\hat{\gamma}} = 0$ for $\hat{\gamma}$ yield the required bayes estimator under the assumed combination of loss function and prior distribution in given by

$$\hat{\gamma}_{(AL,CE)} = \frac{\frac{\psi_2}{2}+n+c_2}{\delta_2+\sum x+\frac{1}{2}} \quad (37)$$

Now, on summarizing the above Bayes' estimators and their special cases, we can have the following table of estimators

Table 1: Bayes' Estimators under Al-Bayyati's loss function and Different Prior Distributions

Prior	Estimator
Extension of Jeffrey's	$\hat{\gamma}_{(AL,EJ)} = \frac{n-2k+c_2+1}{\sum x}$
Hartigan's	$\hat{\gamma}_{(AL,EJ)} = \frac{n-2+c_2}{\sum x}$
Jeffrey's	$\hat{\gamma}_{(AL,EJ)} = \frac{n+c_2}{\sum x}$

Uniform	$\hat{\gamma}_{(AL,EJ)} = \frac{n + c_2 + 1}{\sum x}$
Gamma	$\hat{\gamma}_{(AL,GP)} = \frac{n + \alpha_1 + c_2}{\sum x + \beta_1}$
Gamma-Chi-squares	$\hat{\gamma}_{(AL,GC)} = \frac{n + \alpha_2 + \frac{\psi_1}{2} + c_2 - 1}{\sum x + \frac{1}{2} + \beta_1}$
Gamma-Exponential	$\hat{\gamma}_{(AL,GE)} = \frac{n + \alpha_3 + c_2 + 2}{\sum x + \beta_3 + \delta_1}$
Chi-Squares-Exponential	$\hat{\gamma}_{(AL,CE)} = \frac{\frac{\psi_2}{2} + n + c_2}{\delta_2 + \sum x + \frac{1}{2}}$

Note: Subscript in the form ordered pair in each estimator represents the combination of Al-Bayyati's loss function and prior distribution used in the derivation of Bayes' estimator. First element of the ordered pair represents Al-Bayyati's loss function whereas the second element represent prior distribution.

4.0 SIMULATION STUDY OF AN EXPONENTIAL DISTRIBUTION

In the simulation study, data sets of size n = 50, 100, 200 and 500 have been generated from Exponential distribution with parameter k taking the values 0, 0.5, 1 and 1.5. Whereas, values of another hyper-parameter c_2 are considered to be 1 and 3. We also set $\alpha_1 = 1.5, 3$, $\alpha_2 = 1, 2$, $\alpha_3 = 1, 2$, $\beta_1 = 1, 5$, $\beta_2 = 1.5, 3$, $\beta_3 = 1.5, 3$, $\psi_1 = 0.5, 1$, $\psi_2 = 1.5, 3$, $\delta_1 = 0.5, 1$ and $\delta_2 = 1, 2$. We carried out simulation using R programming language.

The results presented in Tables 2 and 3 displayed the expected an MSE's for estimating the rate parameter γ under the Extension Jeffrey prior distribution. The results in Tables 4 and 5 displayed the expected and MSE's for estimating the rate parameter γ under the Gamma prior distribution. Tables 6 and 7 displayed the expected and MSE's for estimating the rate parameter γ under the Gamma-Chi-squares prior distribution. Tables 8 and 9 displayed the expected and MSE's for estimating the rate parameter γ under the Gamma-Exponential prior distribution. Tables 10 and 11 displayed the expected and MSE's for estimating the rate parameter γ under the Chi-squares-Exponential prior distribution. It was observed that on considering the Hartigan's prior (k =1.5), usually parameter gets under estimated and starts overestimating it as c_2 increases from 1 to 10. The behaviour of Bayesian estimates for all infomative prior distribuitons have better behaviour than other estimates when compared with estimates of non-informative prior distribution for different sample sizes and various rate parameters. Finally, for all parameter values, as the sample size increases the MSE keeps reducing.

Table 2: Bayes' Estimates of Estimators along with their Mean Square Errors under Extension of Jeffrey prior distribution when γ=3

n	k	$c_2 = 1$		$c_2 = 3$		$c_2 = 5$		$c_2 = 10$	
		$\hat{\gamma}$	MSE	$\hat{\gamma}$	MSE	$\hat{\gamma}$	MSE	$\hat{\gamma}$	MSE
100	**0.0**	3.086	0.099	3.141	0.099	3.216	0.103	3.365	0.119
	0.5	3.365	0.119	3.058	0.095	3.174	0.101	3.331	0.109
	1.0	3.005	0.084	3.099	0.099	3.138	0.103	3.300	0.108
	1.5	2.989	0.095	3.066	0.095	3.123	0.095	3.264	0.106
200	**0.0**	3.056	0.048	3.090	0.045	3.101	0.045	3.182	0.054
	0.5	3.023	0.047	3.065	0.046	3.094	0.048	3.169	0.046
	1.0	3.020	0.046	3.036	0.055	3.073	0.048	3.510	0.049
	1.5	2.999	0.046	3.042	0.048	3.062	0.049	3.148	0.049
500	**0.0**	3.014	0.018	3.031	0.019	3.035	0.02	3.067	0.018
	0.5	3.013	0.020	3.030	0.020	3.035	0.017	'3.070	0.020
	1.0	3.002	0.019	3.018	0.017	3.031	0.018	3.056	0.018
	1.5	3.003	0.018	3.016	0.018	3.027	0.018	3.054	0.019

Table 3: Bayes' Estimates of Estimators along with their Mean Square Errors under Extension of Jeffrey prior distribution when γ=5

n	k	$c_2 = 1$		$c_2 = 3$		$c_2 = 5$		$c_2 = 10$	
		$\hat{\gamma}$	MSE	$\hat{\gamma}$	MSE	$\hat{\gamma}$	MSE	$\hat{\gamma}$	MSE
100	**0.0**	5.118	0.265	5.250	0.262	5.023	0.023	5.630	0.353
	0.5	5.098	0.263	5.214	0.260	5.021	0.023	5.540	0.302
	1.0	5.042	0.245	5.151	0.285	5.024	0.023	5.537	0.308
	1.5	4.993	0.271	5.132	0.271	5.019	0.022	5.456	0.274
200	**0.0**	5.017	0.123	5.134	0.138	5.179	0.134	5.299	0.152
	0.5	5.062	0.125	5.089	0.135	5.151	0.130	5.274	0.137
	1.0	5.008	0.122	5.085	0.138	5.142	0.133	5.260	0.138
	1.5	4.993	0.123	5.049	0.133	5.095	0.136	5.237	0.140
500	**0.0**	5.018	0.049	5.051	0.049	5.072	0.048	5.111	0.053
	0.5	5.016	0.053	5.044	0.049	5.053	0.049	5.108	0.054
	1.0	5.003	0.051	5.025	0.051	5.051	0.053	5.114	0.052
	1.5	5.005	0.053	5.019	0.047	5.039	0.049	5.090	0.053

Table 4: Bayes' Estimates of Estimators along with their Mean Square Errors under Gamma prior distribution when γ= 3

n	α_1	β_1	$c_2 = 1$		$c_2 = 3$		$c_2 = 5$		$c_2 = 10$	
			$\hat{\gamma}$	MSE	$\hat{\gamma}$	MSE	$\hat{\gamma}$	MSE	$\hat{\gamma}$	MSE
100	**1.5**	**1**	3.015	0.083	3.066	0.087	3.119	0.099	3.275	0.102
		5	2.702	0.056	2.746	0.058	2.798	0.058	2.927	0.068
	3.0	**1**	3.037	0.085	3.110	0.087	3.170	0.093	3.137	0.100
		5	2.748	0.056	2.785	0.059	2.837	0.066	2.966	0.066
200	**1.5**	**1**	3.005	0.045	3.024	0.047	3.056	0.047	3.140	0.045
		5	2.835	0.034	2.873	0.036	2.906	0.038	2.967	0.038
	3.0	**1**	3.032	0.043	3.059	0.046	3.085	0.047	3.155	0.046
		5	2.866	0.034	2.877	0.034	2.918	0.038	2.981	0.042
500	**1.5**	**1**	3.015	0.016	3.010	0.018	3.028	0.018	3.057	0.018
		5	2.934	0.017	2.941	0.017	2.958	0.017	2.980	0.016
	3.0	**1**	3.015	0.018	3.025	0.018	3.033	0.018	3.075	0.019
		5	2.942	0.016	2.946	0.016	2.969	0.017	2.997	0.017

Table 5: Bayes' Estimates of Estimators along with their Mean Square Errors under Gamma prior distribution when γ= 5

n	α_1	β_1	$c_2 = 1$		$c_2 = 3$		$c_2 = 5$		$c_2 = 10$	
			$\hat{\gamma}$	MSE	$\hat{\gamma}$	MSE	$\hat{\gamma}$	MSE	$\hat{\gamma}$	MSE
100	**1.5**	**1**	4.940	0.211	5.042	0.255	5.106	0.244	5.375	0.266
		5	4.122	0.119	4.203	0.108	4.306	0.121	4.476	0.136
	3.0	**1**	4.965	0.221	5.073	0.259	5.190	0.264	5.429	0.136
		5	4.196	0.115	4.255	0.118	4.360	0.127	4.547	0.131
200	**1.5**	**1**	4.952	0.121	5.013	0.111	5.078	0.113	5.174	0.134
		5	4.538	0.084	4.553	0.080	4.601	0.077	4.725	0.089
	3.0	**1**	5.004	0.114	5.037	0.127	5.093	0.137	5.222	0.130
		5	4.566	0.082	4.588	0.077	4.636	0.085	4.771	0.100
500	**1.5**	**1**	4.977	0.047	5.001	0.055	5.025	0.049	5.074	0.052
		5	4.798	0.041	4.807	0.040	4.843	0.046	4.878	0.042
	3.0	**1**	5.013	0.048	5.027	0.052	5.037	0.053	5.092	0.048
		5	4.804	0.041	4.831	0.040	4.852	0.042	4.907	0.044

Table 6: Bayes' Estimates of Estimators along with their Mean Square Errors under Gamma – Chi-Squares prior distribution when γ= 3

n	α_2	β_2	ψ_1	$c_2=1$		$c_2=3$		$c_2=5$		$c_2=10$	
				$\hat{\gamma}$	MSE	$\hat{\gamma}$	MSE	$\hat{\gamma}$	MSE	$\hat{\gamma}$	MSE
100	1	1.5	1.0	2.812	0.0704	2.880	0.081	2.933	0.078	3.082	0.092
		3.0	0.5	2.700	0.059	2.761	0.065	2.815	0.033	2.946	0.071
	2	1.5	1.0	2.811	0.076	2.879	0.085	2.920	0.078	3.077	0.082
		3.0	0.5	2.688	0.061	2.740	0.062	2.785	0.064	2.928	0.068
200	1	1.5	1.0	2.902	0.042	2.951	0.041	2.964	0.042	3.035	0.048
		3.0	0.5	2.839	0.037	2.789	0.036	2.887	0.038	2.980	0.043
	2	1.5	1.0	2.893	0.038	2.925	0.038	2.961	0.042	3.023	0.042
		3.0	0.5	2.838	0.035	2.881	0.037	2.897	0.040	2.972	0.039
500	1	1.5	1.0	2.957	0.017	2.974	0.019	2.993	0.017	3.025	0.018
		3.0	0.5	2.931	0.016	2.946	0.017	2.956	0.018	2.987	0.017
	2	1.5	1.0	2.969	0.018	2.979	0.017	2.985	0.019	3.014	0.019
		3.0	0.5	2.930	0.017	2.946	0.016	2.960	0.017	2.993	0.017

Table 7: Bayes' Estimates of Estimators along with their Mean Square Errors under Gamma – Chi-Squares prior distribution when γ= 5

n	α_2	β_2	ψ_1	$c_2=1$		$c_2=3$		$c_2=5$		$c_2=10$	
				$\hat{\gamma}$	MSE	$\hat{\gamma}$	MSE	$\hat{\gamma}$	MSE	$\hat{\gamma}$	MSE
100	1	1.5	1.0	4.515	0.158	4.644	0.183	4.745	0.208	4.940	0.019
		3.0	0.5	4.217	0.117	4.324	0.138	4.397	0.147	4.612	0.148
	2	1.5	1.0	4.507	0.168	4.626	0.186	4.714	0.191	4.928	0.205
		3.0	0.5	4.208	0.129	4.322	0.127	4.39	0.13	4.600	0.152
200	1	1.5	1.0	4.753	0.108	4.805	0.097	4.864	0.116	4.972	0.110
		3.0	0.5	4.612	0.088	4.630	0.096	4.700	0.093	4.808	0.099
	2	1.5	1.0	4.739	0.102	4.788	0.109	4.823	0.106	4.961	0.111
		3.0	0.5	4.573	0.090	4.635	0.096	4.691	0.098	4.790	0.097
500	1	1.5	1.0	4.895	0.046	4.937	0.045	4.928	0.045	4.978	0.047
		3.0	0.5	4.820	0.046	4.847	0.046	4.868	0.045	4.919	0.044
	2	1.5	1.0	4.912	0.048	4.917	0.047	4.931	0.046	4.990	0.051
		3.0	0.5	4.830	0.044	4.845	0.043	4.876	0.043	4.913	0.046

Table 8: Bayes' Estimates of Estimators along with their Mean Square Errors under Gamma – Exponential prior distribution when γ=3

n	α_3	β_3	δ_1	$c_2=1$		$c_2=3$		$c_2=5$		$c_2=10$	
				$\hat{\gamma}$	MSE	$\hat{\gamma}$	MSE	$\hat{\gamma}$	MSE	$\hat{\gamma}$	MSE
100	1	1.5	1.0	2.920	0.075	2.973	0.076	3.031	0.083	3.194	0.097
		3.0	0.5	2.859	0.070	2.908	0.068	2.957	0.075	3.094	0.077
	2	1.5	1.0	2.973	0.081	3.022	0.079	3.060	0.080	3.207	0.086
		3.0	0.5	2.873	0.067	2.927	0.070	2.986	0.076	3.120	0.081
200	1	1.5	1.0	2.966	0.042	2.990	0.041	3.024	0.041	3.103	0.045
		3.0	0.5	2.921	0.038	2.944	0.037	2.974	0.040	3.044	0.041
	2	1.5	1.0	2.983	0.043	2.991	0.039	3.033	0.041	3.107	0.047
		3.0	0.5	2.929	0.039	2.9496	0.040	2.993	0.044	3.062	0.041
500	1	1.5	1.0	2.990	0.017	2.998	0.017	3.011	0.016	3.038	0.018
		3.0	0.5	2.967	0.017	2.977	0.016	2.989	0.016	3.021	0.018
	2	1.5	1.0	2.995	0.017	3.007	0.018	3.01	0.017	3.044	0.017
		3.0	0.5	2.972	0.017	2.983	0.017	3.000	0.017	3.036	0.018

Table 9: Bayes' Estimates of Estimators along with their Mean Square Errors under Gamma – Exponential prior distribution when γ= 5

n	α_3	β_3	δ_1	$c_2=1$		$c_2=3$		$c_2=5$		$c_2=10$	
				$\hat{\gamma}$	MSE	$\hat{\gamma}$	MSE	$\hat{\gamma}$	MSE	$\hat{\gamma}$	MSE
100	1	1.5	1.0	4.652	0.178	4.763	0.201	4.829	0.183	5.082	0.198
		3.0	0.5	4.428	0.144	4.543	0.16	4.622	0.154	4.817	0.157
	2	1.5	1.0	4.689	0.169	4.777	0.171	4.883	0.184	5.128	0.214
		3.0	0.5	4.51	0.156	4.601	0.154	4.666	0.149	4.890	0.799
200	1	1.5	1.0	4.819	0.097	4.873	0.103	4.914	0.107	5.030	0.100
		3.0	0.5	4.714	0.097	4.765	0.09	4.797	0.097	4.917	0.099
	2	1.5	1.0	4.854	0.105	4.906	0.113	4.924	0.105	5.070	0.119
		3.0	0.5	4.737	0.096	4.785	0.099	4.817	0.099	4.932	0.103
500	1	1.5	1.0	4.934	0.051	4.935	0.046	4.970	0.051	5.011	0.046
		3.0	0.5	4.885	0.046	4.912	0.045	4.913	0.049	4.966	0.045
	2	1.5	1.0	4.939	0.048	4.950	0.046	4.969	0.049	5.025	0.045
		3.0	0.5	4.874	0.045	4.908	0.041	4.932	0.043	4.969	0.042

Table 10: Bayes' Estimates of Estimators along with their Mean Square Errors under Chi-squares – Exponential prior distribution when γ=3

n	δ_2	ψ_2	$c_2=1$		$c_2=3$		$c_2=5$		$c_2=10$	
			$\hat{\gamma}$	MSE	$\hat{\gamma}$	MSE	$\hat{\gamma}$	MSE	$\hat{\gamma}$	MSE
100	1	1.5	2.983	0.085	2.991	0.075	3.062	0.093	3.214	0.106
		3.0	2.987	0.083	3.030	0.090	3.091	0.086	3.232	0.096
	2	1.5	2.861	0.064	2.932	0.073	2.963	0.077	3.110	0.084
		3.0	2.885	0.080	2.937	0.078	2.995	0.079	3.138	0.079
200	1	1.5	2.968	0.044	2.937	0.078	3.026	0.044	3.106	0.044
		3.0	2.987	0.044	3.020	0.046	3.055	0.049	3.115	0.049
	2	1.5	2.927	0.042	2.958	0.040	2.992	0.043	3.065	0.047
		3.0	2.942	0.042	2.959	0.039	3.008	0.039	3.060	0.043
500	1	1.5	2.992	0.017	3.005	0.019	3.015	0.017	3.036	0.019
		3.0	2.999	0.018	3.012	0.018	3.017	0.016	3.045	0.018
	2	1.5	2.972	0.017	2.985	0.016	2.998	0.017	3.024	0.019
		3.0	2.980	0.018	2.984	0.017	3.001	0.018	3.022	0.017

Table 11: Bayes' Estimates of Estimators along with their Mean Square Errors under Chi-squares – Exponential prior distribution when γ=5

			$c_2 = 1$		$c_2 = 3$		$c_2 = 5$		$c_2 = 10$	
n	δ_2	ψ_2	$\hat{\gamma}$	**MSE**	$\hat{\gamma}$	**MSE**	$\hat{\gamma}$	**MSE**	$\hat{\gamma}$	**MSE**
100	**1**	**1.5**	4.759	0.201	4.837	0.211	4.942	0.204	5.191	0.244
		3.0	4.831	0.219	4.910	0.201	4.986	0.228	5.211	0.023
	2	**1.5**	4.554	0.172	4.660	0.187	4.730	0.164	4.956	0.194
		3.0	4.602	0.160	4.691	0.170	4.732	0.176	4.983	0.201
200	**1**	**1.5**	4.879	0.103	4.930	0.108	4.992	0.127	5.088	0.120
		3.0	4.906	0.110	4.952	0.117	5.008	0.117	5.127	0.120
	2	**1.5**	4.769	0.104	4.831	0.106	4.863	0.106	4.989	0.106
		3.0	4.783	0.096	4.828	0.104	4.874	0.096	5.005	0.1098
500	**1**	**1.5**	4.958	0.051	4.957	0.045	4.988	0.05	5.024	0.0469
		3.0	4.953	0.046	4.980	0.051	4.997	0.051	5.026	0.045
	2	**1.5**	4.907	0.046	4.928	0.048	4.956	0.048	5.000	0.051
		3.0	4.912	0.048	4.935	0.044	4.940	0.047	4.986	0.048

5.0 CONCLUSION

This research work emphasized on the importance of Bayesian approximation using Al-Bayyati's loss function approach. Based on the results presented in Tables 2-11, we see that all the estimated values of the parameters are close to the true values of parameters in Exponential distribution. Bayesian estimates under informative prior distributions proves to be better than the estimates under the non-informative prior distributions proves to be efficient with minimum mean square error.

We conclude that the MSEs based on different prior distributions decreases as the sample size increases. It proves that the obtained estimates for the rate parameter are consistent. We also deduced that the performance of Bayesian estimates under informative prior distributions is better than non-informative prior.

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