Modeling of The Lane Changing Motor Vehicle Traffic On Roads with Two Lanes

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Abstract: *Heterogeneous motor vehicle flow models are developed to take into account of local and major changes such as the increase of motor vehicle types and number, increase of the number of lanes, lane shifting characteristics, overtaking and identification of areas prone to accidents and congestion. Lighthill - Whitham and Richards motor vehicle traffic model, investigated how homogeneous vehicles travels in a one dimensional, single lane road; without curvatures. Petrosyan and Balabanyan, in their headway traffic flow model, demonstrated how overtaking of a lead vehicle is difficulty in a single lane road. Due to the potholes realized in the already existing motor vehicle traffic flow models, there was need to work with heterogeneous traffic flow models so as to take care of dimensional heterogeneity, road heterogeneity and vehicle heterogeneity. This study addresses the short falls of the old models and the challenges posed by the increase in motor vehicle traffic ranging from but not limited to head on collisions, head to back collisions, lane changing difficulties, congestion and pedestrian related accidents not forgetting lack of maneuvering space for negotiating sharp corners; such as the Nyabunde black spot corner, in Kisii, Sachangwan deadly stretch, in Nakuru; both in Kenya where deadly accidents have in the recent past occurred resulting to loss of life and property. The study developed heterogeneous motor vehicle traffic flow models, derived a two ensemble traffic composition formulation and established the solutions to the obtained models numerically. Graphical representation of the traffic flow variables with respect to the distance covered on the highway is done using MATLAB. The information obtained from the study is very useful to civil engineers, mechanical engineers, town planners, computer programmers and upcoming mathematicians not forgetting drivers and riders*.

Key words: Lane, multi – lane, Dual – carriage, Lane function, and Lane changing model

Abbreviations: Two dimension (2D), Lane position (C), and Lighthill-Whitham-Richards (LWR)

1. Introduction

Traffic networks that consist of avenues, highways, lanes, streets and other roadways provide a convenient and an economical conveyance of passengers and goods.The basic activity in transportation is a trip, defined by its origin/destination, departure time/arrival time and travel route. A myriad of trips interact on a traffic network to produce a sophisticated pattern of traffic flows [2]. Nowadays traffic jam has become a major problem in society and economically as far as transportation in developed and developing countries; where our country Kenya as member is considered [8]. In the last few decades much interest has been focused on traffic flow models as the amount of traffic more especially motorcycles continues to increase exponentially [7]. Traffic congestion on motorways in the third world countries is becoming an even more pressing problem, where almost in every weekday morning and every weekday evening including weekends the capacity of many main roads is exceeded [3]. Traffic management is becoming a mounting challenge for many cities and towns across the globe as the world's population grows. Increasing attention as been devoted to the modeling, simulation and visualization of traffic to investigate causes of traffic accidents, congestion and delays, to study the effectiveness of road side hardware, signs and other barriers, to improve policies and guidelines related to traffic regulation and to assist up in development and the design of highways and road systems. The transportation is one of the major pillars supporting live in cities and regions in many large cities. The potentialities of extensive development of the transportation network were exhausted over the last five decades or are approaching completion. That is why optimal planning of transportation, improvement of traffic organization and optimization of the roots of public conveyance take on special importance. Solution of these problems cannot do without mathematical modeling of the transportation systems. In this study the work of [4, 6] is extended into a two lane change model. The derivation of the multi – lane road model is presented as well as the lane shifting model for a road with two lanes. The lane changing model is consists of a system of partial differential equations (PDE). It is almost impossible to find the exact solution of the model as an initial value (IVP). That is why there is demand to find the numerical solution of the model as an initial boundary

ISSN 2250-3153
value problem (IBVP). For numerical solution of the model we describe the derivation of the numerical schemes. In order to implement the numerical scheme. The study develops a computer programming codes and performs computer simulation of the lane changing model using MATLAB with respect to various flow parameters.

2. Literature Review

The most relevant literature, is the one done by [4], in his study of highway capacity of short dense - roads formulated a motor vehicle traffic model. The model only gave the relationship between motor vehicle velocity during free flow and during a traffic jam and could not account for how the road characteristics can influence traffic flow velocity. This occasioned Michalopoulos study later on in addressing this deficiency; which finally led to Michalopoulos traffic flow model. [5] in their investigation of a two-way traffic flow road section developed a traffic flow model containing road characteristics of the road section in question. The most elementary continuum traffic flow model was the first order model developed by [4, 6], based around the assumption that the number of vehicles is conserved between any two points if there are no entrances (sources) or exits (sinks). This produces a continuum model known as the Lighthill-Whitham-Richards (LWR) model.

3. Multi – Lane Road Model

3.1 Definition

A road is a way used for traveling between places, towns, cities and homes. They are usually surfaced by tarmac, asphalt, marram or concrete. Roads in developing countries; both rural and urban are designed to accommodate many and different types of vehicles; traveling in either direction, this has been the case for about 400 years ago. Comparatively, a lane is a single vehicle path. For a single road, one way, a lane width is equal to the road width. The most current roads have at most four lanes on both the RHS and LHS constituting a dual carriage highway. In this context a lane width is a quarter the road width. On a two dimensional, multi - lane road, the road function is derived from the respective velocity components. The velocity components in the respective x and y directions are given as;

$$u = \frac{dx}{dt} \qquad (1)$$

In the y - direction;

$$v = \frac{dy}{dt} \qquad (2)$$

Then, the differential equations representing the lanes is given as;

$$dt = \frac{dx}{u} = \frac{du}{v} \qquad (3)$$

On mathematical manipulation this is expressed as;

$$vdx - udy = 0 \qquad (4)$$

It is clear that, this equation is an exact differential of two independent variables. Let the multi - variate, multi - lane function be represented by the function;

$$\Psi = \psi(x, y) \qquad (5)$$

$\Psi = \psi(x, y)$ is the multi - lane function

Since $\Psi = \psi(x, y)$ is a function of two independent variables then; it is exact differential is given as;

$$\Psi = \psi(x,y) = \frac{\partial \psi(x,y)}{\partial x}dx + \frac{\partial \psi(x,y)}{\partial y}dy \qquad (6)$$

Comparing equation (4) and equation (6); we realize that,

$$\Psi = \psi(x,y) = \frac{\partial \psi(x,y)}{\partial x}dx + \frac{\partial \psi(x,y)}{\partial y}dy = vdx - udy = 0 \qquad (7)$$

Implying that,

$$\Psi = \psi(x,y) = 0 \qquad (8)$$

after taking integrals on both sides, the multi - lane road function is taken to be constant, C; that is to say

$$\Psi = \psi(x,y) = \text{constant, C} \qquad (9)$$

However, the lane function is partially constant and partially varies as y; in the positive y - direction, hence;

$$\Psi = \psi(x,y) = \text{constant, C + ky} \qquad (10)$$

Where C and k are constants; with C representing the lane position from the left hand side and k representing the lane width; in most of the developing countries, the lane-width k = 3.0m, y is the lane position, whose integral values are expressed as; y = 0, 1, 2, 3, 4 for a four lane road. This equation (10) is the multi-lane road function in a 2D.

3.2 The Lane Shifting Traffic Flow Model

Typically, continuum models are based on a system of motor vehicle conservation laws. In this derivation, the study used a dual carriage highway with N lanes on either road; which are numbered as, j = 1, 2, 3, 4,..., N. The lane changing motor vehicle traffic flow model between lane 1 (subscript 1) and lane 2 (subscript 2) based on the extension of LWR model, having a source and a sink; in a generalized form is expressed as:

$$\left.\begin{aligned} \frac{\partial \rho_1}{\partial t} + \frac{\partial (u\rho_1)}{\partial x} &= \frac{\rho_2}{T_2^1} - \frac{\rho_1}{T_1^2} \\ \frac{\partial \rho_2}{\partial t} + \frac{\partial (u\rho_2)}{\partial x} &= \frac{\rho_1}{T_1^2} - \frac{\rho_2}{T_2^1} \end{aligned}\right\} \qquad (11)$$

This is the lane changing that occurs between the first two lanes. For any middle lane j, the lane shifting model becomes;

$$\frac{\partial \rho_j}{\partial t} + \frac{\partial (u\rho_j)}{\partial x} = \frac{\rho_{j-1}}{T_{j-1}^j} - \frac{\rho_j}{T_j^{j-1}} + \frac{\rho_{j+1}}{T_{j+1}^j} - \frac{\rho_j}{T_j^{j+1}} \qquad (12)$$

The lane vehicle exchange between the last two lanes, that is N-1 and N; is given by the equations;

$$\left.\begin{aligned} \frac{\partial \rho_{N-1}}{\partial t} + \frac{\partial (u\rho_{N-1})}{\partial x} &= \frac{\rho_N}{T_N^{N-1}} - \frac{\rho_{N-1}}{T_{N-1}^N} \\ \frac{\partial \rho_N}{\partial t} + \frac{\partial (u\rho_N)}{\partial x} &= \frac{\rho_{N-1}}{T_{N-1}^N} - \frac{\rho_N}{T_N^{N-1}} \end{aligned}\right\} \qquad (13)$$

Where the subscripts, j = 1, 2, 3, 4, ..., N-1 and N refers to the number of lanes. The quantities ρ_j, v_j and q_{j} = ρ_j X v_j are the motor vehicle densities, vehicle velocities and flux in j^{th} lane respectively. At last $T_j^k(\rho_j, \rho_k)$ is the vehicle transition from the j^{th} to the k^{th} lane; such that; |j - k| = 1.

4. Solution of the Lane Changing Model

In particular, a macroscopic, lane changing motor vehicle traffic flow model for a road with two lanes as given by the system of equations (13); were numerically solved as illustrated below;

$$\left.\begin{aligned}\frac{\partial\rho_1}{\partial t}+\frac{\partial(u\rho_1)}{\partial x}=\frac{\rho_2}{T_2^1}-\frac{\rho_1}{T_1^2}\\ \frac{\partial\rho_2}{\partial t}+\frac{\partial(u\rho_2)}{\partial x}=\frac{\rho_1}{T_1^2}-\frac{\rho_2}{T_2^1}\end{aligned}\right\} \qquad (14)$$

In obtaining the numerical solutions of the coupled system of equations representing the lane changing traffic flow model, the study took the model as an initial boundary value problem (IBVP) by inserting initial and boundary conditions. Finite difference method was used in finding the solution. Density is a function of time, t, lane position and space x. In order to develop the numerical scheme; we discretize the time derivatives using the forward difference in time; and the space derivatives by the central difference in space.

The discretization in time and in space, for lane 1 (subscript 1)

$$\frac{\rho_{1,i}^{n+1}-\rho_{1,i}^{n}}{k}+\frac{u(\rho_{1,i+1}^{n}-\rho_{1,i-1}^{n})}{2h}=\frac{\rho_{2,i}^{n}}{T_2^1}-\frac{\rho_{1,i}^{n}}{T_1^2} \qquad (15)$$

and for lane 2 (subscript 2) is given as;

$$\frac{\rho_{2,i}^{n+1}-\rho_{2,i}^{n}}{k}+\frac{u(\rho_{2,i+1}^{n}-\rho_{2,i-1}^{n})}{2h}=\frac{\rho_{1,i}^{n}}{T_1^2}-\frac{\rho_{2,i}^{n}}{T_2^1} \qquad (16)$$

These equations are Lax Friedrich Finite difference scheme for our IBVP equation.

Taking the time step, k and step length, h as h = k= 1; u = 80km/hr and the transitions as T_2^1 = 20 percent; T_1^2 = 10 percent, the numerical scheme for lane 1 becomes;

$$\rho_{1i}^{n+1}=\rho_{1i}^{n}-80\rho_{1i+1}^{n}+80\rho_{1i-1}^{n}+5\rho_{2i}^{n}-10\rho_{1i}^{n} \qquad (17)$$

the numerical scheme for lane 2 becomes

$$\rho_{2i}^{n+1}=\rho_{2i}^{n}-80\rho_{2i+1}^{n}+80\rho_{2i-1}^{n}+5\rho_{1i}^{n}-10\rho_{2i}^{n} \qquad (18)$$

The lane changing traffic flow model is solved at four time levels that is n = 0, n = 1, n = 2, n = 3. Taking u = 80km/hr into the above scheme, at the time level n = 0, the nodal points in the x - direction as i = 1, 2, 3, ...5. A system of algebraic equations are obtained. Using the initial condition ρ (x, 0) = 21/0.1km and boundary conditions as ρ (0, t) = 5 cos 2x + 21, the system of equations generated from the numerical scheme above is solved by matrix laboratory. The results are tabulated in table 1.

Table 1: The lane 1 changing Solutions values

Grid point(i, lane 1, n)	n = 0	n = 1	n = 2	n = 3
(1, 1, n)	25.997000	232.013000	2064.877	17821.333
(2, 1, n)	25.988300	231.972000	1990.468	15330.853
(3, 1, n)	25.973000	230.797000	1960.533	14470.788
(4, 1, n)	25.951000	229.639000	1914.110	13509.641
(5, 1, n)	25.924000	228.423000	1905.244	12998.374

It is seen from the results in Table 1 that for a given value of i, ρ_{1i}^{n} values tends to increases to infinity as n increases to infinity. Also for a given value of n, ρ_{1i}^{n} values decreases to its initial density as i increases to infinity. Using the same numerical procedure, lane 2 algebraic system of equations results are generated and recorded as shown in table 2. The results in Table 1 are represented graphically in 2D in figure 1.

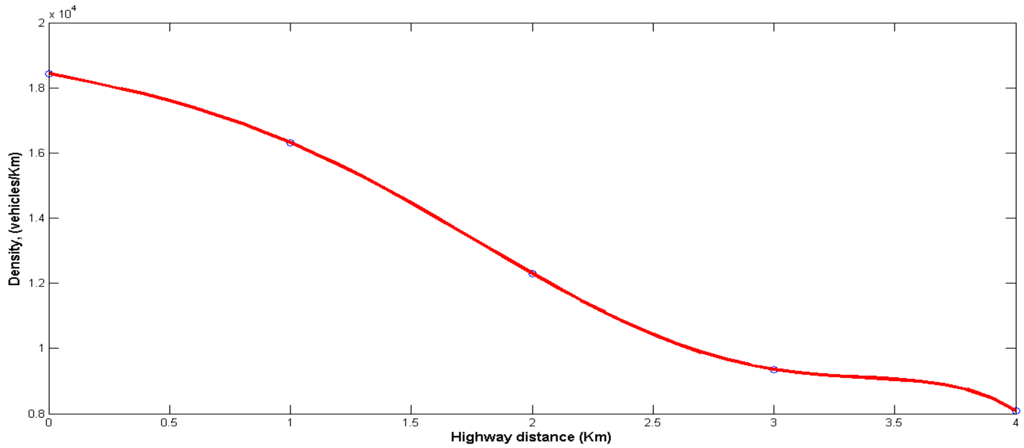

Figure 1: Lane one changing solution values

Table 2: The Lane 2 Changing Solutions Values

Grid point(i, lane 2, n)	n = 0	n = 1	n = 2	n = 3
(1, 2, n)	15.997000	93.028000	1033.67200	18442.472
(2, 2, n)	15.988300	62.032000	1023.55200	16315.128
(3, 2, n)	15.973000	60.932000	718.88000	12303.928
(4, 2, n)	15.951000	50.884000	625.74400	9358.464
(5, 2, n)	15.924000	42.041000	506.25600	8091.496

The results in Table 2 are represented graphically in 2D in figure 2. It is seen from the results in Table 2 that for a given value of i, ρ_{2i}^{n} values tends to increases to infinity as n increases to infinity. Also for a given value of n, ρ_{2i}^{n} values decreases to its initial density as i tends to increase to infinity.

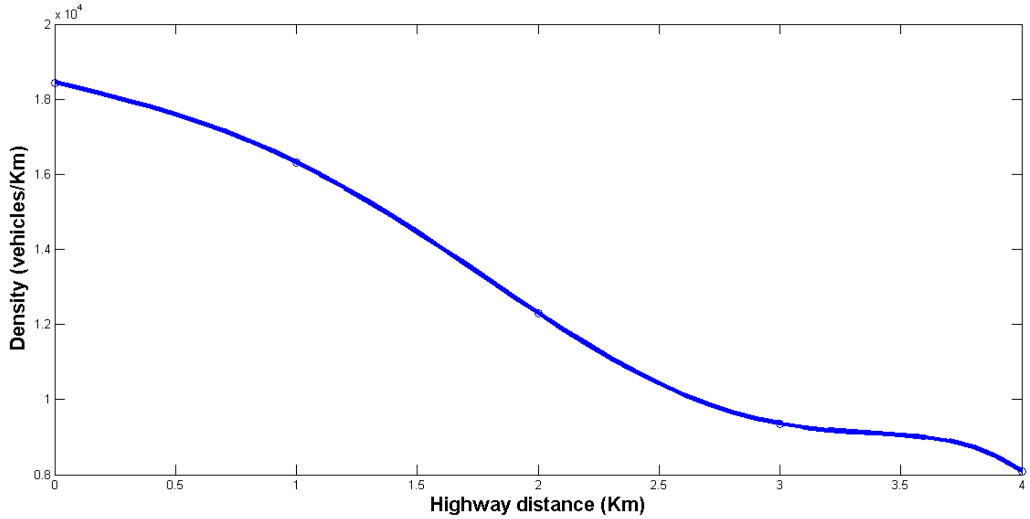

Figure 2: Lane two changing solution values

Additionally, the numerical values for dynamic density, velocity and flux at different time levels for lane 2, between the nodes i = 4, i = 5, are as illustrated in the table 3;

Table 3: Variation of Density, Velocity and Flux with Time, t

Grid point(i, lane 2, n)	**Time, t**	**Density, ρ**	**Velocity, u**	**Flux, q**
(5, 2, 0)	0.000	1.08000	79.22000	85.536
(5, 2, 1)	6.000	1.92500	61.80000	119.00
(5, 2, 2)	12.000	4.12000	34.50000	142.14
(5, 2, 3)	18.000	7.80000	16.65600	130.00
(5, 2, 4)	24.000	16.0000	7.500000	120.00

Table 3 shows that when vehicles are few, traffic density is lower, the flux is also lower; the velocity is high as there is maximum vehicle - road interaction. As the number of vehicles increases such that many vehicles passes the fifth nodal point at a relatively high velocity and density; the flow rate is high. At maximum density called jam density, vehicles cannot move; the flux is lower. The results from table 4.5 can be represented graphically in 2D in figure 3 to show the variation of density, ρ with velocity, u and with flux, q, respectively.

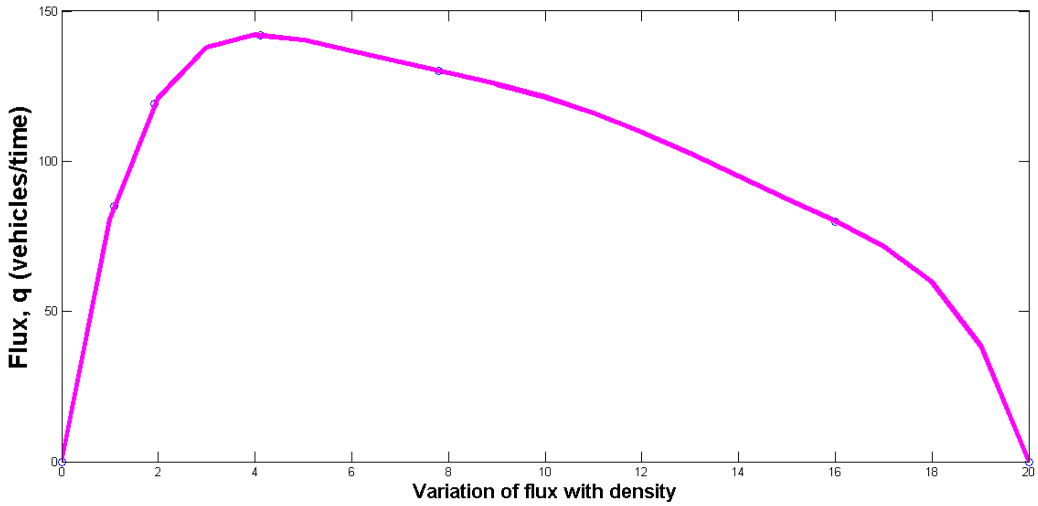

Figure 3: Variation of flux with density

5 CONCLUSION AND RECOMMENDATIONS

5.1 CONCLUSION

Based on the objectives, the study developed heterogeneous motor vehicle traffic flow models focusing on two dimensional heterogeneity and multi - lane road heterogeneity aspects. It established the 2 - ensemble traffic composition formulation using vehicle heterogeneity characteristics in two dimensions. It determined numerical schemes founded on finite difference method to solve the model equations. The method obtained a finite system of linear algebraic equations from the PDE by discretizing the generated PDEs and coming up with the numerical schemes analogous to the equations. The thesis solved the equations subject to the given initial and boundary conditions as per the geometry of the traffic problems as an IBVP. MATLAB software was used to generate solution values in this study and produce 2D graphical representation of the traffic flow variables at different points on the highway. The finite difference technique, used as a solution method in the study, basically involves replacing the partial derivatives occurring in the partial differential equation as well as in the boundary and initial conditions by their corresponding finite difference approximations and then solving the resulting linear algebraic system of equations by a standard iterative procedure. The numerical values of the dependent variables are obtained at the points of intersection of the parallel lines, called mesh points or nodal points.

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