M^x/G(a,b)/1 WITH VACATION INTERRUPTED, OPTIONAL RESERVICE AND BALKING

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Abstract- In this paper, a single server bulk service queuing system with interrupted vacation, request for Re-service and Balking is considered. At a service completion, if the server finds at least 'a' customers waiting for service say ξ, he serves a batch of min (ξ ,b) customers, whereb $\geq$ a. At the completion of an essential service, the leaving batch of customers may request for a re-service with probability π. However, the re-service is rendered only when the number of customers waiting in the queue is less than 'a'. If no request for re-service is made after the completion of an essential service and numbers of customers in the queue is less than then 'a' the server will avail a vacation of a random length. If the queue size reaches 'a' during the vacation, the server terminates the vacation abruptly and resumes for primary service. On completion of the vacation the server remains in the system (dormant period) until the queue length reaches 'a'. Customers arrive in batches to the system and are served on a first come-first served basis. Also assuming that the batch arrival units may decide not to join the system (balks) by estimating the duration of waiting time for a service to get completed or by witnessing the long length of the queue. Analytical treatment of this model is obtained by the supplementary variable technique.

***Index Terms*-:** General Bulk service, interrupted vacation, Re-service, balking

I. INTRODUCTION

In recent years, Bulk service queueing system with server vacations have been developed for a wide range applications in production, communication systems, bank services and etc. Li and Tian (2007) studied the discrete-time GI/Geo/1 queue with working vacation and vacation interruption. Ji-Hong Li et al (2008) studied GI/M/1 queue with working vacations and vacation interruption. Zhang and Shi (2009) provided a study on the M/M/1 queue with Bernoulli-Schedule-Controlled vacation and vacation interruption. Mian Zhang and ZhengtingHou (2010) studied an M/G/1 queue with working vacations and vacation interruption. Yutaka BABA (2010) studied the M/PH/1 queue with working vacations and vacation interruption. Mian Zhang and ZhengtingHou (2011) studied an MAP/G/1 queue with working vacations and vacation interruption.

Madan and Baklizi (2002) considered an M/G/1 queueing model, in which the server performs first essential service to all arriving customers. As soon as the first service is over, they may leave the system with the probability (1- π) and second optional service is provided with probability π. Madan et al (2004) analyzed a single server bulk arrival queue, in which the leaving batch of customers might opt for re-service.. Arumuganathan and JudethMalliga (2006) analyzed a bulk queue with repair of service station and set up time..S.Jayakumar and R.Arumuganathan (2011) discussed a bulk service queue with multiple vacation and request for re-service.

In many real life situations, the arriving customers may be discouraged due to long queue, and decide not to join the queue and leave the system at once. This behavior of customers is referred as balking. Sometimes customers get impatient after joining the queue and leave the system without getting service. This behavior of customer recognizes as reneging. In the last few years, we see studies on queues with balking and reneging gaining significant importance. Queuing system with impatient customers is a familiar phenomenon we come across in many real life situations. Customers may be discouraged to join a queue due to long waiting time or service times or for other constraints and may leave the queue without joining and this is known as balking. We see applications of queue with balking in emergency services in hospitals dealing serious patients, communication systems, production and inventory system and many more. A queue with balking was initially studied by Haight (1957). Since then, extensive amount of work has been done on queuing systems related to impatient customers. Queues with balking have been studied by authors like Altman and Yechiali (2006), Ancker et al. (1963), Choudhury and Medhi (2011), in the last few years.

II. MATHEMATICAL MODEL.

- Let X be the group size random variable of the arrival, λ be the Poisson arrival rate, g_k be the probability that k customers arrive in a batch and X(z) be its PGF.
- The service follows a general (arbitrary) distribution with cumulative distribution function S(.) and density function s(x). Let $\tilde{S}(\theta)$denote the Laplace -Stieltjes transform of S and $S^0(t)$ denote the remaining service time of a batch at an arbitrary time t.
- The server's thevacation time follows a general (arbitrary) distribution with cumulative distribution function$V(.)$ and density function $v(x)$. Let $\tilde{V}(\theta)$ denote the Laplace -Stieltjes transform of V and $V^0(t)$ denote the remaining service time of a batch at an arbitrary time t.

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- The re-service follows a general (arbitrary) distribution with cumulative distribution function R(.) and density function r(x). Let $\tilde{R}(\theta)$denote the Laplace -Stieltjes transform of R and $R^0(t)$ denote the remaining re- service time at an arbitrary time t.
- We assume that $(1-a_1)(0 \leq a_1 \leq 1)$ is the probability that an arriving customer balks during the period when the server is busy and$(1-a_2)(0 \leq a_2 \leq 1)$ is the probability that an arriving customer balks during the period when the server is on vacation
- N_s(t) and N_q(t) are the number of customers in the service and number of customers in the queue respectively.

The different states of the server at time t are defined as follows

$$C(t) = \begin{cases} 0, \text{if the server is busy with service} \\ 1, \text{if the server is on vacatio} \\ 2, \text{if the server is busy with Re} - \text{service} \\ 3, \text{if the server is on idle period} \end{cases}$$

To obtain the system of Equations, the following state probabilities are defined:

$P_{i,n}(x,t)dt = P\{N_s(t) = i, N_q(t) = n, x \leq S^0(t) \leq x + dt, C(t) = 0\}, a \leq i \leq b, n \geq 0,$

$Q_n(x,t)dt = P\{N_q(t) = n, x \leq V^0(t) \leq x + dt, C(t) = 1\}, n \geq 0$

$R_n(x,t)dt = P\{N_q(t) = n, x \leq R^0(t) \leq x + dt, C(t) = 2\}, n \geq 0$

$T_n(t)dt = P\{N_q(t) = n, C(t) = 3\}, 0 \leq n \leq a-1.$

Now, the following system equations are obtained for the queueing system, using supplementary variable technique:

$T_0(t+\Delta t) = T_0(t)(1-\lambda\Delta t) + Q_0(0,t)\Delta t$

$$T_n(t+\Delta t) = T_n(t)(1-\lambda\Delta t) + Q_n(0,t)\Delta t + \sum_{k=1}^{n} T_{n-k}(t)\lambda g_k \Delta t, \qquad 1 \leq n \leq a-1,$$

$$P_{i,0}(x-\Delta t, t+\Delta t) = P_{i,0}(x,t)(1-\lambda\Delta t) + \lambda(1-a_1)P_{i,0}(x,t) + \sum_{m=a}^{b} P_{m,i}(0,t)s(x)\Delta t$$
$$+R_i(0)s(x) + \sum_{k=0}^{a-1} Q_k(x,t)\lambda g_{i-k} s(x)\Delta t + \sum_{m=0}^{n-1} T_m(t)\lambda g_{i-m}\, s(x)\Delta t, ; a \leq i \leq b$$

$$P_{i,j}(x-\Delta t, t+\Delta t) = P_{i,j}(x,t)(1-\lambda\Delta t) + \lambda(1-a_1)P_{i,j}(x,t) + a_1 \sum_{k=1}^{j} P_{i,j-k}(x,t)\lambda g_k \Delta t,$$
$$a \leq i \leq b-1; \text{j} \geq 1$$

$$P_{b,j}(x-\Delta t, t+\Delta t) = P_{b,j}(x,t)(1-\lambda\Delta t) + \lambda(1-a_1)P_{b,j}(x,t) + \sum_{m=a}^{b} P_{m,b+j}(0,t)s(x)\Delta t$$
$$+R_{b+j}(0)s(x) + a_1 \sum_{k=1}^{j} P_{b,j-k}(x,t)\lambda g_k \Delta t + \sum_{m=0}^{n-1} T_m(t)\lambda g_{b+j-m}\, s(x)\Delta t$$
$$+ \sum_{k=0}^{a-1} Q_k(x,t)\lambda g_{b+j-k} s(x)\Delta t; \qquad j \geq 1$$

$$Q_0(x-\Delta t, t+\Delta t) = Q_0(x,t)(1-\lambda\Delta t) + \lambda(1-a_2)Q_0(x,t)$$
$$+(1-\pi)\sum_{m=a}^{b} P_{m,0}(0)v(x)\Delta t + R_0(0)v(x);$$

$$Q_n(x-\Delta t, t+\Delta t) = Q_n(x,t)(1-\lambda\Delta t) + \lambda(1-a_2)Q_n(x,t) + (1-\pi)\sum_{m=a}^{b} P_{m,0}(0)v(x)\Delta t$$
$$+\sum_{k=1}^{n} Q_{n-k}(x,t)\lambda g_k \Delta t + R_n(0)v(x); \qquad 1 \leq n \leq a-1$$

$$R_0(x-\Delta t, t+\Delta t) = R_0(x,t)(1-\lambda\Delta t) + \lambda(1-a_1)R_0(x,t) + \pi \textstyle\sum_{m=a}^{b} P_{m,0}(0,t)r(x)$$

$$R_n(x-\Delta t, t+\Delta t) = R_n(x,t)(1-\lambda\Delta t) + \lambda(1-a_1)R_n(x,t) + \pi \sum_{m=a}^{b} P_{m,n}(0,t)r(x)$$
$$+\lambda a_1 \textstyle\sum_{k=1}^{n} R_{n-k}(x,t)\, g_k;\ 1 \leq n \leq a-1,$$

$$R_n(x-\Delta t, t+\Delta t) = R_n(x,t)(1-\lambda\Delta t) + \lambda(1-a_1)R_n(x,t) + \lambda\sum_{k=1}^{n} R_{n-k}(x,t)\, g_k;\ \ n \geq a.$$

III. STEADY STATE QUEUE SIZE DISTRIBUTION

From the above equations, the steady state queue size equations are obtained as follows:

$$0 = -\lambda T_0 + Q_0(0) \quad (1)$$

$$0 = -\lambda T_n + Q_n(0) + \textstyle\sum_{k=1}^{n} T_{n-k}\lambda\, g_k \quad ,1 \leq n \leq a-1 \quad (2)$$

$$-P_{i,0}^{'}(x) = -\lambda P_{i,0}(x) + \lambda(1-a_1)P_{i,0}(x) + \sum_{m=a}^{b} P_{m,i}(0)s(x)\Delta t$$

$$+\textstyle\sum_{k=0}^{a-1} Q_k(x)\lambda g_{i-k}s(x) + \sum_{m=0}^{n-1} T_m\lambda g_{i-m}\, s(x) + R_i(0)s(x);\, a \leq i \leq b \quad (3)$$

$$-P_{i,j}^{'}(x) = -\lambda P_{i,j}(x) + \lambda(1-a_1)P_{i,j}(x) + a_1\textstyle\sum_{k=1}^{j} P_{i,j-k}(x)\,\lambda g_k; \qquad a \leq i \leq b\, , \mathrm{j} \geq 1 \quad (4)$$

$$-P_{b,j}^{'}(x) = -\lambda P_{b,j}(x) + \lambda(1-a_1)P_{b,j}(x) + a_1\sum_{k=1}^{j} P_{b,j-k}(x)\,\lambda g_k + \sum_{m=a}^{b} P_{m,b+j}(0)s(x)$$

$$+\textstyle\sum_{m=0}^{n-1} T_m\lambda g_{b+j-m}\, s(x) + \sum_{k=0}^{a-1} Q_k(x)\lambda g_{b+j-k}s(x) + R_{b+j}(0)s(x);\, j \geq 1 \quad (5)$$

$$-Q_0^{'}(x) = -\lambda Q_0(x) + \lambda(1-a_2)Q_0(x) + (1-\pi)\textstyle\sum_{m=a}^{b} P_{m,0}(0)v(x) + R_0(0)v(x); \quad (6)$$

$$-Q_n^{'}(x) = -\lambda Q_n(x) + \lambda(1-a_2)Q_n(x) + (1-\pi)\sum_{m=a}^{b} P_{m,0}(0)v(x)$$

$$+a_2\textstyle\sum_{k=1}^{n} Q_{n-k}(x)\lambda g_k + R_n(0)v(x); \qquad 1 \leq n \leq a-1 \quad (7)$$

$$-R_0^{'}(x) = -\lambda R_0(x) + \lambda(1-a_1)R_0(x) + \pi\textstyle\sum_{m=a}^{b} P_{m,0}(0)r(x) \quad (8)$$

$$-R_n^{'}(x) = -\lambda R_n(x) + \lambda(1-a_1)R_n(x) + \pi\textstyle\sum_{m=a}^{b} P_{m,n}(0)r(x) + \lambda a_1\sum_{k=1}^{n} R_{n-k}(x)\, g_k;\, 1 \leq n \leq a-1, \quad (9)$$

$$-R_n^{'}(x) = -\lambda R_n(x) + \lambda(1-a_1)R_n(x) + \lambda\textstyle\sum_{k=1}^{n} R_{n-k}(x)\, g_k;\ \ n \geq a. \quad (10)$$

The Laplace-Stieltjes transforms of $P_{i,n}(x)$ and $Q_j(x)$ are defined as:
$\tilde{P}_{i,n}(\theta) = \int_0^{\infty} e^{-\theta x} P_{i,n}(x)dx$; $\tilde{Q}_j(\theta) = \int_0^{\infty} e^{-\theta x} Q_j(x)dx$and $\widetilde{R}_n(\theta) = \int_0^{\infty} e^{-\theta x} R_n(x)dx$
Taking Laplace-Stieltjes transform on both sides of the equations (1) to (10), we get

$$\theta\tilde{P}_{i,0}(\theta) - P_{i,0}(0) = \lambda\tilde{P}_{i,0}(\theta) - \lambda(1-a_1)\tilde{P}_{i,0}(\theta)$$

$$-\left[\textstyle\sum_{m=a}^{b} P_{m,i}(0) + R_i(0) + \sum_{k=0}^{a-1} Q_k(x)\lambda g_{i-k} + \sum_{m=0}^{n-1} T_m\lambda g_{i-m}\right]\tilde{S}(\theta); \quad a \leq i \leq b, \quad (11)$$

$$\theta\tilde{P}_{i,j}(\theta) - P_{i,j}(0) = \lambda\tilde{P}_{i,j}(\theta) - \lambda(1-a_1)\tilde{P}_{i,j}(\theta)\tilde{S}(\theta) - \lambda a_1\textstyle\sum_{k=1}^{j}\tilde{P}_{i,j-k}(\theta)\, g_k; \quad \mathrm{a} \leq \mathrm{i} < b, j \geq 1 \quad (12)$$

$$\theta\tilde{P}_{b,j}(\theta) - P_{b,j}(0) = \lambda\tilde{P}_{b,j}(\theta) - \lambda(1-a_1)\tilde{P}_{b,j}(\theta) - a_1\sum_{k=1}^{j}\tilde{P}_{b,j-k}(\theta)\,\lambda g_k$$

$$-\textstyle\sum_{m=a}^{b} P_{m,b+j}(0)\tilde{S}(\theta) + R_{b+j}(0)\tilde{S}(\theta) - \sum_{m=0}^{n-1} T_m\lambda g_{b+j-m}\,\tilde{S}(\theta) - \sum_{k=0}^{a-1}\tilde{Q}_k(\theta)\lambda g_{b+j-k}\tilde{S}(\theta);\ j \geq 1 \quad (13)$$

$$\theta\tilde{Q}_0(\theta) - Q_0(0) = \lambda\tilde{Q}_0(\theta) - \lambda(1-a_2)\tilde{Q}_0(\theta) - (1-\pi)\textstyle\sum_{m=a}^{b} P_{m,0}(0)\tilde{V}(\theta) - R_0(0)\tilde{V}(\theta); \quad (14)$$

$$\theta\tilde{Q}_n(\theta) - Q_n(0) = \lambda\tilde{Q}_n(\theta) - \lambda(1-a_2)\tilde{Q}_n(\theta) - (1-\pi)\sum_{m=a}^{b} P_{m,n}(0)\tilde{V}(\theta) - \lambda a_2\sum_{k=1}^{j}\tilde{Q}_{n-k}(\theta)\, g_k - R_n(0)\tilde{V}(\theta);$$

$$1 \leq n \leq a-1 \quad (15)$$

$$\theta\tilde{R}_0(\theta) - R_0(0) = \lambda\tilde{R}_0(\theta) - \lambda(1-a_1)\tilde{R}_0(\theta) - \pi\textstyle\sum_{m=a}^{b} P_{m,0}(0)\,\tilde{R}(\theta); \quad (16)$$

$$\theta\tilde{R}_n(\theta) - R_n(0) = \lambda\tilde{R}_n(\theta) - \lambda(1-a_1)\tilde{R}_n(\theta) - \pi\textstyle\sum_{m=a}^{b} P_{m,0}(0)\,\tilde{R}(\theta) - \lambda a_1\sum_{k=1}^{j}\tilde{R}_{n-k}(\theta)\, g_k;\ \ 1 \leq n \leq a-1 \quad (17)$$

$$\theta\tilde{R}_n(\theta) - R_n(0) = \lambda\tilde{R}_n(\theta) - \lambda(1-a_1)\tilde{R}_n(\theta) - \lambda a_1\textstyle\sum_{k=1}^{j}\tilde{R}_{n-k}(\theta)\, g_k;\ \ n \geq a. \quad (18)$$

IV. SYSTEM SIZE DISTRIBUTION

To obtain the system size distribution let us define PGF's as follows:
$\tilde{P}_i(z,\theta) = \sum_{j=0}^{\infty}\tilde{P}_{i,j}(\theta)z^j$;$P_i(z,0) = \sum_{j=0}^{\infty} P_{i,j}(0)z^j;\, a \leq i \leq b,$
$\tilde{Q}(z,\theta) = \sum_{j=0}^{a-1}\tilde{Q}_j(\theta)z^j$;$Q(z,0) = \sum_{j=0}^{a-1} Q_j(0)z^j$
$\tilde{R}(z,\theta) = \sum_{n=0}^{\infty}\tilde{R}_n(\theta)z^n$; $R(z,0) = \sum_{n=0}^{\infty} R_n(0)z^n;$
$T(z) = \sum_{j=0}^{a-1} T_n z^n$ (19)

The probability generating function P(z) of the number of customers in the queue at an arbitrary time epoch of the proposed model can be obtained using the following equation

$$\mathrm{P(z)} = \sum_{\mathrm{i=a}}^{\mathrm{b-1}} \tilde{P}_i(z,0) + \tilde{P}_b(z,0) + \tilde{Q}(z,0) + \tilde{R}(z,0) + T(z) \tag{20}$$

In order to find the following $\tilde{P}_i(z,\theta), \tilde{P}_b(z,\theta)$, $\tilde{Q}(z,\theta)$ and $\tilde{R}(z,\theta)$sequence of operations are done.

Multiply the equations (14) byz^0 (15) by $z^n(1 < n < a-1)$ and summing up from $n = 0\ to a-1$ and by using (20), we get

$$\left[\theta - a_2\left(\lambda - \lambda\xi(z)\right)\right]\tilde{Q}(z,\theta) = Q(z,0) - \tilde{V}(\theta)\sum_{n=0}^{a-1}\left[\sum_{m=a}^{b}(1-\pi)P_{m,n}(0) + R_n(0)\right]z^n \tag{21}$$

Where $\xi(z) = \sum_{k=1}^{a-1} g_k z^k$

Multiply the equations (16) byz^0,(17) by $z^n(1 < n < a-1)$, (18)by $z^n(n > a)$ and summing up from $n = 0\ to a-1$ and by using (20), we get

$$\tilde{R}(z,\theta) = \frac{\left(\tilde{R}\left(a_1\left(\lambda - \lambda X(z)\right)\right) - \tilde{R}(\theta)\right)\pi\sum_{n=0}^{a-1} p_n z^n}{\left(\theta - \alpha\left(\lambda - \lambda X(z)\right)\right)} \tag{22}$$

Multiply the equations (11) byz^0 , (12) byz^j $(j > a)$ and summing up from $n = 0\ to\infty$ and by using (20), we get

$$\left[\theta - a_1\left(\lambda - \lambda X(z)\right)\right]\tilde{P}_i(z,\theta) = P_i(z,0) - \tilde{S}(\theta)\left[\sum_{m=a}^{b} P_{m,i}(0) + R_i(0) + \sum_{k=0}^{a-1} Q_k(x,t)\lambda g_{i-k} + \sum_{m=0}^{n-1} T_m(t)\lambda g_{i-m}\right]$$

$$a \le i \le b-1 \tag{23}$$

Multiply the equations (11) byz^0 (13) z^j $(j > a)$ and summing up from $j = 0\ to\infty$ and by using (20), we get

$$z^b\left[\theta - a_1\left(\lambda - \lambda X(z)\right)\right]\tilde{P}_b(z,\theta) = P_b(z,0)\left(z^b - \tilde{S}(\theta)\right)$$

$$-\tilde{S}(\theta)\left[\begin{array}{c} \sum_{m=a}^{b-1} P_m(z,0) - \sum_{m=a}^{b}\sum_{j=0}^{b-1} P_{m,j}(0)z^j \\ +R(z,0) - \sum_{n=0}^{b-1} R_n(0)z^n \\ +\lambda\left(T(z)X(z) - \sum_{m=0}^{a-1}\left(T_m z^m \sum_{\mathrm{j=1}}^{\mathrm{b-m-1}} \mathrm{g_j z^j}\right)\right) \\ +\lambda\left(X(z)\sum_{i=0}^{a-1}\tilde{Q}_i(\theta)z^i - \sum_{i=0}^{a-1}\left(\tilde{Q}_i(\theta)z^i \sum_{\mathrm{j=1}}^{\mathrm{b-i-1}} \mathrm{g_j z^j}\right)\right) \end{array}\right] \tag{24}$$

By substituting$\theta = a_2\left(\lambda - \lambda\xi(z)\right)$ in the equations (21), we get

$$Q(z,0) = \tilde{V}\left(a_2\left(\lambda - \lambda\xi(z)\right)\right)\sum_{n=0}^{a-1}\left[(1-\pi)\sum_{m=a}^{b} P_{m,n}(0) + R_n(0)\right]z^n; \tag{25}$$

By substituting$\theta = a_1\left(\lambda - \lambda X(z)\right)$ in the equations (22) & (24), we get

$$R(z,0) = \tilde{R}\left(a_1\left(\lambda - \lambda X(z)\right)\right)\pi\sum_{n=0}^{a-1}\sum_{m=a}^{b} P_{m,n}(0)\, z^n, \tag{26}$$

$$P_i(z,0) = \tilde{S}\left(a_1\left(\lambda - \lambda X(z)\right)\right)\left[\sum_{m=a}^{b} P_{m,i}(0) + R_i(0) + \sum_{k=0}^{a-1} Q_k\left(\lambda - \lambda X(z)\right)\lambda g_{i-k} + \sum_{m=0}^{n-1} T_m\lambda g_{i-m}\right]$$

$$\mathrm{a \le i \le b-1} \tag{27}$$

$$P_b(z,0) = \frac{\tilde{S}\left(a_1\left(\lambda - \lambda X(z)\right)\right)f(z)}{Z^b - \tilde{S}\left(a_1\left(\lambda - \lambda X(z)\right)\right)} \tag{28}$$

where

$$f(z)=\frac{\sum_{m=a}^{b-1}P_m(z,0)-\sum_{m=a}^{b}\sum_{j=0}^{b-1}P_{m,j}(0)z^j+R(z,0)-\sum_{n=0}^{b-1}R_n(0)z^n+\lambda\left(T(z)X(z)-\sum_{m=0}^{a-1}\left(T_m z^m\sum_{j=1}^{b-m-1}g_j z^j\right)\right)}{+\lambda\left(X(z)\sum_{i=0}^{a-1}\tilde{Q}_i\left(a_2(\lambda-\lambda X(z))\right)z^i-\sum_{i=0}^{a-1}\left(\tilde{Q}_i\left(a_2(\lambda-\lambda X(z))\right)z^i\sum_{j=1}^{b-i-1}g_j z^j\right)\right)} \quad (29)$$

From the equation (21) & (24), we have

$$\tilde{Q}(z,\theta)=\frac{\left(\tilde{V}\left(a_2(\lambda-\lambda\xi(z))\right)-\tilde{V}(\theta)\right)\sum_{n=0}^{a-1}[(1-\pi)\sum_{m=a}^{b}P_{m,n}(0)+R_n(0)]z^n}{\left(\theta-a_2(\lambda-\lambda\xi(z))\right)} \quad (30)$$

From the equation (22) & (25), we have

$$\tilde{P}_i(z,\theta)=\frac{1}{\left(\theta-a_1(\lambda-\lambda X(z))\right)}\begin{bmatrix}\left(\tilde{S}\left(a_1(\lambda-\lambda X(z))\right)-\tilde{S}(\theta)\right)\left(\sum_{m=a}^{b}P_{m,i}(0)+R_i(0)+\sum_{m=0}^{n-1}T_m\lambda g_{i-m}\right)\\ \tilde{S}\left(a_1(\lambda-\lambda X(z))\right)\sum_{k=0}^{a-1}\tilde{Q}_k(\lambda-\lambda X(z))\lambda g_{i-k}\\ -\tilde{S}(\theta)\sum_{k=0}^{a-1}\tilde{Q}_k(\theta)\lambda g_{i-k}\end{bmatrix}$$

$$a\le i\le b-1 \quad (31)$$

From the equation (23) & (26), we have

$$\tilde{P}_b(z,\theta)=\frac{\begin{bmatrix}\left(\tilde{S}\left(a_1(\lambda-\lambda X(z))\right)-\tilde{S}(\theta)\right)g(z)+R(z,0)-\sum_{n=0}^{b-1}R_n(0)z^n\\ +\lambda\sum_{k=0}^{a-1}z^i\left(\tilde{Q}_k\left(a_2(\lambda-\lambda X(z))\right)\tilde{S}\left(a_1(\lambda-\lambda X(z))\right)-\tilde{Q}_k(\theta)\tilde{S}(\theta)\right)\left(X(z)-\sum_{j=1}^{b-i-1}g_j z^j\right)\\ -\lambda\sum_{i=0}^{a-1}z^{i-b}\left(\tilde{Q}_i\left(a_2(\lambda-\lambda X(z))\right)-\tilde{Q}_i(\theta)\right)\tilde{S}\left(a_1(\lambda-\lambda X(z))\right)\tilde{S}(\theta)\left(X(z)-\sum_{j=1}^{b-i-1}g_j z^j\right)\end{bmatrix}}{\left(\theta-a_1(\lambda-\lambda X(z))\right)\left(Z^b-\tilde{S}\left(a_1(\lambda-\lambda X(z))\right)\right)} \quad (32)$$

where

$$g(z)=\sum_{m=a}^{b-1}P_m(z,0)-\sum_{m=a}^{b}\sum_{j=0}^{b-1}P_{m,j}(0)z^j+\lambda\left(T(z)X(z)-\sum_{m=0}^{a-1}\left(T_m z^m\sum_{j=1}^{b-m-1}g_j z^j\right)\right)$$

Let $p_i=\sum_{m=a}^{b}P_{m,i}(0)$, $r_i=R_i$, $i\ge 0$

Using the Equations (21), (22) and (23) in the Equation (14), the probability generating function of the queue size P(z) at an arbitrary time epoch is obtained as

$$P(z)=\frac{\begin{pmatrix}\sum_{i=a}^{b-1}\left((z^b-1)\tilde{S}\left(a_1(\lambda-\lambda X(z))\right)\lambda E_1+\left(\tilde{S}\left(a_1(\lambda-\lambda X(z))\right)-1\right)z^bE_3-\left(z^b-\tilde{S}\left(a_1(\lambda-\lambda X(z))\right)\right)\lambda E_2\right)\\ -\left(\tilde{S}\left(a_1(\lambda-\lambda X(z))\right)-1\right)\sum_{j=1}^{b-1}(1-\pi)p_j z^j+\left(\tilde{S}\left(a_1(\lambda-\lambda X(z))\right)-1\right)\lambda E_4+\pi\left(\tilde{R}\left(a_1(\lambda-\lambda X(z))\right)-1\right)\sum_{j=1}^{a-1}p_j z^j\\ +\left(z^b-\tilde{S}\left(a_1(\lambda-\lambda X(z))\right)\right)a_1(-\lambda+\lambda X(z))T(z)+\lambda\sum_{i-0}^{a-1}z^i f(\tilde{S},\tilde{Q},z)-\lambda\sum_{i=0}^{a-1}z^{i-b}\phi(\tilde{S},\tilde{Q},z)\end{pmatrix}}{-a_1(\lambda-\lambda X(z))\left(Z^b-\tilde{S}\left(a_1(\lambda-\lambda X(z))\right)\right)}$$

$$+\frac{\left(\tilde{V}\left(a_2(\lambda-\lambda\xi(z))\right)-1\right)\sum_{n=0}^{a-1}(1-\pi)p_n z^n}{-a_2(\lambda-\lambda\xi(z))} \quad (33)$$

On further simplification, we get

$P(z)$

$$= \frac{a_2\left(-\lambda+\lambda\xi(z)\right)\begin{pmatrix} \sum_{i=a}^{b-1}\left((z^b-1)\tilde{S}\left(a_1(\lambda-\lambda X(z))\right)\lambda E_1+\left(\tilde{S}\left(a_1(\lambda-\lambda X(z))\right)-1\right)z^b E_3-\left(z^b-\tilde{S}\left(a_1(\lambda-\lambda X(z))\right)\right)\lambda E_2\right) \\ -\left(\tilde{S}\left(a_1(\lambda-\lambda X(z))\right)-1\right)\sum_{j=1}^{b-1}p_j z^j+\left(\tilde{S}\left(a_1(\lambda-\lambda X(z))\right)-1\right)\lambda E_4+\pi\left(\tilde{R}\left(a_1(\lambda-\lambda X(z))\right)-1\right)\sum_{j=1}^{a-1}p_j z^j \\ -\pi\sum_{j=1}^{a-1}p_j z^j\left(\tilde{S}\left(a_1(\lambda-\lambda X(z))\right)-1\right)+\left(z^b-\tilde{S}\left(a_1(\lambda-\lambda X(z))\right)\right)a_1(-\lambda+\lambda X(z))T(z) \\ +\lambda\sum_{i=0}^{a-1}z^i f(\tilde{S},\tilde{Q},z)-\lambda\sum_{i=0}^{a-1}z^{i-b}\phi(\tilde{S},\tilde{Q},z) \end{pmatrix} + \left(a_1(-\lambda+\lambda X(z))\left(Z^b-\tilde{S}\left(a_1(\lambda-\lambda X(z))\right)\right)\left(\tilde{V}\left(a_2(\lambda-\lambda\xi(z))\right)-1\right)\sum_{n=0}^{a-1}(1-\pi)p_n z^n\right)}{a_1 a_2(-\lambda+\lambda\xi(z))(-\lambda+\lambda X(z))\left(Z^b-\tilde{S}\left(a_1(\lambda-\lambda X(z))\right)\right)} \tag{34}$$

where

$$f(\tilde{S},\tilde{Q},z)=\left(\tilde{Q}_i a_2(\lambda-\lambda X(z))\tilde{S}\left(a_1(\lambda-\lambda X(z))\right)-\tilde{Q}_i(\theta)\tilde{S}(\theta)\right)\left(X(z)-\sum_{j=1}^{b-i-1}g_j z^j\right)$$

$$\phi(\tilde{S},\tilde{Q},z)=\left(\tilde{Q}_i\left(a_2(\lambda-\lambda X(z))\right)-\tilde{Q}_i(0)\right)\tilde{S}\left(a_1(\lambda-\lambda X(z))\right)\left(X(z)-\sum_{j=1}^{b-i-1}g_j z^j\right)$$

$$E_1=\sum_{k=0}^{a-1}\tilde{Q}_k\left(a_2(\lambda-\lambda X(z))\right)g_{i-k} \quad ; \quad E_2=\sum_{k=0}^{a-1}\tilde{Q}_k(\theta)\lambda g_{i-k}$$

$$E_3=p_i+r_i+\sum_{m=0}^{n-1}T_m\lambda g_{i-m} \quad ; \quad E_4=\left(T(z)X(z)-\sum_{m=0}^{a-1}\left(T_m z^m\sum_{j=1}^{b-m-1}g_j z^j\right)\right)$$

The probability generating function P(z) has to satisfy P(1) = 1.
In order to satisfy the condition, applying L'Hospital's rule and evaluating $\lim_{z\to\infty}$ P(z)and equating the expression to 1, $b-\lambda a_1 E(X)[E(S)+\pi E(R)]>0$is obtained.

Define 'ρ' as $\frac{\lambda a_1 E(X)[E(S)+\pi E(R)]}{b}$. Thus ρ<1 is the condition to be satisfied for the existence of steady state for the model.

V. PERFORMANCE MEASURES

In this section, some useful performance measures of the proposed model like, expected number of customers in the queue E(Q), expected length of idle period E(I), expected length of busy period E(B) are derived which are useful to find the total average cost of the system. Also, probability that the server is on setup work P(U), probability that the server is on vacation P(V) and probability that the server is busy P(B) are derived.

EXPECTED QUEUE LENGTH

The expected queue length E(Q) (i.e. mean number of customers waiting in the queue) at an arbitrary time epoch, is obtained by differentiating P(z) at z=1 and given by

$$\lim_{z\to 1}P(z)=E(Q)$$

$$E(Q)=\frac{\begin{bmatrix}\sum_{i=a}^{b-1}[2(T11)(f1)-(f7)(f8)]-\sum_{j=0}^{b-1}d_j[2(T11)(f2)-(f7)(f9)] \\ \sum_{n=0}^{a-1}[2(T11)\lambda(f5-f6)-\lambda(f7(f11)]+2(T11)(f4)-(f7)(f11) \\ \pi[2f14T11-f7f13]\sum_{i=0}^{a-1}p_i\end{bmatrix}}{12(T11)^2}+\frac{\sum_{n=0}^{a-1}(1-\pi)p_n[f3-f10]}{(T27)^2} \tag{35}$$

where

$S1=\lambda a_1 E(X)E(S)$; $S2=\lambda a_1 X''(1)E(S)+\lambda^2 {a_1}^2 E^2(X)E(S^2)$;
$S3=\lambda a_1 X'''(1)E(S)+3\lambda^2 {a_1}^2 E(X)E(X^2)E(S^2)+\lambda^3 {a_1}^3 E^2(X)E(S^3)$;

$$V1=\lambda a_2E(X)E(V);\;\; \mathrm{G1}=1-\sum_{j=1}^{b-i-1} g_j\,;\mathrm{G2}=\mathrm{E(X)}-\sum_{j=1}^{b-i-1} jg_j\,;$$

$$\mathrm{G3}=\mathrm{E}(X^2)-\sum_{j=1}^{b-i-1} j(j-1)g_j\,;\mathrm{G4}=\mathrm{T}(1)-\sum_{m=0}^{a-1}\sum_{j=1}^{b-m-1} T_m g_j\,;$$

$$\mathrm{G5}=\sum_{m=0}^{a-1}\sum_{j=1}^{b-m-1}(m+j)(m+j-1)T_m g_j;\;\; \mathrm{G6}=\sum_{m=0}^{a-1}\sum_{j=1}^{b-m-1}(\mathrm{m}+\mathrm{j})T_m g_j\,;$$

$\mathrm{G7}=E(X)\mathrm{T}'(1);\;\; \mathrm{G8}=E(X)\mathrm{T}(1);\;\; \mathrm{G9}=E(X^2)\mathrm{T}(1)$

$$\mathrm{G10}=p_i+r_i+\sum_{m=0}^{n-1}T_m\lambda g_{i-m}\;;\;\; \mathrm{T1}=\left(\sum_{k=0}^{a-1}\tilde{Q}_k\left(a_2\big(\lambda-\lambda X(z)\big)\right)g_{i-k}\right)_{z=1};$$

$$\mathrm{T2}=\left(\frac{\mathrm{d}}{\mathrm{dz}}\left(\sum_{k=0}^{a-1}\tilde{Q}_k\left(a_2\big(\lambda-\lambda X(z)\big)\right)g_{i-k}\right)\right)_{z=1};\;\mathrm{T3}=\left(\sum_{k=0}^{a-1}\tilde{Q}_k\left(a_2\big(\lambda-\lambda X(z)\big)\right)\right)_{z=1};$$

$$\mathrm{T4}=\left(\frac{\mathrm{d}}{\mathrm{dz}}\left(\sum_{k=0}^{a-1}\tilde{Q}_k\left(a_2\big(\lambda-\lambda X(z)\big)\right)\right)\right)_{z=1};\;\;\mathrm{T5}=\left(\frac{\mathrm{d}^2}{\mathrm{dz}^2}\left(\sum_{k=0}^{a-1}\tilde{Q}_k\left(a_2\big(\lambda-\lambda X(z)\big)\right)\right)\right)_{z=1}$$

$$\mathrm{T6}=\sum_{k=0}^{a-1}\tilde{Q}_k(0)\,g_{i-k}\,;\mathrm{T7}=\frac{\mathrm{d}}{\mathrm{dz}}\left(\sum_{k=0}^{a-1}\tilde{Q}_k(0)\right);\;\; T8=\sum_{k=0}^{a-1}\tilde{Q}_k(0)\,;$$

$$\mathrm{T9}=\left(\frac{\mathrm{d}^3}{\mathrm{dz}^3}\left(\sum_{k=0}^{a-1}\tilde{Q}_k\left(a_2\big(\lambda-\lambda X(z)\big)\right)\right)\right)_{z=1};\mathrm{T10}=\left(\frac{\mathrm{d}^2}{\mathrm{dz}^2}\left(\sum_{k=0}^{a-1}\tilde{Q}_k\left(a_2\big(\lambda-\lambda X(z)\big)\right)g_{i-k}\right)\right)_{z=1}$$

$T11=\lambda a_1E(X)(b-S1);\;\; T12=\big(-\lambda+\lambda\,\xi(z)\big);\;\; d_n=(1-\pi)p_n+r_n$

$\xi(1)=\sum_{k=1}^{a-1}g_k$ and $\xi'(1)=\sum_{k=1}^{a-1}kg_k$;

$$f1(\lambda,b,S,R,\tilde{Q})=3\lambda b[(b-1)(T2+T1S1)+G10+2T2S1+T1S2]+\lambda b(b-1)(b-2)[T1-T6]+G10[3b(b-1)+3bS2+S3]+\lambda T6S3;$$

$f2(\lambda,X,S)=3j[S2+(j-1)S1]+S3;$

$f3\big(\lambda,\xi,\tilde{V}\big)=[\tilde{V}(\lambda-\lambda\,\xi(z)-1]\big(nT12-\lambda\xi'(1)\big);$

$$f4(\lambda,b,S,R)=3\lambda S1[G9+2G7+T''(1)-G5]+3\lambda S2[G8+T'(1)-G6]+\lambda G4S3+\lambda G8[3b(b-1)-S2]+3\lambda(b-S1)[2G7+3G9];$$

$$f5\big(S,G,\tilde{Q}\big)=[T4+S1T3](3i(i-1)G1+6IG2+3G3)+[T5+2T4S2+T3S2](3iG1+3G2)+[T9+3T5S1+3T4S2+T3S3]G1;$$

$$f6\big(S,G,\tilde{Q}\big)=3(i-b)[(i-b-1)G1T4+T5S1+2T4(S1G1+G2)]+3T5(S1G1+G2)+3T4(S2G1+G3+2S1G1)+T9G1;$$

$f7(\lambda,X,b,S)=3\lambda a_1[X^{'''''\ '\ '}(1)(b-S1)+X^{'\ '}(1)(b(b-1)-S2)]$

$f8\big(\lambda,b,S,G,\tilde{Q}\big)=\lambda[b(2T2+T1(b-1)+2S1)-(b-1)T6+T6S2]+G10[2bS1+S2];$

$f10(\lambda,X,S)=2jS1+S2;$

$f11(\lambda,S,G)=2\lambda S1[G8+T'(1)-G6+\lambda S2G4]+2\lambda(b-S1)G8;$

$f12\big(S,G,\tilde{Q}\big)=[S1T8+T4](2iG1+G2)+(t5+2T4S1+T3S2)G1-2T4G1(i-b+S1)-2T4G2-T5G1;$

$f13(\lambda,b,S,R)=2R1b;$

$f12(\lambda,b,S,R)=3[(b(b-1)+2bi)R1+bR2]$

PARTICULAR CASES

In this section, an existing model is deduced as particular case of the proposed model.

Case(i)

If no request for re-service (i.e. π=0) and if no balking (i.e.,$a_1=1,a_2=1$) then the Equation (32) reduces to

$$P(z)=\frac{a_2\left(-\lambda+\lambda\xi(z)\right)\begin{pmatrix}\sum_{i=a}^{b-1}\left((z^b-1)\tilde{S}\left(\lambda-\lambda X(z)\right)\lambda E_1+\left(\tilde{S}\left(\lambda-\lambda X(z)\right)-1\right)z^bE_3-\left(z^b-\tilde{S}\left(\lambda-\lambda X(z)\right)\right)\lambda E_2\right)\\ -\left(\tilde{S}\left(\lambda-\lambda X(z)\right)-1\right)\sum_{j=1}^{b-1}p_jz^j+\left(\tilde{S}\left(\lambda-\lambda X(z)\right)-1\right)\lambda E_4\\ +\left(z^b-\tilde{S}\left(\lambda-\lambda X(z)\right)\right)\left(-\lambda+\lambda X(z)\right)T(z)+\lambda\sum_{i-0}^{a-1}z^i f\left(\tilde{S},\tilde{Q},z\right)-\lambda\sum_{i=0}^{a-1}z^{i-b}\phi\left(\tilde{S},\tilde{Q},z\right)\end{pmatrix}\\ +\left(\left(-\lambda+\lambda X(z)\right)\left(Z^b-\tilde{S}\left(\lambda-\lambda X(z)\right)\right)\left(\tilde{V}\left(\lambda-\lambda\xi(z)\right)-1\right)\sum_{n=0}^{a-1}p_nz^n\right)}{\left(-\lambda+\lambda\xi(z)\right)\left(-\lambda+\lambda X(z)\right)\left(Z^b-\tilde{S}\left(a_1\left(\lambda-\lambda X(z)\right)\right)\right)}$$

which is the PGF of queue size distribution of Analysis of a batch arrival general bulk service queueing system with interrupted vacation of M.Haridoss (2012)

Case(ii)

If no request for re-service (i.e. π=0) and if the server doesn't avail any vacation(i.e.,$\tilde{V}\left(a_2\left(\lambda-\lambda\xi(1)\right)\right)=1$), and if no balking (i.e.,$a_1=1, a_2=1$) then the Equation (32) reduces to

$$P(z)=\frac{\begin{pmatrix}\sum_{i=a}^{b-1}\left((z^b-z^i)\ p_i\left(\tilde{S}\left(\lambda-\lambda X(z)\right)-1\right)\right)+\lambda\sum_{i=a}^{b-1}\left(\left(\tilde{S}\left(\lambda-\lambda X(z)\right)-1\right)z^b\sum_{m=0}^{a-1}T_m g_{i-m}\right)\\ -\left(\tilde{S}\left(\lambda-\lambda X(z)\right)-1\right)\sum_{j=1}^{a-1}p_jz^j+\left(z^b-\tilde{S}\left(\lambda-\lambda X(z)\right)\right)\left(-\lambda+\lambda X(z)\right)T(z)\\ +\lambda\left(\left(\tilde{S}\left(\lambda-\lambda X(z)\right)-1\right)\left(T(z)X(z)-\sum_{m=0}^{a-1}\left(T_m z^m\sum_{j=1}^{b-m-1}g_jz^j\right)\right)\right)\end{pmatrix}}{-\left(\lambda-\lambda X(z)\right)\left(Z^b-\tilde{S}\left(\lambda-\lambda X(z)\right)\right)}$$

which is the PGF of queue size distribution of M^X /G /1 queueing system without vacation and it coincides with the result of M^X /G /1 queueing system without modified vacation and constant arrival rate of Balasubramanian et al (2011).

VI. Conclusion

An M^X /G(a,b)/1 queue with vacation interrupted, request for Re-service and balking service has been studied. The PGF of queue size at an arbitrary time epoch is obtained . Some performance measures are also derived.

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