Construction and Properties of Ternary Cantor set

[1]Paramacutty Paramadevan

[1]Department of Mathematics, Eastern University Sri Lanka.
paramadevanp@esn.ac.uk

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Abstract: A remarkable mathematical construct, the Ternary Cantor set is distinguished by its special qualities, such as being an uncountable set of measure zero, a perfect set, and a nowhere dense set. Despite its relatively straightforward construction, this set has extensive applications in computer science, physics, and mathematics. Among other areas in which it is important are chaotic dynamical systems, fractal theory, and set theory. Moreover, the Cantor set has been extended to higher dimensions and used in a variety of mathematical environments, including fractal, metric, and topological spaces. With an explanation of its recursive construction and a focus on its essential qualities, this study attempts to present the Cantor set in a way that is easily understood by novices. Furthermore, a number of significant theorems about the Cantor set are explored to provide a deeper understanding of its structure and significance in mathematics.

Keywords: uncountable set, measure, nowhere dense, fractal theory, chaotic dy- namical systems.

Introduction

Cantor set is a set possess uncountable many points in the closed interval [0, 1] and was discovered in 1874 by Henry John Stephen Smith. Later on, after the first research article published on Cantor ternary set by Georg Cantor, a German mathematician, in 1883, Cantor set become famous and started to play important roles in many area of mathematics such as set theory, measure theory to real analysis, fractal theory, topology and chaotic dynamical systems and in many area of computer science. The interplay between ternary arithmetic and the iterative construction of the Cantor set has implications for signal processing, information theory and the broader study of non-linear dynamics. Cantor's original construction involved removing the middle third of a line segment repeatedly, producing a set with remarkable properties. The classical construction of the Cantor set involves a process known as iterated removal of intervals. This approach, while conceptually simple, has inspired numerous variations and extensions. Researchers have explored modifications to the removal process considering different fractions and patterns leading to rich variety of Cantor like sets.

Even though the Cantor set can be define in many ways in various literature[1], the most accepted and comprehensive form [2,3] is as follows:

$$C = [0,1] \,/ \bigcup_{n=0}^{\infty} \bigcup_{n=1}^{3n-1} \left(\frac{3k+1}{3^{n+1}}, \frac{3k+2}{3^{n+1}} \right),$$

where, every middle third is removed as the open interval $\left(\frac{3k+1}{3^{n+1}}, \frac{3k+2}{3^{n+1}}\right)$ from the closed interval $\left[\frac{3k+0}{3^{n+1}}, \frac{3k+3}{3^{n+1}}\right] = \left[\frac{k+0}{3^n}, \frac{k+1}{3^n}\right]$ surrounding it. While in some literatures Canter set defined quite different form such as:

$$C = \bigcap_{n=0}^{\infty} \bigcup_{k=0}^{3^{n-1}-1} \left(\left[\frac{3k+0}{3^n}, \frac{3k+1}{3^n}\right] \cup \left[\frac{3k+2}{3^n}, \frac{3k+3}{3^n}\right] \right),$$

where, the middle third $\left(\frac{3k+1}{3^n}, \frac{3k+2}{3^n}\right)$ of the foregoing closed interval

$\left[\frac{k+0}{3^{n-1}}, \frac{k+1}{3^{n-1}}\right] = \left[\frac{3k+0}{3^n}, \frac{3k+3}{3^n}\right]$ is removed by intersecting with $\left[\frac{3k+0}{3^n}, \frac{3k+1}{3^n}\right] \cup \left[\frac{3k+2}{3^n}, \frac{3k+3}{3^n}\right]$.

Let us explore the construction process of the Cantor set in the next section.

Construction of ternary cantor set

In this section, let us explore the iterative process to form the Cantor set by recursion method. At first trisect the closed interval [0,1] at two points 1/3, 2/3 and remove the open interval (1/3,2/3), which is called the *middle third*, and obtained the set $C_1 = [0,1/3] \cup [2/3,1]$. Again by trisecting each of the closed intervals [0, 1/3] and [2/3,1] and removing the open middle thirds (1/9,2/9) and (7/9,8/9), respectively, we obtain

$C_2 =$ [0,1/9]∪[2/9,3/9]∪[6/9,7/9]∪[8/9,9/9]. Repeatedly, by removing the middle third of each of the above four intervals, we obtain C_3= [0, 1/27] ∪ [2/27, 3/27] ∪ [6/27, 7/27] ∪ [8/27, 9/27] ∪ [18/27, 19/27] ∪ [20/27, 21/27] ∪ [24/27, 25/27] ∪ [26/27, 27/27].

By continuing in the same manner recursively, we can obtain sets $C_1, C_2, C_3, \dots$.

Then the Ternary Cantor set, denoted by C,is the intersection of all C_1, C_2, C_3 ...

$$ie.\ C = \bigcap_{k=1}^{\infty} C_K$$

Alternatively, as discussed in [4], if S is a subset of the set of real numbers, then we define

$$\frac{1}{3}s = \left\{\frac{1}{3}s : s \in S\right\} and\ S + \frac{1}{3} = \left\{s + \frac{1}{3} : s \in S\right\}.$$

$$Let\ C_0 = [0,1],$$

$$then\ C1 = \frac{1}{3}\ C_0 \cup \left(\frac{1}{3}C_0 + \frac{2}{3}\right) = \left[0,\frac{1}{3}\right] \cup \left[\frac{2}{3}\ 1\right],$$

$C_2 = \frac{1}{3}\ C_1 \cup \left(\frac{1}{3}C_1 + \frac{2}{3}\right) = \left[0,\frac{1}{9}\right] \cup \left[\frac{2}{9},\frac{3}{9}\right] \cup \left[\frac{6}{9},\frac{7}{9}\right] \cup \left[\frac{8}{9},\frac{9}{9}\right]$ by continuing, we can obtain

$$C_{k+1} = \frac{1}{3}C_k \cup \left(\frac{1}{3}C_k + \frac{2}{3}\right).$$

Since $C = \cap_{k=1}^{\infty} C_K, we\ have\ C = \frac{1}{3}C \cup \left(\frac{1}{3}C + \frac{2}{3}\right).$

In these ways the Cantor set can be generated recursively.

Simply, the graphical representation of the elements of the Cantor set can be represented as follows:

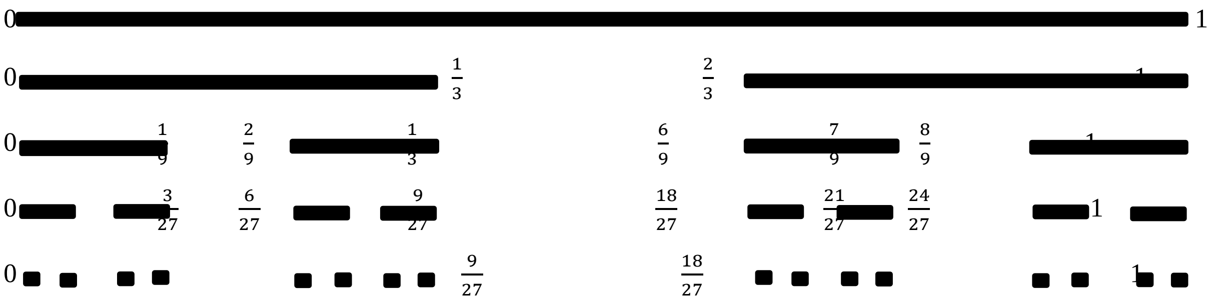

Figure (i)

In the above Figure (i), each line segment indicates real numbers in the respective closed intervals. The first line shows that, initially we consider all real numbers in the closed interval [0, 1]. Second line shows two disjoint closed intervals [0, 1/3] and [2/3, 1] after the removal of the middle third (1/3,2/3) of [0,1]. Meantime, third line shows the four closed intervals remaining once the middle thirds (1/9, 2/9) and (7/9, 8/9) of (0, 1/3) and (2/3, 1) respectively are removed and the iterative process continuous infinitely may times.

In the next section let us discuss some properties of Ternary Cantor set.

Some properties of ternary cantor set

In this section let us explore some basic properties of Ternary Cantor set.

Property 1: C is non-empty
We can observe that during each step findingC_k, the end points of C_{k-1} remain in C_k. Therefore, C contains all the end points, hence C is non-empty.

Property 2: C is closed
Since each C_k is closed, as it is a finite union of closed intervals and $C = \cap C_K$ is a countable intersection of closed sets and is therefore closed.

Property 3: C is perfect
We have seen that the Cantor set is closed and it is proved in Theorem 6 it has no isolated points. These are the conditions for a set to be perfect[5].

Theorems related with ternary cantor set

Before get in to the theorem let us start with the following lemma:

Lemma .1

Every real number in [0, 1] can be expressed of the form

$$\sum_{k=1}^{\infty} \frac{a_k}{2^k}\ , where\ a_k \in \{0,1,2\}.$$

Proof: Define a function $f : [0,1,2]^N \to [0,1]$ such that

$f(a_1, a_2, a_3, \ldots\ldots) = f\left(\sum_{k=1}^{\infty} \frac{a_k}{2^k}\right) = c \in [0,1]$, where the sequence $(a_1, a_2, a_3, \ldots\ldots)$ represents the series $\sum_{k=1}^{\infty} \frac{a_k}{3^k}$.

Clearly f is a well-defined function and the sequence of partial sum $s_n = \sum_{k=1}^{n} \frac{a_k}{3^k}$

is increasing in n and bounded above by

$$2\sum_{k=1}^{\infty} \frac{1}{3^k} = 2.\frac{\frac{1}{3}}{1-\frac{1}{3}} = 1.$$

Therefore, the infinite series is convergent in [0, 1].

First, let us prove f is onto.

Let $c \in [0,1)$ and a_1 be the largest integer such that $\frac{a_1}{3} \leq c$. Since $0 \leq c < 1$

we have $0 \leq a_1 \leq 2\ and\ 0 \leq c - \frac{a_1}{3} < \frac{1}{3}$.

Let a_2 be the largest integer such that $\frac{a_2}{3} \leq c - \frac{a_1}{3} < \frac{1}{3}$. We must have $0 \leq a_2 \leq 2\ and\ 0 \leq c - \frac{a_1}{3} - \frac{a_2}{3^2} < \frac{1}{3^2}$.

By continuing in this way we obtain a sequence of $a_i \in \{0,1,2\}$ such that

$$0 \leq c - \sum_{k=1}^{n} \frac{a_k}{3^k} < \frac{1}{3^n}.$$

This implies,

$$c = lim \sum_{k=1}^{n} \frac{a_k}{3^k} = \sum_{k=1}^{\infty} \frac{a_k}{3^k}.$$

Further, $c = 1$, we take all $a_j = 2$ then

$$\sum_{k=1}^{n}\frac{2}{3^k}=\frac{2}{3}\sum_{k=0}^{n-1}\frac{1}{3^k}=1-\frac{1}{3^n}.\ Hence\ \sum_{k=1}^{\infty}\frac{2}{3^k}=1$$

Thus we proved $\forall c\in[0,1]$, there exists a sequence $(a_1,a_2,a_3,\ldots..)$ such $f(a_{1,}a_2,a_3,\ldots.)=$

$$\sum_{k=1}^{\infty}\frac{a_k}{3^k}=c, that\ is\ f\ is\ onto.$$

However, this mapping is not one to one. Let we explore in which situation it is not one to one.

Suppose we have two different sequences $(a_1,a_2,a_3,\ldots.)$ and $(b_1,b_2,b_3,\ldots)$ such that

$$\sum_{k=1}^{\infty}\frac{a_k}{3^k}=\sum_{k=1}^{\infty}\frac{b_k}{3^k}.$$

Since these are different sequences there exists some k such that $a_k\neq b_k$.

Suppose j is the smallest integer such that $a_j\neq b_j$.

Without loss of generality as- same $a_j>b_j$ then we have

$$\frac{a_j-b_j}{3^j}+\sum_{k=j+1}^{\infty}\frac{a_k}{3^k}=\sum_{k=j+1}^{\infty}\frac{b_k}{3^k}.$$

Since $\frac{a_j-b_j}{3^j}\geq\frac{1}{3^j}$ and $\sum_{k=j+1}^{\infty}\frac{b_k}{3^k}\leq\sum_{k=j+1}^{\infty}\frac{2}{3^k}=\frac{1}{3^n}$.

The only one possibility to have this inequalities and the previous equation are that

$a_j-b_j=1$ and $a_k=0, b_k=2$ for $k\geq j+1$.

That is the sequence a_k has terminal 0's and b_k has terminal 2's. Except this situation the ternary expansion of $c\in\{0,1\}$ is unique.

Theorem 1. Each number of the Cantor set can be expressed as a series, called

Ternary expansion, of the form $\sum_{k=1}^{\infty}\frac{a_k}{3^k}, where\ a_k\in\{0,2\}$.

$proof$. Note that in the proof of the above lemma, $a_j=b_j+1$ then we have either $a_j=1$ or $b_j=1$. From this we can conclude that different sequence in $\{0, 2\}^{\mathrm{N}}$ give different values of c in the cantor set C ⊆ [0, 1].

Theorem 2: Every real number in [0, 1] can be expressed in the scale of 2, called

Binary expansion, of the form $\sum_{k=1}^{\infty}\frac{b_k}{2^k}, where\ b_k\ \in\{0,1\}$.

$Proof$. Define a map $f: C\rightarrow[0,1]$ such that $f(x)=\sum_{k=1}^{\infty}\frac{\frac{a_k}{2}}{2^k}, where\ x\in C$

has unique expression such a way that $x=\sum_{k=1}^{\infty}\frac{a_k}{3^k}, a_k\in\{0,2\}$.

We can see that if x has ternary expansion $(a_1,a_2,a_3,\ldots), then\ f(x)$ has binary expansion $\left(\frac{a_1}{2},\frac{a_2}{2},\frac{a_3}{2},\ldots\right)$. That is the binary expansion become an element of [0,1]. Thus the function is well-defined and clearly this function is on to since every sequence in $\{0,1\}^{\mathrm{N}}$ can be obtained in this manner.

But this mapping is not one to one since $\frac{1}{3}=(0,2,2,2,\ldots)$, then $f\left(\frac{1}{3}\right)=(0,1,1,1,\ldots)=\frac{1}{2}$, while $\frac{2}{3}=(2,0,0,0,\ldots)$, then $f\left(\frac{2}{3}\right)=(1,0,0,0,\ldots)=\frac{1}{2}$.

Theorem 3: The Canter set is non-denumerable.

$Proof$. By Theorem 1, every number in the Cantor set has ternary expansion of the form

$$\sum_{k=1}^{\infty} \frac{a_k}{3^k}, where\ a_k \in \{0,2\}.$$

While Theorem 2, every number in [0, 1] has binary expansion of the form

$\sum_{k=1}^{\infty} \frac{b_k}{2^k}, where\ b_k \in \{0,1\}$ with the assumption that not all b_k after certain terms are 0.

Now we construct a one to one correspondence between these two expansions such that $b_k = 0$ when ever $a_k = 0$ and $b_k = 1$ when ever $a_k = 2$. Then, since the set of all real numbers in [0, 1] is non-denumerable so also the Cantor set.

Theorem 4: The Cantor set is measurable.

$Proof.$ We can obtain a sequence, when see what is removed from each iteration. At first, the middle third of [0, 1], interval of length $\frac{1}{3}$ is removed. During the second iteration, two middle third intervals each of length $\frac{1}{9}$ are removed. In the next iteration, four middle third intervals each has length $\frac{1}{27}$ is removed. In general at the k^{th} step 2^{k-1} intervals with length $\frac{1}{3^k}$ is removed. Therefore, measure of C, $1 - \sum_{n=1}^{\infty} \frac{2^{n-1}}{3^n} = 0$.

Theorem 5: The Cantor set is nowhere dense.

$Proof.$ It is enough to proof C does not contain any open intervals.

Let O be an arbitrary open interval in [0, 1] with length l. By Archimedean property,

choose k such that $\frac{1}{3^k} < l$. We know that each C_k has 2^k disjoint closed intervals each has length $\frac{1}{3^k}$. Let I be one of such interval. Therefore, length of I is less than the length of the open interval O. Hence $O \nsubseteq I \subseteq C_k$, thus $O \nsubseteq C$.

Theorem 6: The Cantor set has no isolated points.

$Proof.$ It is enough to proof every point c of C is a limit point of C. Which is

equivalent to show for any arbitrary ϵ, we can find $t \in C$ such that $0 < |c - t| < \epsilon$

By Archimedean property, choose k such that $\frac{1}{3^k} < l$. Since $c \epsilon C, c \in C_{K.}$ for some k.

That is, there is a closed interval $I \in C$ with length $\frac{1}{3^k}$

When we consider $I \cap C_{k+1}$, it contains two component intervals in which one, say

I_0 must contain c with an end point t. Thus $0 < |c - t| \leq \frac{1}{3^k} < \epsilon$.

Conclusion

The Cantor set has many interesting properties and consequences in the field of pure mathematics, computer science and engineering. One of the interesting property addressed in this research work is that it is a uncountable set with measure zero. In this study the non-denumerable property has been proved using the fact that every number in the Cantor set has ternary expansion in which only the integers 0, 2 involve and every real number in [0,1] has binary expansion. While the classical construction method is well established, alternative approaches also to constructing Cantor set have been proposed in various literatures. These include probabilistic methods, symbolic dynamics and number theory connections. Exploring this alternative avenues not only deepens our understanding of Cantor set but also opens up new perspective for future research. In conclusion, the construction of the Cantor set has been a central theme in mathematical research for over a century. The rich tapestry of approaches, generalizations, and applications underscores the enduring significance of this fundamental mathematical concept. As researchers continue to explore new directions and address open questions, the literature on the construction of Cantor set is likely to evolve and expand contributing to the broader landscape of mathematical knowledge.

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