Properties of Nαg^*s Closed Sets in Topological Spaces

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Abstract- In this paper, we investigate Nαg^*s closure and Nαg^*s interior. Also we study NT^*αgs space and its relation to various other spaces.

Index Terms- nano toplogy, Nαg^*s closed set, NT^* αgs space.

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I. Introduction

Lellis Thivagar etal [3] introduced a nano topological space with respect to a subset X of an universe which is defined in terms of lower and upper approximations of X. The elements of a nano topological space are called nano open sets. 'Nano' is a Greek word which means 'very small'. The topology studied here is given the name nano topology as it has utmost five elements. Certain weak forms of nano sets were studied by various authors. Njastad [7] and Mashour etal [6] investigated the concept of α open and α closed sets respectively in topological spaces. The generalized closed (briefly g closed) sets were analyzed by Levine [5]. Arya and Nour [1] introduced and studied weaker forms of closed sets namely, generalized semi closed (briefly gs closed) sets using open sets. T.D. Rayanagoudar[8] introduce a new class of closed sets, called αg^* semi closed (briefly αg^*s closed) sets using gs open sets in topological spaces.
In this paper we study Nαg^*s closure and Nαg^*s interior and some applications of Nαg^*s closed sets.

II. PRELIMINARIES

Definition 2.1: [3] Let $\mathcal{U}$ be a non-empty finite set of objects called the universe and R be an equivalence relation on $\mathcal{U}$ named as the indiscernibility relation. Elements belonging to the same equivalence class are said to be indiscernible with one another. The pair $(\mathcal{U}, R)$ is said to be the approximation space. Let X⊆ $\mathcal{U}$.

(i) The lower approximation of X with respect to R is the set of all objects, which can be for certain classified as X with respect to R and it is denoted by $L_R(X)$. That is, $L_R(X) = \bigcup_{x\epsilon\mathcal{U}}\{R(x): R(x) \subseteq X\}$, where R(x) denotes the equivalence class determined by x.

(ii) The upper approximation of X with respect to R is the set of all objects, which can be possibly classified as X with respect to R and it is denoted by $U_R(X)$. That is, $U_R(X)$= $\bigcup_{x\epsilon\mathcal{U}}\{R(x): R(x) \cap X \neq \emptyset\}$.

(iii) The boundary region of X with respect to R is the set of all objects, which can be classified neither as X nor as not-X with respect to R and it is denoted by $B_R(X)$. That is $B_R(X)$=$U_R(X) - L_R(X)$.

Definition 2.2:[3] let $\mathcal{U}$ be the universe, R be an equivalence relation on $\mathcal{U}$ and τ_R(X)= $\{\mathcal{U}, \varphi, L_R(X), U_R(X), B_R(X)\}$, where X⊆ $\mathcal{U}$. τ_R(X) satisfies the following axioms.

i) $\mathcal{U}$ and $\varphi\epsilon\tau_R(X)$.
ii) The union of the elements of any subcollection of τ_R(X) is in τ_R(X).
iii) The intersection of the elements of any finite subcollection of τ_RX) is in τ_R(X).

That is, τ_R(X) forms a topology on $\mathcal{U}$ called as the nano topology on $\mathcal{U}$ with respect to X. We call $(\mathcal{U}, \tau_R$(X)) as the nano topological space. The elements of τ_R(X) are called as nano-open sets. A set A is said to be nano closed if its complement is nano-open.

Definition 2.3: [3] If $(\mathcal{U}, \tau_R$(X)) is a nano topological space with respect to X where X⊆ $\mathcal{U}$ and if A⊆ $\mathcal{U}$, then the nano interior of A is defined as the union of all nano-open subsets of A and it is denoted by Nint(A). That is, Nint(A) is the largest nano-open subset of A.

The nano closure of A is defined as the intersection of all nano closed sets containing A and it is denoted by Ncl(A). That is, Ncl(A) is the smallest nano closed set containing A.

Definition 2.4: A nano subset A of a nano topological space $(\mathcal{U}, \tau_R(X))$ is called a

1. Nano pre closed if Ncl Nint(A) $\subseteq A$
2. Nano semi closed if Nint Ncl(A) $\subseteq A$
3. Nano α closed if Ncl Nint Ncl(A) $\subseteq A$
4. Nano semi pre closed if Nint Ncl Nint(A) $\subseteq A$
5. Nano regular closed if Ncl Nint(A) $=A$

For a nano subset A of $(\mathcal{U}, \tau_R(X))$ the intersection of all nano pre closed. (nano semi closed, nano α closed, nano semi pre closed) sets of $(\mathcal{U}, \tau_R(X))$ containing A is called nano pre closure of A (nano semi closure of A, nano α closure of A, nano semi pre closure of A) and is denoted by Npcl(A) (Nscl(A), Nαcl(A), Nspcl(A)).

Definition 2.5: A nano subset A of a nono topological space $(\mathcal{U}, \tau_R(X))$ is called a

1. Nano generalized closed (briefly Ng closed) if Ncl(A) $\subseteq U$, whenever A$\subseteq U$ and U is nano open in $\mathcal{U}$.
2. Nano generalized semi closed (briefly Ngs closed) if Nscl(A) $\subseteq U$ whenever A$\subseteq U$ and U is nano open in $\mathcal{U}$.
3. Nano α generalized regular closed (briefly Nαgr closed) if Nαcl(A) $\subseteq U$ whenever A$\subseteq U$ and U is nano regular open in $\mathcal{U}$.
4. Nano α generalized semi closed (briefly Nαgs closed) if Nαcl(A) $\subseteq U$ whenever A$\subseteq U$ and U is nano semi open in $\mathcal{U}$.
5. Nano α generalized closed (briefly Nαg closed) if Nαcl(A) $\subseteq U$ whenever A$\subseteq U$ and U is nano open in $\mathcal{U}$.
6. Nano generalized semi pre closed (briefly Ngsp closed) if Nspcl(A) $\subseteq U$ whenever A$\subseteq U$ and U is nano open in $\mathcal{U}$.
7. Nano generalized pre closed (briefly Ngp closed) if Npcl(A) $\subseteq U$ whenever A$\subseteq U$ and U is nano open in $\mathcal{U}$.
8. Nano g^* pre closed (briefly Ng^*p closed) if Npcl(A) $\subseteq U$ whenever A$\subseteq U$ and U is Ng open in $\mathcal{U}$.
9. Nano generalized pre regular closed (briefly Ngpr closed) if Npcl(A) $\subseteq U$ whenever A$\subseteq U$ and U is nano regular open in $\mathcal{U}$.
10. Nano semi generalized closed (briefly Nsg closed) if Nscl(A) $\subseteq U$ whenever A$\subseteq U$ and U is nano semi open in $\mathcal{U}$.
11. Nano $g^{\#}\alpha$ closed (briefly N$g^{\#}\alpha$ closed) if Nαcl(A) $\subseteq U$ whenever A$\subseteq U$ and U is Ng open in $\mathcal{U}$.
12. Nano $g^{\#}s$ closed (briefly N$g^{\#}s$ closed) if Nscl(A) $\subseteq U$ whenever A$\subseteq U$ and U is Nαg open in $\mathcal{U}$.

The complements of the above mentioned nano closed sets are respective nano open sets.

Definition 2.6: A nano topological space $(\mathcal{U}, \tau_R(X))$ is said to be a

1. Nano semi $T_{\frac{1}{2}}$ space if every Nsg closed set is nano semi closed.
2. N$T_b$space if every Ngs closed set of $(\mathcal{U}, \tau_R(X))$ is nano closed in $(\mathcal{U}, \tau_R(X))$.
3. $N_\alpha T_b$ space if every Nαg-closed set of $(\mathcal{U}, \tau_R(X))$ is nano closed in $(\mathcal{U}, \tau_R(X))$.
4. NT_b^*- space if every Ng^*s- closed set of $(\mathcal{U}, \tau_R(X))$ is nano closed in $(\mathcal{U}, \tau_R(X))$.
5. T_b^{**}- space if every Ng^*s- closed set of $(\mathcal{U}, \tau_R(X))$ is Nα closed in $(\mathcal{U}, \tau_R(X))$.

III. 3. N$\boldsymbol{\alpha g}^*$S CLOSURE AND N$\boldsymbol{\alpha g}^*$S INTERIOR

Definition:3.1 A subset A of a nano topological space $(\mathcal{U}, \tau_R(X))$ is said to be Nαg^* semi closed (briefly Nαg^*s closed) set if Nαcl(A) ⊆U, whenever A⊆U and U is Ngs open in $(\mathcal{U}, \tau_R(X))$.

Definition: 3.2 For a subset A of $(\mathcal{U}, \tau_R(X))$, Nαg^*s - closure of A, denoted by Nαg^*scl(A) and is defined as Nαg^*scl(A) = $\cap \{G: A \subseteq G, G \text{ is } N\alpha g^*s - \text{closed in } (\mathcal{U}, \tau_R(X))\}$.

Theorem :3.3 For any $x \epsilon X$, $x\epsilon$Nαg^*scl(A) if and only if A∩ $V \neq \varphi$ for every Nαg^*s- open set V containing x.

Proof: Let $x\epsilon$Nαg^*scl(A). Suppose there exists an Nαg^*s – open set V containing x such that V∩A=ϕ. Then A⊆ X-V. Since X-V is Nαg^*s - closed, Nαcl(A) ⊆ X-V. This implies x∉ Nαg^*scl(A). which is a contradicition. Hence V∩A≠ ϕ for every Nαg^*s- open set V containing x.

Conversely, let A∩V≠ ϕ for every Nαg^*s - open set V containing x. To prove that $x\epsilon$Nαg^*scl(A). Suppose x∉ Nαg^*scl(A). Then there exists a Nαg^*s- closed set G containing A such that x ∉G. Then x∈ $X - G$ and X-G is Nαg^*s- open. Also (X-G)∩ $A = \phi$,which is a contradiction to the hypothesis. Hence $x \in$ Nαg^*scl(A).

Theorem :3.4 If A⊆ X, then A ⊆ Nαg^*scl(A) ⊆ $Ncl(A)$.

Proof: Since every nano closed set is Nαg^*s- closed, the proof follows.

Remark:3.5 Both containment relations in the theorem 3.3 may be proper as seen from the following example.

Example 3.6: Let $\mathcal{U}=\{a,b,c\}$, $\mathcal{U}/R=\{\{a,b\},\{c\}\}$, X = $\{a,b\}$, τ_R(X) = $\{\mathcal{U},\varphi,\{a,b\}\}$. Let E=$\{a\}$, Nαg^*cl(E)= $\{a,c\}$, Ncl(E) = $\mathcal{U}$. So, the containment may be proper.

Theorem: 3.7 Let E and F be subsets of $(\mathcal{U},\tau_R(X))$, then

a) Nαg^*scl (ϕ)= ϕ.
b) Nαg^*scl (X)= X.
c) Nαg^*scl (E) is Nαg^*s- closed set in $(\mathcal{U},\tau_R(X))$.
d) If E⊆ F, then Nαg^*scl (E) ⊆ Nαg^*scl (F).
e) Nαg^*scl (EUF)=Nαg^*scl (E)U Nαg^*scl (F).
f) Nαg^*scl [Nαg^*scl (E)] = Nαg^*scl (E).

Proof: The proof of a), b), c) and d) follow from the definition 3.1.

e) To prove that Nαg^*scl(E) U Nαg^*scl(F) ⊆ Nαg^*scl(EUF)

we have Nαg^*scl (E) ⊆ Nαg^*scl (EUF) and

Nαg^*scl (F) ⊆ Nαg^*scl (EUF).

Therefore Nαg^*scl (E) U Nαg^*scl (F) ⊆ Nαg^*scl (EUF)……… (1)

Now, we prove Nαg^*scl (EUF) ⊆Nαg^*scl (E)U Nαg^*scl (F).

Let x be any point such that x∉ Nαg^*scl (E)U Nαg^*scl (F).

Then there exists Nαg^*s- closed sets A and B such that E ⊆A and F ⊆ B, x∉ A and x ∉ B. Then x ∉ AUB, EUF ⊆AUB and AUB is Nαg^*s- closed set. Thus x ∉Nαg^*scl (EUF).

Therefore we have Nαg^*scl (EUF) ⊆ Nαg^*scl (E)U Nαg^*scl (F)………. (2)

Hence from (1) and (2), Nαg^*scl (EUF)=Nαg^*scl (E)U Nαg^*scl (F).

f) Let A be Nαg^*s- closed set containing E. Then by definition Nαg^*scl(E) ⊆ A.

Since A is Nαg^*s- closed set and contains Nαg^*scl (E) and is contained in every Nαg^*s- closed set containing E, it follows that Nαg^*scl [Nαg^*scl (E)] ⊆ Nαg^*scl (E). Therefore Nαg^*scl [Nαg^*scl (E)] = Nαg^*scl (E).

Theorem: 3.8 Nαg^*s- closure is a Kuratowski closure operator on $(\mathcal{U},\tau_R(X))$.

Proof: Follows from the theorem 3.6

Remark:3.9 A subset A is Nαg^*s- closed if and only if Nαg^*scl (A) = A.

Proof: Let A be Nαg^*s- closed set in $(\mathcal{U},\tau_R(X))$. Since A⊆ A and A is Nαg^*s- closed set, A∈ $\{G: A \subseteq G, G\ is\ \text{N}\alpha g^*\text{s} - \text{closed set}\}$ which implies that ∩ $\{G: A \subseteq G, G\ is\ \text{N}\alpha g^*\text{s} - \text{closed set}\} \subseteq A$. That is N$\alpha g^*$scl ($A$) ⊆ A. Note that A ⊆ N$\alpha g^*$scl ($A$) is always true. Hence A= Nαg^*scl (A).

Conversely, suppose that Nαg^*scl (A) =A. Since A⊆ A and A is Nαg^*s-closed set, therefore A must be a nano closed set. Hence A is Nαg^*s- closed.

Definition :3.10 For a subset A of $(\mathcal{U},\tau_R(X))$, Nαg^*s -interior of A, denoted by Nαg^*sint(A) is defined as Nαg^*sint(A)= U $\{G: G \subseteq A\ and\ G\ is\ \text{N}\alpha g^*\text{s} - \text{open in } (\mathcal{U},\tau_R(X))\}$. That is N$\alpha g^*$sint(A) is the union of all Nαg^*s-open sets contained in A.

Theorem: 3.11 Let A be a nano subset of $(\mathcal{U},\tau_R(X))$. Then Nαg^*sint(A) is the largest Nαg^*s -open subset of $(\mathcal{U},\tau_R(X))$ contained in A if A is Nαg^*s- open.

Proof: Let A be a nano subset of $(\mathcal{U},\tau_R(X))$ *and* be Nαg^*s-open. Then Nαg^*sint(A)=U$\{G: G \subseteq A\ and\ G\ is\ \text{N}\alpha g^*\text{s} - \text{open in } (\mathcal{U},\tau_R(X))\}$ .Since A⊆ A and A is Nαg^*s -open , A= Nαg^*sint(A) is the largest Nαg^*s -open subset of $(\mathcal{U},\tau_R(X))$ contained in A.

The converse of the above theorem need not be true as seen from the following example.

Example: 3.12 Let $\mathcal{U}=\{a,b,c,d\}$, $\mathcal{U}/R=\{\{a\},\{c\},\{b,d\}\}$, X = $\{a,b\}$, τ_R(X) = $\{\mathcal{U},\varphi,\{a\},\{a,b,d\},\{b,d\}\}$

Let A=$\{b,c,d\}$, Nαg^*sint(A)= $\{b,d\}$ is Nαg^*s open but A= $\{b,c,d\}$ is not Nαg^*s open.

Remark:3.13 For any subset A of $(\mathcal{U},\tau_R(X))$, Nint (A) ⊆Nαg^*sint(A) ⊆A.

Remark:3.14 For a subset A of $(\mathcal{U},\tau_R(X))$, Nαg^*sint(A)≠Nint(A) as seen from the following example.

Example: 3.15 Refer Example:3.6

Let A= $\{a,c\}$, Nαg^*sint(A)= $\{a\}$. Nint (A)= ϕ.

Hence Nαg^*sint(A) ≠ Nint(A).

Remark:3.16 Since every Nα- open set is Nαg^*s- open set, every Nα- interior point of a set A is Nαg^*s- interior point of A. Thus in general, N$\alpha int(A)$ ⊆ Nαg^*sint(A).
In general N$\alpha int(A)$ ≠ Nαg^*sint(A) as seen from the following example.

Example: 3.17 Refer Example:3.6
Let A= $\{a, c\}$, Nαg^*sint(A)= $\{a\}$, Nαint (A)= ϕ.
Hence Nαg^*sint(A) ≠ Nαint(A).

Theorem: 3.18 If A⊆ B, then Nαg^*sint(A) ⊆ Nαg^*sint(B).
Proof: Suppose that A⊆B. We know that Nαg^*sint(A)⊆ A. Also we have A⊆B, which implies Nαg^*sint(A) ⊆ B, Nαg^*sint(A) is nano open set which is contained in B. But Nαg^*sint(B) is the largest nano open set contained in B. Therefore Nαg^*sint(B) is larger than Nαg^*sint(A). That is Nαg^*sint(A)⊆ Nαg^*sint(B)

Remark:3.19 Nαg^*sint(A)= Nαg^*sint(B) does not imply that A=B. This is shown by the following example.

Example: 3.20 Refer Example:3.12
Let A=$\{a, b\}$, B=$\{a\}$ Nαg^*sint(A)= $\{a\}$= Nαg^*sint(B). But A≠B.

Theorem: 3.21 For any nano subset A of X, the following results are true:

1) Nαg^*sint (ϕ)= ϕ.
2) Nαg^*sint (X)= X.
3) If A ⊆ B then Nαg^*sint(A) ⊆ Nαg^*sint(B).
4) Nαg^*sint (A) is the largest Nαg^*s -open set contained in A.
5) Nαg^*sint ($A \cap B$) = Nαg^*sint (A) ∩ Nαg^*sint (B).
6) Nαg^*sint (AUB) ⊇ Nαg^*sint (A)U Nαg^*sint (B).
7) Nαg^*sint [Nαg^*sint (A)] = Nαg^*sint (A)

Proof: Proof follows from the definition 3.10

Theorem: 3.22 A is Nαg^*s-open if and only if Nαg^*sint (A)= A.

Remark:3.23 For any nano subset A of $(\mathcal{U}, \tau_R(\mathrm{X}))$, Nint(A) ⊆ N$\alpha g^*$sint ($A$) ⊆ A.

Theorem: 3.24 For any nano subset A of $(\mathcal{U}, \tau_R(\mathrm{X}))$, [$\mathcal{U}$ - Nαg^*sint (A)] = Nαg^*scl($\mathcal{U}$ -A).
Proof: Let x∈ $\mathcal{U}$ - Nαg^*sint (A). Then x ∉Nαg^*sint (A). That is every Nαg^*s-open set G containing x is such that G⊄ A. This implies, every Nαg^*s-open set G containing x intersects $\mathcal{U}$ -A. That is, G∩ ($\mathcal{U}$ -A) ≠ ϕ. Then by theorem 3.3, x∈ Nαg^*scl($\mathcal{U} -$ A) and therefore [$\mathcal{U}$ - Nαg^*sint (A)] ⊆ Nαg^*scl($\mathcal{U} -$ A).

Conversely, let x∈ Nαg^*scl($\mathcal{U} -$ A) . Then every Nαg^*s- open set G containing x intersects $\mathcal{U}$ -A. That is, G∩ $(\mathcal{U} - A) = \phi$ .That is, every Nαg^*s – open set G containing x is such that G⊄ A. Then by definition 3.10, x∉Nαg^*sint (A). That is x∈ [$\mathcal{U}$ - Nαg^*sint (A)], and so Nαg^*scl($\mathcal{U} -$ A) ⊆ [$\mathcal{U} -$ Nαg^*sint (A)]. Thus [$\mathcal{U}$ - Nαg^*sint (A)] = Nαg^*scl($\mathcal{U} -$ A).

Remark:3.25 For any A ⊆X. we have

1. [$\mathcal{U}$ - Nαg^*scl($\mathcal{U} -$ A)]=[Nαg^*sint (A)]
2. [$\mathcal{U}$ - Nαg^*sint($\mathcal{U} -$ A)]=[Nαg^*scl (A)]

Taking complement in the above theorem 3.23 and by replacing A by $\mathcal{U}$ -A in theorem 3.23, the above results follow.

Definition: 3.26 A nano subset A of a topological space$(\mathcal{U}, \tau_R(\mathrm{X}))$ is called Nαg^*s-neighbourhood (briefly Nαg^*s -nbd) of a point x of $\mathcal{U}$ if there exists a Nαg^*s- open set U such that x ∈ $U \subseteq A$.

Definition:3.27 Let $(\mathcal{U}, \tau_R(\mathrm{X}))$ be a nano topological space and A be a nano subset of $\mathcal{U}$. A nano subset N of $\mathcal{U}$ is said to be Nαg^*s-neighbourhood of A if there exists a Nαg^*s -open set G such that A ⊆G⊆ N.

Theorem: 3.28 Let A be a nano subset of a nano topological space $(\mathcal{U}, \tau_R(\mathrm{X}))$. Then A is Nαg^*s -open if and only if A contains a Nαg^*s nbd of each of its points.

Proof: Let A be a Nαg^*s -open set in $(\mathcal{U},\tau_R(X))$. Let x ϵA, which implies x∈ A⊆ A. Thus A is Nαg^*s nbd of x. Hence A contains a Nαg^*s nbd of each of its points.

Conversely, suppose A contains a Nαg^*s nbd of each of its points. For every x∈ A there exists a Nαg^*s neighbourhood N_x of x such that x∈ N_x⊆ A. By the definition of Nαg^*s nbd of x, there exists a Nαg^*s -open set G_x such that x∈ G_x⊆ N_x⊆ A. Now we shall prove that A=∪ {G_x: x ∈A}. Let x∈A. Then there exists Nαg^*s -open set G_x such that x ∈ G_x. Therefore,

x∈ ∪ {G_x: x ∈A} which implies A⊆ ∪ {G_x: x ∈A}. Now let y∈∪ {G_x: x ∈A} so that y∈some G_x for some x∈ A and hence y∈ A.So∪ {G_x: x ∈A}⊆ A Hence A=∪ {G_x : x∈ A}. Also each G_x is a Nαg^*s-open set and hence A is a Nαg^*s -open set.

Theorem: 3.29 If A is a Nαg^*s - closed set of $(\mathcal{U},\tau_R(X))$ and x∈ X-A, then there exists a Nαg^*s nbd N of x such that N∩A= ϕ.

Proof: If A is a Nαg^*s -closed set in X, then X-A is a Nαg^*s -open set. By theorem 3.27, $\mathcal{U}$ -A contains a Nαg^*s nbd of each of its points, which implies that, there exists a Nαg^*s nbd of N of x such that N⊆ $\mathcal{U}$ -A. That is, no point of N belongs to A and hence N∩A≠ ϕ.

Theorem: 3.30 Let $(\mathcal{U},\tau_R(X))$ be a nano topological space. If A is a Nαg^*s -closed subset of $(\mathcal{U},\tau_R(X))$ and x ∈ $\mathcal{U}$-A then there exists a Nαg^*s -neighbourhood N of X such that A∩N≠ ϕ.

Proof: Since A is Nαg^*s - closed, $\mathcal{U}$ -A is Nαg^*s -open set in $(\mathcal{U},\tau_R(X))$. By theorem 3.27, $\mathcal{U}$ -A contains a Nαg^*s –neighbourhood of each of its points. Hence there exists a Nαg^*s -neighbourhood of N of x such that N ⊆ $\mathcal{U}$ -A. Then N∩A≠ ϕ.

Definition:3.31 Let $(\mathcal{U},\tau_R(X))$ be a nano topological space and A be a nano subset of $(\mathcal{U},\tau_R(X))$. Then a point x∈ $\mathcal{U}$ is called a Nαg^*s – limit point of A if and only if every Nαg^*s nbd of x contains a point of A distinct from x. That is [N-{x}]∩ A≠ ϕ, for every αg^*s nbd N of x. Also equivalently if and only if every Nαg^*s –open set G containing x contains a point of A other than x.

Theorem: 3.32 Let A and B be nano subsets of $(\mathcal{U},\tau_R(X))$. Then A⊆ B implies Nαg^*sd (A) ⊆ Nαg^*sd (B).

Proof: Let x∈ Nαg^*sd (A) implies x is a Nαg^*s -limit point of A. That is every Nαg^*s nbd of x contains a point of A other than x. Since A⊆B, every Nαg^*s nbd of x contains a point of B other than x. Consequently x is a Nαg^*s -limit point of B. That is x∈ Nαg^*sd (B). Therefore Nαg^*sd (A) ⊆ Nαg^*sd (B).

Therem:3.33 Let $(\mathcal{U},\tau_R(X))$ be a nano topological space and A be a nano subset of $(\mathcal{U},\tau_R(X))$. Then A is Nαg^*s -closed if and only if Nαg^*s d(A) ⊆ A.

Proof: Let A be Nαg^*s - closed set. That is $\mathcal{U}$ -A is Nαg^*s -open set. Now we prove that Nαg^*sd(A) ⊆ A. Let x∈ Nαg^*sd (A). This implies x is a Nαg^*s -limit point of A. That is every Nαg^*s nbd of x contains a point of A different from x. Now suppose x∉ A so that x ∈ $\mathcal{U}$-A, which is Nαg^*s -open and by definition by Nαg^*s- open sets, there exists a Nαg^*s nbd N of x such that N⊆ $\mathcal{U}$ -A. From this we conclude that N contains no point of A, which is a contradiction. Therefore x∈ A and hence Nαg^*sd (A) ⊆ A.

Conversely, assume that Nαg^*sd (A) ⊆ A. We will prove that A is a αg^*s-closed set in $(\mathcal{U},\tau_R(X))$ or $\mathcal{U}$ -A is Nαg^*s-open set. Let x be an arbitrary point of $\mathcal{U}$ -A, so that x∉ A which implies that x∉ Nαg^*sd (A). That is there exists a Nαg^*s nbd N of x which consists of only points of $\mathcal{U}$ -A. This means that $\mathcal{U}$ -A is Nαg^*s- open and hence A is Nαg^*s-closed set in $(\mathcal{U},\tau_R(X))$.

Therem:3.34 Let $(\mathcal{U},\tau_R(X))$ be a nano topological space. Then every Nαg^*s-derived set in $(\mathcal{U},\tau_R(X))$ is Nαg^*s-closed set.

Proof: Let A be a nano subset of $(\mathcal{U},\tau_R(X))$ and Nαg^*sd (A) be Nαg^*s-derived set of A. By theorem 3.33, A is Nαg^*s-closed if and only if Nαg^*sd (A) ⊆ A. Hence Nαg^*sd (A) is Nαg^*s-closed if and only if Nαg^*sd (Nαg^*sd (A)) ⊆Nαg^*sd (A). That is every Nαg^*s-limit point of Nαg^*sd (A) belongs to Nαg^*sd (A).

Now let x be a Nαg^*s limit point of Nαg^*sd (A). That is x∈ (Nαg^*sd Nαg^*sd (A)). So that there exists a Nαg^*s-open set G containing x such that $\{G-\{x\}\}\cap$ Nαg^*s$d(A)\neq\phi$ which implies $\{G-\{x\}\}\cap A)\neq\phi$, because every N$\alpha g^*$s nbd of an element of N$\alpha g^*$sd(A) has at least one point of A. Hence x is a Nαg^*s- limit point of A. That is x belongs to Nαg^*sd(A).

Thus x∈ Nαg^*sd(Nαg^*sd(A)) implies x ∈Nαg^*sd(A). Therefore Nαg^*sd(A) is Nαg^*s- closed set in $(\mathcal{U},\tau_R(X))$.

IV. APPLICATIONS OF N$\boldsymbol{\alpha g}^*$S CLOSED SETS

In this section we introduce four new spaces namely; N$T^*_{\alpha gs}$ spaces, $N^*T_{\alpha gs}$ spaces, $N_{\alpha gs}T^*$spaces, and $N_{\alpha gs}T^*_{\frac{1}{2}}$ spaces as applications in nano topological spaces and study some of their properties.

Definition: 4.1 A nano technological space $(\mathcal{U},\tau_R(X))$ is said to be a N$T^*_{\alpha gs}$ space if every Nαg^*s closed set is nano semi closed.

Example: 4.2 Let $\mathcal{U}$= $\{a,b,c\}$, $\mathcal{U}$/R=$\{\{a\},\{b,c\}\}$, X = $\{a\}$, τ_R(X) = $\{\mathcal{U},\varphi,\{a\}\}$

Here every Nαg^*s closed set is nano semi closed. Hence $(\mathcal{U},\tau_R(X))$ is N$T^*_{\alpha gs}$ space.

Theorem: 4.3 If a space $(\mathcal{U},\tau_R(X))$ is N$T^*_{\alpha gs}$ - space, then every singleton of $(\mathcal{U},\tau_R(X))$ is either Ngs-closed or nano semi open.

Proof: Let x∈X and suppose that{x} is not a Ngs-closed set of $(\mathcal{U}, \tau_R(X))$. This implies $\{x\}^c$ is not Ngs -open. So $\mathcal{U}$ is the only Ngs -open set containing $\{x\}^c$. Then $\{x\}^c$ is Nαg^*s - nano closed set of $(\mathcal{U}, \tau_R(X))$. Since $(\mathcal{U}, \tau_R(X))$ is a N$T^*_{\alpha gs}$ -space, $\{x\}^c$ is nano semi-closed or equivalently {x} is nano semi -open in $(\mathcal{U}, \tau_R(X))$.

Theorem: 4.4 Every NT_b- space is N$T^*_{\alpha gs}$ -space but not conversely.
Proof: Let $(\mathcal{U}, \tau_R(X))$ be a NT_b- space. Let A be a Nαg^*s- closed set of $(\mathcal{U}, \tau_R(X))$. By theorem, "Every Nαg^*s closed set is Ngs-closed set", A is Ngs-closed set. Since $(\mathcal{U}, \tau_R(X))$ is NT_b-space, A is nano closed and so it is nano semi-closed in $(\mathcal{U}, \tau_R(X))$. Therefore $(\mathcal{U}, \tau_R(X))$ is N$T^*_{\alpha gs}$ -space.

Example: 4.5 Refer Example: 4.2
$(\mathcal{U}, \tau_R(X))$ is N$T^*_{\alpha gs}$ space but not a NT_b space.

Theorem:4.6 Every NT^*_b-space is N$T^*_{\alpha gs}$ but not conversely.
Proof: Let $(\mathcal{U}, \tau_R(X))$ be a NT^*_b -space. Let A be a Nαg^*s - closed set of $(\mathcal{U}, \tau_R(X))$. By theorem, "Every Nαg^*s -closed set is Ng^*s-closed set". A is Ng^*s- closed set. Since $(\mathcal{U}, \tau_R(X))$ is T^*_b-space, A is nano closed and so it is nano semi-closed in $(\mathcal{U}, \tau_R(X))$. Therefore $(\mathcal{U}, \tau_R(X))$ is N$T^*_{\alpha gs}$- space.

Example: 4.7 Refer Example: 4.2
$(\mathcal{U}, \tau_R(X))$ is N$T^*_{\alpha gs}$- space but not a NT^*_b space.

Theorem: 4.8 Every NT^{**}_b-space is N$T^*_{\alpha gs}$- space but not conversely.
Proof: Let $(\mathcal{U}, \tau_R(X))$ be a NT^{**}_b -space. Let A be a Nαg^*s - closed set of $(\mathcal{U}, \tau_R(X))$. By theorem, "Every Nαg^*s -closed set is Ng^*s-closed set". A is Ng^*s- closed set. Since $(\mathcal{U}, \tau_R(X))$ is NT^{**}_b-space, A is Nα -closed. Since every Nα -closed set is nano semi-closed, then A is nano semi closed in $(\mathcal{U}, \tau_R(X))$. Therefore $(\mathcal{U}, \tau_R(X))$ is N$T^*_{\alpha gs}$- space.

Example: 4.9 Let $\mathcal{U}=\{a,b,c,d\}$, $\mathcal{U}$/R=$\{\{a\},\{c\},\{b,d\}\}$, X = $\{a,b\}$, $\tau_R(X) = \{\mathcal{U}, \varphi, \{a\}, \{b,d\}\}$, $(\mathcal{U}, \tau_R(X))$ is N$T^*_{\alpha gs}$ space but not NT^{**}_b-space.

Theorem: 4.10 Every semi- N$T_{\frac{1}{2}}$ - space is N$T^*_{\alpha gs}$- space but not conversely.
Proof: Let $(\mathcal{U}, \tau_R(X))$ be a semi –N$T_{\frac{1}{2}}$ space. Let A be a Nαg^*s – closed set of $(\mathcal{U}, \tau_R(X))$. By theorem, "Every Nαg^*s –closed set is Nsg – closed set". A is Nsg –closed set. Since $(\mathcal{U}, \tau_R(X))$ is semi –N$T_{\frac{1}{2}}$ space, A is nano semi – closed in $(\mathcal{U}, \tau_R(X))$. Therefore $(\mathcal{U}, \tau_R(X))$ is N$T^*_{\alpha gs}$– space.

Example: 4.11 Refer Example: 4.9
$(\mathcal{U}, \tau_R(X))$ is N$T^*_{\alpha gs}$- space but not semi- N$T_{\frac{1}{2}}$ space.

Theorem: 4.12 Every $N_\alpha T_b$– space is is $NT^*_{\alpha gs}$ – space but not conversely.
Proof: Let $(\mathcal{U}, \tau_R(X))$ be a $N_\alpha T_b$ - space. Let A be a Nαg^*s -closed set of $(\mathcal{U}, \tau_R(X))$. By theorem, "Every Nαg^*s -closed set is Nαg – closed set". A is Nαg – closed set. Since $(\mathcal{U}, \tau_R(X))$ is $N_\alpha T_b$ – space, A is nano closed. Since every nano closed set is nano semi-closed, then A is nano semi-closed in $(\mathcal{U}, \tau_R(X))$. Therefore $(\mathcal{U}, \tau_R(X))$ is $NT^*_{\alpha gs}$– space.

Example: 4.13 Refer Example: 4.2
$(\mathcal{U}, \tau_R(X))$ is $NT^*_{\alpha gs}$ space but not $N_\alpha T_b$ space.

Definition: 4.14 A nano topological space $(\mathcal{U}, \tau_R(X))$ is said to be a $N^*T_{\alpha gs}$ - space if every Nαg^*s -closed set is Nα -closed.

Example: 4.15 Refer Example: 4.9
$(\mathcal{U}, \tau_R(X))$ is $N^*T_{\alpha gs}$ space.

Theorem: 4.16 Every $N^*T_{\alpha gs}$ – space is $NT^*_{\alpha gs}$ – space.
Proof: Let $(\mathcal{U}, \tau_R(X))$ be a $N^*T_{\alpha gs}$- space. Let A be a Nαg^*s - closed set of $(\mathcal{U}, \tau_R(X))$. Then A is Nα – closed set. Since every Nα -closed set is nano semi –closed, then A is nano semi closed in $(\mathcal{U}, \tau_R(X))$. Therefore $(\mathcal{U}, \tau_R(X))$ is $NT^*_{\alpha gs}$ space.

Theorem: 4.17 Every NT_b- space is $N^*T_{\alpha gs}$- space but not conversely.

Proof: Let $(\mathcal{U}, \tau_R(\mathrm{X}))$ be a NT_b- space. Let A be a Nαg^*s -closed set of $(\mathcal{U}, \tau_R(\mathrm{X}))$. By theorem, "Every N$\alpha g^*$s -closed set is Ngs–closed set". A is Ngs – closed set. Since $(\mathcal{U}, \tau_R(\mathrm{X}))$ is NT_b – space, A is nano closed and so it is Nα – closed in $(\mathcal{U}, \tau_R(\mathrm{X}))$. Therefore $(\mathcal{U}, \tau_R(\mathrm{X}))$ is
$N^*T_{\alpha gs}$ – space.

Example: 4.18 Refer Example: 4.2
$(\mathcal{U}, \tau_R(\mathrm{X}))$ is $N^*T_{\alpha gs}$ space but not NT_b space.

Theorem: 4.19 Every NT_b^* - space is $N^*T_{\alpha gs}$- space but not conversely.
Proof: Let $(\mathcal{U}, \tau_R(\mathrm{X}))$ be a NT_b^* - space. Let A be a Nαg^*s -closed set of $(\mathcal{U}, \tau_R(\mathrm{X}))$. By theorem, "Every N$\alpha g^*$s -closed set is N$g^*$s – closed set". A is N$g^*$s – closed set. Since $(\mathcal{U}, \tau_R(\mathrm{X}))$ is NT_b^* – space, A is nano closed and so it is Nα – closed in $(\mathcal{U}, \tau_R(\mathrm{X}))$. Therefore $(\mathcal{U}, \tau_R(\mathrm{X}))$ is $N^*T_{\alpha gs}$ – space.

Example: 4.20 Refer Example: 4.2
$(\mathcal{U}, \tau_R(\mathrm{X}))$ is $N^*T_{\alpha gs}$ space but not NT_b^*space.

Theorem:4.21 A space $(\mathcal{U}, \tau_R(\mathrm{X}))$ is $N^*T_{\alpha gs}$ – space if and only if every singleton of $(\mathcal{U}, \tau_R(\mathrm{X}))$ is either Ngs- closed or Nα – open.
Proof: Let x ∈ $\mathcal{U}$ and suppose that {x} is not a Ngs- closed set of $(\mathcal{U}, \tau_R(\mathrm{X}))$. This implies $\{x\}^c$ is not Ngs open. So $\mathcal{U}$ is the only Ngs open set containing $\{x\}^c$ .Then $\{x\}^c$ is Nαg^*s -closed set of $(\mathcal{U}, \tau_R(\mathrm{X}))$. Since $(\mathcal{U}, \tau_R(\mathrm{X}))$ is a $N^*T_{\alpha gs}$ -space $(\mathcal{U}, \tau_R(\mathrm{X}))$ is a $N^*T_{\alpha gs}$– space $\{x\}^c$ is Nα -closed or equivalently {x} is Nα – open in $(\mathcal{U}, \tau_R(\mathrm{X}))$. Conversely, let A be a Nαg^*s -closed set of $(\mathcal{U}, \tau_R(\mathrm{X}))$. Trivially A⊆ Nαcl (A). Let x∈ Nαcl (A). By hypothesis, {x} is either Ngs –closed or Nα-open.
Case:(i) Suppose {x} is Ngs –closed. By theorem, "A nano subset A is an Nαg^*s –closed set in $(\mathcal{U}, \tau_R(\mathrm{X}))$ if and only if Nαcl(A)-A contains no non-empty Ngs-closed set in $(\mathcal{U}, \tau_R(\mathrm{X}))$" Nαcl(A)-A does not certain any non-empty Ngs –closed set in $(\mathcal{U}, \tau_R(\mathrm{X}))$. If x ∉A, then {x}⊆ Nαcl(A)-A. But this is not possible according to the theorem, as A is Nαg^*s -closed set. Therefore x∈ A.
Case: (ii) Suppose {x} is Nα-open. Since x∈Nαcl (A). then {x}∩ A≠ ϕ. So x∈ A. Thus in any case x∈A. So Nαcl (A) ⊆ A. Therefore A=Nαcl(A) or equivalently A is Nα-closed. Therefore $(\mathcal{U}, \tau_R(\mathrm{X}))$ is $N^*T_{\alpha gs}$ –space.

Definition: 4.22 A topological space $(\mathcal{U}, \tau_R(\mathrm{X}))$ is said to be a $N_{\alpha gs}T^*$-space if every nano semi-closed is Nαg^*s -closed.

Example: 4.23 Refer Example: 4.2
$(\mathcal{U}, \tau_R(\mathrm{X}))$ is $N_{\alpha gs}T^*$ space.

Theorem:4.24 Every NT_b- space is $N_{\alpha gs}T^*$–- space but not conversely.
Proof: Let $(\mathcal{U}, \tau_R(\mathrm{X}))$ be a NT_b-space. Let A be nano semi-closed set of $(\mathcal{U}, \tau_R(\mathrm{X}))$. Then A is Ngs – closed set in $(\mathcal{U}, \tau_R(\mathrm{X}))$. Since $(\mathcal{U}, \tau_R(\mathrm{X}))$ is NT_b-space, A is nano closed and so it is Nαg^*s –closed in $(\mathcal{U}, \tau_R(\mathrm{X}))$. Therefore$(\mathcal{U}, \tau_R(\mathrm{X}))$ is $N_{\alpha gs}T^*$- space.

Example: 4.25 Refer Example: 4.2
$(\mathcal{U}, \tau_R(\mathrm{X}))$ is $N_{\alpha gs}T^*$ space but not NT_b space.

Theorem: 4.26 If a space $(\mathcal{U}, \tau_R(\mathrm{X}))$ is $N_{\alpha gs}T^*$– space, then every singleton of $(\mathcal{U}, \tau_R(\mathrm{X}))$ is either Nsg- open or Nαg^*s -closed.
Proof: Let x∈X and suppose that {x} is not a Nsg-open set of $(\mathcal{U}, \tau_R(\mathrm{X}))$. Then $\{x\}^c$ is not Nsg- closed. By theorem, we have {x} is semi-closed. Since $(\mathcal{U}, \tau_R(\mathrm{X}))$ is $N_{\alpha gs}T^*$-space, {x} is Nαg^*s -closed in $(\mathcal{U}, \tau_R(\mathrm{X}))$.

Definition: 4.27 A topological space $(\mathcal{U}, \tau_R(\mathrm{X}))$ is said to be a $N_{\alpha gs}T^*_{\frac{1}{2}}$-space if every N$\alpha g^*s$ closed set is nano closed.

Example: 4.28 Refer Example: 4.9
$(\mathcal{U}, \tau_R(\mathrm{X}))$ is $N_{\alpha gs}T^*_{\frac{1}{2}}$ space.

Theorem: 4.29 Every NT_b -space is $N_{\alpha gs}T^*_{\frac{1}{2}}$ space but not conversely.
Proof: Let $(\mathcal{U}, \tau_R(\mathrm{X}))$ be a NT_b-space. Let A be a Nαg^*s –closed set of $(\mathcal{U}, \tau_R(\mathrm{X}))$. By theorem, "Every N$\alpha g^*$s –closed set is Ngs-closed set", A is Ngs-closed set. Since (($\mathcal{U}, \tau_R(\mathrm{X}))$ is NT_b -space, A is nano closed in $(\mathcal{U}, \tau_R(\mathrm{X}))$. Therefore $(\mathcal{U}, \tau_R(\mathrm{X}))$ is $N_{\alpha gs}T^*_{\frac{1}{2}}$ space.

Example: 4.30 Refer Example: 4.9
$(\mathcal{U}, \tau_R(\mathrm{X}))$ is $N_{\alpha gs}T^*_{\frac{1}{2}}$ space but not NT_b space.

Theorem: 4.31 Every NT_b^*-space is $N_{\alpha gs}T_{\frac{1}{2}}^*$ space but not conversely.

Proof: Let $(\mathcal{U}, \tau_R(X))$ be a NT_b^*-space. Let A be a Nαg^*s –closed set of $(\mathcal{U}, \tau_R(X))$. By theorem, "Every Nαg^*s –closed set is Ng^*s -closed set", A is Ng^*s-closed set. Since $(\mathcal{U}, \tau_R(X))$ is NT_b^*-space, A is nano closed in $(\mathcal{U}, \tau_R(X))$. Therefore $(\mathcal{U}, \tau_R(X))$ is $N_{\alpha gs}T_{\frac{1}{2}}^*$ space.

Example: 4.32 Refer Example: 4.9
$(\mathcal{U}, \tau_R(X))$ is $N_{\alpha gs}T_{\frac{1}{2}}^*$ space but not NT_b^* space.

Theorem: 4.33 Every $N_{\alpha gs}T_{\frac{1}{2}}^*$ -space is $NT_{\alpha gs}^*$-space but not conversely.

Proof: Let $(\mathcal{U}, \tau_R(X))$ be a $N_{\alpha gs}T_{\frac{1}{2}}^*$ space. Let K be a Nαg^*s –closed set of $(\mathcal{U}, \tau_R(X))$. Then K is nano closed in $(\mathcal{U}, \tau_R(X))$. So K is nano semi-closed in $(\mathcal{U}, \tau_R(X))$. Therefore $(\mathcal{U}, \tau_R(X))$ is $NT_{\alpha gs}^*$-space.

Example: 4.34 Refer Example: 4.2
$(\mathcal{U}, \tau_R(X))$ is $NT_{\alpha gs}^*$ space but not $N_{\alpha gs}T_{\frac{1}{2}}^*$ space.

Theorem: 4.35 Every $N_{\alpha gs}T_{\frac{1}{2}}^*$-space is $N^*T_{\alpha gs}$ -space but not conversely.

Proof: Let $(\mathcal{U}, \tau_R(X))$ be a $N_{\alpha gs}T_{\frac{1}{2}}^*$-space. Let K be a N$\alpha g^*$s –closed set of $(\mathcal{U}, \tau_R(X))$. Then K is closed in $(\mathcal{U}, \tau_R(X))$. So K is Nα-closed in $(\mathcal{U}, \tau_R(X))$. Therefore $(\mathcal{U}, \tau_R(X))$ is $N^*T_{\alpha gs}$-space.

Example: 4.36 Refer Example: 4.2
$(\mathcal{U}, \tau_R(X))$ is $N^*T_{\alpha gs}$ space but not $N_{\alpha gs}T_{\frac{1}{2}}^*$ space.

Theorem: 4.37 If a space $(\mathcal{U}, \tau_R(X))$ is $N_{\alpha gs}T_{\frac{1}{2}}^*$ -space, then every singleton of $(\mathcal{U}, \tau_R(X))$ is either Ngs –closed or open.

Proof: Let x∈X and suppose that {x} is not a Ngs- closed set of $(\mathcal{U}, \tau_R(X))$. This implies $\{x\}^c$ is not a Ngs -open. So $\mathcal{U}$ is the only Ngs-open set containing $\{x\}^c$. Then $\{x\}^c$ is Nαg^*s closed set of $(\mathcal{U}, \tau_R(X))$. Since $(\mathcal{U}, \tau_R(X))$ is a $N_{\alpha gs}T_{\frac{1}{2}}^*$ space,$\{x\}^c$ is closed or equivalently {x} is open in $(\mathcal{U}, \tau_R(X))$

Example: 4.38 Refer Example: 4.2
In $(\mathcal{U}, \tau_R(X))$ every singleton is either Ngs closed or Ngs open but $\{b\}$ is Nαg^*s closed but not closed.

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