Reversible Index-Proper Splitting of Matrices

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Abstract- The theory of matrix splitting is one of the most useful tool for finding an iterative solution of a linear system of equations. In this article, we introduce the notion of reversible index-proper splitting for singular matrices. Further we derive several convergence results for different subclasses of reversible index-proper splitting.

Index Terms- Drazin inverse; Reverse index-proper splitting; Reversible index-proper splitting; Convergence theorem; Comparison theorem.

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I. INTRODUCTION

Consider a system of linear equations of the form

$$Ax = b, \tag{1.1}$$

where $A \in \mathbb{R}^{n\times n}$ is singular, $b \in \mathbb{R}^n$ is given and $x \in \mathbb{R}^n$ is unknown. In some situation people pay more attention towards the Drazin inverse solution $x = A^D b$ of the system (1.1) (see,[10, 12, 13, 14] for instance.). Here, A^D is the Drazin inverse of A ([2]). For large n, iterative methods are the standard approach to compute the solution of (1.1) when the system is consistent and the co-efficient matrix A is sparse. An additive decomposition $A = U - V$ is called a *splitting*. A splitting $A = U - V$ is called an *index-proper splitting* ([6]) if $R(A^k) = R(U^k)$ and $N(A^k) = N(U^k)$, where $k = ind(A)$ (see, the next section for its definition), $R(A)$ and $N(A)$ stand for the range space of A and the null space of A respectively. It reduces to *index splitting* ([12]) if $ind(U) = 1$. When $k = 1$, then an index-proper splitting becomes a p*roper splitting* ([3]). The approximate solution x^{i+1} is generated as follows

$$x^{i+1} = U^D V x^i + U^D b, i = 0,1,...., \tag{1.2}$$

Where x^0 is the given initial vector and U^D is the Drazin inverse of U. The above iterative method is convergent to the unique solution $A^D b$ for each x^0 if and only if the spectral radius of $U^D V$ is strictly less than 1. More on index-proper splitting can be found in the recent articles [4, 5, 6, 7, 8].

Applications of Drazin inverse lie in many areas such as singular differential and difference equations, Markov chain, cryptography, iterative methods, multi-body dynamics and optimal control. Therefore the computation of the Drazin inverse and its properties has been an area of active research. Here, only a few articles on the Drazin inverse are mentioned, but there is a vast amount of literature on it. (See the references [1, 4, 5, 6,7, 8,12] and the references cited therein.)

The aim of this article is to suggest a method for solution of square singular linear systems using iteration method by introducing a new splitting called *reverse index-proper splitting*. The definition of this splitting is motivated by the idea of reverse splitting ([15,16]) and reverse proper splitting ([9]). The proposed iterative method associated with these splittings is convergent if and only if the spectral radius of the iteration matrix is less than one. In this article, we provide some theoretical convergence results when the corresponding iteration matrix has spectral radius one.

The organization of this paper is as follows. In Section 2, we list all relevant definitions, notation and some earlier results which we use throughout the paper. Section 3 contains the main results of this article concerning the convergence results for different subclass of reverse index-proper splitting.

II. Preliminaries

Throughout this article, we will deal with $\mathbb{R}^n$ equipped with its standard cone $\mathbb{R}^n_+$, and all our matrices are real square matrices of order n unless stated otherwise. We denote the transpose, the null space and the range space of A by A^T, $N(A)$ and $R(A)$, respectively. A is said to be *non-negative* (i.e., $A \geq 0$) if all the entries of A are non-negative, and $B \geq C$ for matrices B and C, if $B - C \geq 0$. We also use these notation and nomenclature for vectors. Let L, M be complementary subspaces of $\mathbb{R}^n$. Then $P_{L,M}$ stands for the projection of $\mathbb{R}^n$ onto L along M. So, $P_{L,M}B = B$ if and only if $R(B) \subseteq L$, and $BP_{L,M} = B$ if and only if $N(B) \supseteq M$. The *spectral radius* of a matrix A is denoted by $\rho(A)$, and is equal to the maximum of the moduli of the eigenvalues of A. For any two matrices A and B, we have $\rho(AB) = \rho(BA)$. The *index* of A is the least nonnegative integer k such that rank (A^{k+1}) = rank (A^k), and we denote it by $ind(A)$. Then $ind(A) = k$ if and only if $R(A^k) \oplus N(A^k) = \mathbb{R}^n$. Also, for $l \geq k$, $R(A^l) = R(A^k)$ and $N(A^l) = N(A^k)$. The *Drazin inverse* of a matrix $A \in \mathbb{R}^{n\times n}$ is the unique solution $X \in \mathbb{R}^{n\times n}$ satisfying the equations: $A^k = A^k XA$, $X = XAX$ and $AX = XA$, where k is the index of A. It is denoted by A^D. When $k = 1$, then the Drazin inverse is said to be the *group inverse* and is denoted by $A^\#$. While Drazin inverse exists for all matrices, the group inverse does not. It exists if and only if $ind(A) = 1$. If A is nonsingular, then of course, we have $A^{-1} = A^D = A^\#$. Some well-known properties of A^D ([1]) are follows: $R(A^k) = R(A^D)$; $N(A^k) = N(A^D)$; $AA^D = P_{R(A^k),N(A^k)} = A^D A$. In particular, if $x \in R(A^k)$ then $x = A^D Ax$. We list certain results to be used in the sequel. The next two theorems deal with nonnegativity and spectral radius, and the first one is known as Perron ─Frobenius theorem which states that:

Theorem 2.1 (Theorem 2.20, [11])
Let $A \geq 0$. Then A has a nonnegative real eigenvalue equal to its spectral radius.

Theorem 2.2 (Theorem 2.21, [11])
Let $A \geq B \geq 0$. Then $\rho(A) \geq \rho(B)$.

Theorem 2.3 (Theorem 3.16, [11])
Let $X \geq 0$. Then $\rho(X) < 1$ if and only if $(I - X)^{-1}$ exists and $(I - X)^{-1} = \sum_{k=0}^{\infty} X^k \geq 0$.

The following theorem gives some of the properties of an index-proper splitting.

Theorem 2.4 (Theorem 3.2, [6])
Let $A = U - V$ be an index-proper splitting. Then
(a) $AA^D = UU^D = A^D A$;
(b) $I - U^D V$ is invertible;
(c) $A^D = (I - U^D V)^{-1} U^D$.

Different subclasses of an index-proper splittings are recalled next.

Definition 2.5 *An index-proper splitting* $A = U - V$ *is called:*
(a) an index-proper regular splitting [1], if $U^D \geq 0$ and $V \geq 0$;
(b) an index-proper weak regular splitting [1], if $U^D \geq 0$ and $U^D V \geq 0$;

(c) an index-proper nonnegative splitting [4] if $U^DV \geq 0$.
One can find that there exist some splitting which are not included in Definition 2.5.

For example, let $A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 4 & 0 \end{pmatrix}$ and $U = (1/2)A$. Then the index of A and U are 1. Also $R(A) = R(U)$ and $N(A) = N(U)$. Hence, $A = U - V$ is an index-proper splitting. We can see that $V \ngeq 0$, $U^D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0.5000 & 0 \\ 0 & 0.5000 & 0 \end{pmatrix} \geq 0$ but

$$U^DV = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & -1 & 0 \end{pmatrix} \ngeq 0.$$

The above example shows that the splitting is not included in the Definition 2.5, so that we cannot discuss its convergence according to the theories of [4,5, 7]. Motivated by the definition of reverse proper splitting introduced by Mishra [9] and reverse splitting for nonsingular matrices [16], here we propose another splitting called reverse index-proper splitting of square singular matrices.

III. Main Results

We begin with the following definition.

Definition 3.1 A splitting $A = U - V$ is called a reverse index-proper splitting of A, if $R(V^k) = R(U^k)$ and $N(V^k) = N(U^k)$

Example 3.2 *Let* $A = \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ 0 & -2 & 0 \end{pmatrix}$ *and* $U = 2A$. *Then the index of* U *and* V *are 2. Also,* $R(U^2) = R(V^2)$ *and* $N(U^2) = N(V^2)$. *Hence,* $A = U - V$ *is a reverse index-proper splitting of* A.

A few properties of a reverse index-proper splitting are presented next.

Theorem 3.3 Let $A = U - V$ *be a reverse index-proper splitting. Then*
(a) $VV^D = UU^D = V^DV$;
(b) $I - U^DA$ is invertible;
(c) $V^D = (I - U^DA)^{-1}U^D$.

Proof.
(a) $VV^D = P_{R(V^k),N(V^k)} = P_{R(U^k),N(U^k)} = UU^D$.
(b) Let $(I - U^DA)x = 0$. Then $x = U^DAx \in R(U^D) = R(U^k) = R(V^D) = R(V^k)$.
Therefore,

$$\begin{aligned} x &= U^DAx \\ &= U^D(U - V)x \\ &= (U^DUx - U^DVx) \\ &= (I - U^DV)x. \end{aligned}$$

Which implies $U^D Vx = 0$. Pre-multiplying $U^D Vx = 0$ by U, we get $Vx = 0$. So, $x \in N(V) \subseteq N(V^k)$. Hence $x \in R(V^k) \cap N(V^k) = \{0\}$. Thus, $I - U^D A$ is invertible.

(c) Suppose $X = (I - U^D A)^{-1} U^D$. Let $x \in R(V^k)$. Then

$$\begin{aligned} XVx &= (I - U^D A)^{-1} U^D Vx \\ &= (I - U^D A)^{-1} U^D (Ux - Ax) \\ &= (I - U^D A)^{-1} (U^D Ux - U^D Ax) \\ &= (I - U^D A)^{-1} (x - U^D Ax) \\ & (I - U^D A)^{-1} (I - U^D A)x \\ &= x. \end{aligned}$$

If $y \in N(V^k) = N(U^k)$, then $Xy = (I - U^D A)^{-1} U^D y = 0$. Hence, $X = (I - U^D A)^{-1} U^D = V^D$.

A relation between the eigenvalues of $U^D A$ and $V^D A$ is established by a lemma given below.

Lemma 3.4 *Let* $A = U - V$ *be a reverse index-proper splitting. Let* $\mu_i, 1 \le i \le s$ *and* λ_j, $1 \le j \le s$ *be the eigenvalues of the matrices* $U^D A (AU^D)$ *and* $V^D A (AV^D)$, *respectively. Then for every* i, *there exists* j *such that* $\mu_i = \dfrac{\lambda_j}{1 + \lambda_j}$ *and for every* j, *there exists* i *such that* $\lambda_j = \dfrac{\mu_i}{1 - \mu_i}$.

Proof. Since $A = U - V$ is a reverse proper splitting of A, by Theorem 3.3 (c) $V^D = (I - U^D A)^{-1} U^D$. Therefore, $V^D A = (I - U^D A)^{-1} U^D A$. Let μ_i be an eigenvalue of $U^D A$ with an eigenvector x for some i. Then $\mu_i \ne 1$ and

$$\begin{aligned} V^D Ax &= (I - U^D A)^{-1} U^D Ax \\ &= \mu_i (I - U^D A)^{-1} x \\ &= \frac{\mu_i}{1 - \mu_i} x \end{aligned}$$

Thus $\dfrac{\mu_i}{1 - \mu_i}$ is an eigenvalue of $V^D A$, and so there exists some j such that $\lambda_j x = \dfrac{\mu_i}{1 - \mu_i} x$, where λ_j is an eigenvalue of $V^D A$. Therefore, the two matrices $U^D A$ and $V^D A$ have the same set of eigenvectors with their eigenvalues related by $\lambda_j = \dfrac{\mu_i}{1 - \mu_i}$, for $i = 1, 2, \ldots, n$. Again, $U = V + A$ so that $U = V - (-A)$ is a proper splitting of U. Similarly, we have $U^D A = (I + V^D A)^{-1} V^D A$.

If λ_j is an eigenvalue of $V^D A$ for some j with an eigenvector y, then $\lambda_j \ne -1$ and

$$\begin{aligned} U^D Ay &= (I + V^D A)^{-1} V^D Ay \\ &= \lambda_j (I + V^D A)^{-1} y \end{aligned}$$

$$= \frac{\lambda_j}{1+\lambda_j} y.$$

Hence $\frac{\lambda_j}{1+\lambda_j}$ is an eigenvalue of $U^D A$, and so there exists some i such that $\mu_i = \frac{\lambda_j}{1+\lambda_j}$, for $j = 1,2,\ldots n$.

A few more properties of a reverse index-proper splitting are discussed next.

Theorem 3.5 Let $A = U - V$ *be a reverse index-proper splitting. Suppose that* $U^D \geq 0$ *and* $A \geq 0$. *Then the following are equivalent:*

(a) $V^D \geq 0$;
(b) $V^D A \geq 0$;
(c) $\rho(U^D A) \leq 1$.

Proof. (a)$\Rightarrow$(b): The condition $V^D \geq 0$ and $A \geq 0$ imply $V^D A \geq 0$.

(b)$\Rightarrow$ (c): We have $U^D A \geq 0$ and $V^D A \geq 0$. Let λ and μ be any nonnegative eigenvalues of $V^D A$ and $U^D A$, respectively. Let $f(\lambda) = \frac{\lambda}{1+\lambda}, \lambda \geq 0$. Then f is strictly increasing function. Then by Lemma 3.4, $\mu = \frac{\lambda}{1+\lambda}$. So, μ attains its maximum when λ is maximum. But λ is maximum when $\lambda = \rho(V^D A)$. As a result, the maximum value of μ is $\rho(U^D A)$. Hence, $\rho(U^D A) = \frac{\rho(V^D A)}{1+\rho(V^D A)} < 1$.

(c)$\Rightarrow$(a): Let $\rho(U^D A) < 1$. Then by Theorem 2.3, we have $(I - U^D V)^{-1} = \sum_{k=0}^{\infty} (U^D A)^k \geq 0$. Since $A \geq 0$, so $V^D = (I - U^D V)^{-1} U^D = \sum_{k=0}^{\infty} (U^D A)^k \geq 0$.

Next we introduce a subclass of a reverse index-proper splitting called reversible index-proper splitting and the definition is as follows.

Definition 3.6 *A reverse index-proper splitting* $A = U - V$ *is called a reversible index-proper splitting of* A *if* $\lambda_i(U^D A) \geq 0$ *for each* $i = 1,2,\ldots,n$.

Example 3.7 *Let* $A = \begin{pmatrix} 2 & 0 & -1 \\ -1 & 2 & 0 \\ -1 & 2 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 2 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} = U - V$. *Then the index of* U *and* V *are 1. Also* $R(U) = R(V)$ *and* $N(U) = N(V)$. *Hence* $A = U - V$ *is a reverse index-proper splitting of* A,
$U^D A = \begin{pmatrix} 1 & 0 & -0.5000 \\ -0.5000 & 1 & 0 \\ -0.5000 & 1 & 0 \end{pmatrix}$ *and* $\lambda_i(U^D A) = \{0, 1.5, 0.5\}$ *for* $i = 1,2,3$. *So* $\lambda_i(U^D A) \geq 0$. *Hence,* A *possess a reversible index-proper splitting.*

A convergence result for a reversible index-proper splitting is presented next.

Theorem 3.8 *If* $A = U - V$ *is a reversible index-proper splitting of* A *and* $\rho(U^D A) < 1$, *then* $\rho(U^D V) < 1$.

Proof. Since $A = U - V$ is a reversible index-proper splitting, so $\lambda_i(U^D A) \geq 0$ for $i = 1,2,\ldots,n.$ By Theorem 3.3, we have $V = U(I - U^D A)$. So, $U^D V = U^D U(I - U^D A)$. We then have $U^D V + U^D A = U^D U$ and $\lambda_i(U^D V) + \lambda_i(U^D A) = \lambda_i(U^D U) \leq 1$ for $i = 1,2,\ldots,n$. The condition $\rho(U^D A) < 1$ means that $0 \leq \lambda_i(U^D A) < 1$ for $i = 1,2,\ldots,n$. Hence $\rho(U^D V) < 1$.

Unlike index-proper regular splitting, index-proper weak regular splitting and index-proper nonnegative splitting, we now introduce different subclass of a reversible index-proper splitting.

Definition 3.9 A reversible index-proper splitting $A = U - V$ is called:
(a) a reversible index-proper regular splitting, if $U^D \geq 0$ and $A \geq 0$,
(b) a reversible index-proper weak regular splitting, if $U^D \geq 0$ and $U^D A \geq 0$,
(c) a reversible index-proper nonnegative splitting, if $U^D A \geq 0$.

Example 3.10 *Let* $A = \begin{pmatrix} 2 & 1 \\ 6 & 3 \end{pmatrix} = U - V$, *where* $U = \begin{pmatrix} 3 & 1.5 \\ 9 & 4.5 \end{pmatrix}$ *and* $V = \begin{pmatrix} 1 & 0.5 \\ 3 & 1.5 \end{pmatrix}$. *Then,* $A = U - V$ *is a reverse index-proper splitting. Also* $U^D = \begin{pmatrix} 0.0533 & 0.0267 \\ 0.1600 & 0.0800 \end{pmatrix} \geq 0$, $U^D A = \begin{pmatrix} 0.2667 & 0.1333 \\ 0.8000 & 0.4000 \end{pmatrix} \geq 0$ *and* $\lambda_i(U^D A) = \{0, 0.6667\}$. *Hence,* $A = U - V$ *is a reversible index-proper weak regular splitting.*

Example 3.11 *Let* $A = \begin{pmatrix} -1 & -1 \\ -2 & -2 \end{pmatrix}$. *Set* $U = 2A$. *Then,* $A = U - V$ *is a reverse index-proper splitting. We can observe that* $U^D = \begin{pmatrix} -0.0556 & -0.0556 \\ -0.1111 & -0.1111 \end{pmatrix} \leq 0, U^D A = \begin{pmatrix} 0.1667 & 0.1667 \\ 0.3333 & 0.3333 \end{pmatrix} \geq 0$ *and* $\lambda_i(U^D A) = \{0, 0.5\}$. *Hence,* $A = U - V$ *is a reversible index-proper nonnegative splitting.*

Now we present a convergence theorem for a reversible index-proper regular splitting.

Theorem 3.12 *Let* $A = U - V$ *be a reversible index-proper regular splitting with* $V^D \geq 0$. *Then the following conditions holds.*
(a) $V^D \geq U^D$;
(b) $\rho(V^D A) \geq \rho(U^D A)$;
(c) $\rho(U^D A) = \dfrac{\rho(V^D A)}{1 + \rho(V^D A)} < 1$.

Proof. (i) By Theorem 3.3 (c), we have $V^D = (I - U^D A)^{-1} U^D$. So $U^D A V^D = V^D - U^D$. Since A possess reversible index-proper regular splitting and $V^D \geq 0$, so $V^D - U^D \geq 0$. Hence $V^D \geq U^D$.
(ii) By post multiplying A in (i) and then Theorem 2.2 yields $\rho(V^D A) \geq \rho(U^D A)$.
(iii) Since $A = U - V$ is a reversible index-proper regular splitting and $V^D \geq 0$, so $V^D A \geq 0$ and $U^D A \geq 0$. Let λ and μ be any nonnegative eigenvalues of $V^D A$ and $U^D A$, respectively. Let $f(\lambda) = \dfrac{\lambda}{1 + \lambda}, \lambda \geq 0$. Then f is strictly increasing function.

Then by Lemma 3.4, $\mu = \frac{\lambda}{1+\lambda}$. So, μ attains its maximum when λ is maximum. But λ is maximum when $\lambda = \rho(V^D A)$. As a result, the maximum value of μ is $\rho(U^D A)$. Hence, $\rho(U^D A) = \frac{\rho(V^D A)}{1+\rho(V^D A)} < 1$.

As an immediate consequence of Theorem 3.8 and Theorem 3.12, we obtain the following result.

Corollary 3.13 *Let* $A = U - V$ *be a reversible index-proper regular splitting. If* $V^D \geq 0$, *then* $\rho(U^D V) < 1$.

Comparison theorems between the spectral radii of matrices are useful tools in analysis of rate of convergence of iterative methods. An accepted rule for preferring one iteration scheme to another is to choose the scheme having the smaller spectral radius.

Theorem 3.14 *Let* $A = U_1 - V_1 = U_2 - V_2$ *be two reversible index-proper regular splitting such that* $V_2^D \geq V_1^D \geq 0$. *If* $A \geq 0$, *then* $\rho(U_1^D A) \leq \rho(U_1^D A) < 1$.

Proof. By Theorem 3.12, we have $\rho(U_i^D A) < 1$ for $i = 1,2$. Since $V_2 \geq V_1$ and $A \geq 0$, then $V_2 A \geq V_1 A \geq 0$. So by Theorem 3.12, we have $\rho(V_2^D A) \geq \rho(V_1^D A) \geq 0$. Hence,

$$\frac{\rho(V_1^D A)}{1+\rho(V_1^D A)} \leq \frac{\rho(V_2^D A)}{1+\rho(V_2^D A)},$$

i.e.,

$$\rho(U_1^D A) \leq \rho(U_1^D A) < 1.$$

The next result discusses convergence of a reversible index-proper weak regular splitting.

Theorem3.15 Let $A = U - V$ *be a reversible index-proper weak regular splitting. Then* $(a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (d) \Rightarrow (e) \Rightarrow (f) \Rightarrow (g)$, *where*

(a) $V^D U \geq 0$;

(b) $\rho(U^D A) = \frac{\rho(V^D U) - 1}{\rho(V^D U)}$;

(c) $\rho(U^D A) < 1$;

(d) $(I - U^D A)^{-1} \geq 0$;

(e) $V^D A \geq 0$;

(f) $V^D A \geq U^D A$;

(g) $\rho(U^D A) = \frac{\rho(V^D A)}{1+\rho(V^D A)}$.

Proof. Since A has a reversible index-proper weak regular splitting, so A possess reverse index-proper splitting with $U^D \geq 0, U^D A \geq 0$.

(a) $\Rightarrow$ (b): Since $U^D A \geq 0$, then by theorem, there exists a nonnegative vector $x\,(x \neq 0)$ such that $U^D Ax = \rho(U^D A)x$. Hence $x \in R(U^D) = R(U^k)$, so that $x = U^D Ux$. By Theorem 3.3, we have $V^D = (I - U^D A)^{-1} U^D$ which implies $V^D U =$ $(I - U^D A)^{-1} U^D U$. Then

$$V^D Ux = (I - U^D A)^{-1} U^D Ux$$
$$= (I - U^D A)^{-1} x$$
$$= \frac{1}{1-\rho(U^D A)} x.$$

So, $\frac{1}{1-\rho(U^D A)} \geq 0$ and is an eigenvalue of $V^D U$. Hence $0 \leq \frac{1}{1-\rho(U^D A)} \leq \rho(V^D U)$ implies $\rho(U^D A) \leq \frac{\rho(V^D U - 1)}{\rho(V^D U)}$. Again, the condition $V^D U \geq 0$ yields existence of a nonnegative vector $y\,(y \neq 0)$ such that $V^D Uy = \rho(V^D U) y$. Also $U^D U = (I - U^D A) V^D Uy$ which implies $U^D A V^D Uy = V^D Uy - y$. Hence, $U^D Ay = \frac{\rho(V^D U - 1)}{\rho(V^D U)} y$. So $\frac{\rho(V^D U - 1)}{\rho(V^D U)} \leq \rho(U^D A)$.

(b) $\Rightarrow$ (c): Obvious.

(c) $\Rightarrow$ (d): The condition $U^D A \geq 0$ and Theorem 2.3 yields $(I - U^D A)^{-1} = \sum_{k=0}^{\infty} (U^D A)^k \geq 0$.

(d)$\Rightarrow$(e): By Theorem 3.3, we also have $V^D = (I - U^D A)^{-1} U^D$. Since, $(I - U^D A)^{-1} \geq 0, U^D A \geq 0$, then $V^D A = (I - U^D A)^{-1} U^D A$ implies $V^D A \geq 0$.

(e)$\Rightarrow$(f): Since $V^D A = (I - U^D A)^{-1} U^D A$. So $(I - U^D A) V^D A = U^D A$, i.e., $V^D A - U^D A = U^D A V^D A$. Again $V^D A \geq 0$ and $U^D A \geq 0$ implies $V^D A - U^D A \geq 0$. Hence $V^D A \geq U^D A$.

(f) $\Rightarrow$ (g): The proof of this is similar to that of Theorem 3.12 $(b) \Rightarrow (c)$.

A convergence result for the reversible index-proper nonnegative splitting is proved next.

Theorem 3.16 *Let* $A = U - V$ *be a reversible index-proper nonnegative splitting. Then* $V^D A \geq 0$ *if and only if* $\rho(U^D A) = \frac{\rho(V^D A)}{1+\rho(V^D A)} < 1$.

Proof. Suppose that $V^D A \geq 0$. The fact $A = U - V$ is a reversible index-proper nonnegative splitting yields that A possess a reverse proper splitting with $U^D A \geq 0$. Now the proof is similar to that of Theorem 3.5 (c).

Conversely, assume that $\rho(U^D A) < 1$. Then by Theorem 2.3, we get $(I - U^D A)^{-1} = \sum_{k=0}^{\infty} (U^D A)^k \geq 0$. Since $V^D A = (I - U^D A)^{-1} U^D A$, so, we have $V^D A = \sum_{k=1}^{\infty} (U^D A)^k \geq 0$.

As an immediate consequence of Theorem 3.8 and Theorem 3.16, we obtain the following results.

Corollary 3.17 *Let* $A = U - V$ *be a reversible index-proper nonnegative splitting. If* $V^D A \geq 0$, *then* $\rho(U^D V) < 1$.

Theorem 3.18 *Let* $A = U - V$ *be a reversible splitting with* $V^D U \geq 0$. *Then* $\rho(U^D A) = \frac{\rho(V^D U) - 1}{\rho(V^D U)} < 1$.

Conversely, if $\rho(U^D A) \leq 1$, then $V^D U \geq 0$.

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