A Review of Multiple Approaches for Binomial Theorem

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Abstract: The Binomial Theorem has since long been of paramount importance in mathematics in varied forms. It was known to the ancients and in the hands of Leibniz, Newton, Euler, Galois, and others, it became an essential tool in both algebra and analysis. In this paper we have collected the Multiple Proofs of Binomial Theorem.

Keywords:

Binomial distribution; Independent and identically distributed random variables; Principle of induction, Uniqueness Theorem, Fundamental theorem of calculus, Mean value theorem. Laplace transforms etc.

Introduction:

Binomial Theorem is a crucial result of mathematics which provides augmentation of positive integer powers of sums of two expressions.

Statement is as follows:

Binomial theorem, for all $n \geq 1$ and $a, b \in R$

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} \qquad ...1$$

The Binomial Theorem has played a crucial role in the development of mathematics, algebraic or analytic, pure or applied [1]. It was very important in the development of the calculus, in a variety of ways and has certainly been much important in the development of number theory. It plays a dominant role in function field arithmetic [1]. In fact, it almost appears as function field arithmetic (and a large chunk of arithmetic in general) but a commentary on this amazing result. In turn, function field arithmetic has recently returned the favour by shedding new light on the Binomial Theorem. It is our purpose here to recall the history of the Binomial Theorem [1].

History of the Binomial Theorem:

As per known information, trace of Binomial Theorem started in 4th century B.C., in which Euclid found formulae for $(a + b)^2$. Later, 3rd century B.C., Pingala Indian mathematician presented a formula which is currently known as 'Pascals triangle' which gives binomial coefficients in a triangle [1]. After that, in 10th century A.D., Halayudha Indian mathematician and Persian mathematician al-Karaji derived similar results as did by the 13th century Chinese mathematician Yang Hui. It is notable that al-Karaji appears to have used mathematical induction in his studies. Indeed, binomial coefficients, appearing in Pascal's triangle, seem to have been widely known in antiquity [1].

Multiple Theorem's for Proof of Binomial Theorem:

1. **Binomial Theorem using the uniqueness theorem for the initial value problems in linear ordinary differential equations**.

Theorem 1: For a positive integer n and real s, the following holds [2].

$$(1+s)^n = 1 + \sum_{k=1}^{n} \frac{n!}{k!(n-k)!} s^k \qquad 1.1$$

Proof: Observe that if (1.1) holds for all $s > -1$ then taking limit as $s \to -1+$ on both sides of (1.1), we get the result for $s = -1$ as well. Also, if (1.1) holds for all $s > -1$ then it holds for all $s < -1$ since if that is the case then we can write:

$$(1+s)^n = s^n(1+\tfrac{1}{s})^n = s^n \sum_{k=0}^{n} \frac{n!}{k!(n-k)!}\frac{1}{s^k} = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} \; s^{n-k} = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} \; s^k \quad 1.2$$

Thus, it is sufficient to prove (1.1) for $s > -1$. So, consider the initial value problem.

$$x'(s) - \frac{n}{(1+s)}x(s) = 0,\; x(0) = 0, s > -1 \qquad 1.3$$

Which has zero function as its unique solution for $s > -1$.

Let $\emptyset(s) = (1+s)^n - 1 - \sum_{k=1}^{n} \frac{n!}{k!(n-k)!} s^k$ for $s > -1$, differentiating it with respect to s, and multiplying throughout by $(1 + s)$, we get:

$$(1+s)\emptyset'(s) = n(1+s)^n - (1+s)\sum_{k=1}^{n} \frac{n!}{k!(n-k)!} ks^{k-1}$$

$$= n(1+s)^n - \sum_{k=1}^{n} \frac{n!}{(k-1)!(n-k)!}(s^{k-1} + s^k)$$

$$= n(1+s)^n - n - \sum_{k=2}^{n} \frac{n!}{(k-1)!\,(n-k)!} s^{k-1} - \sum_{k=1}^{n} \frac{n!}{(k-1)!\,(n-k)!} s^k$$

$$= n(1+s)^n - n - \sum_{k=1}^{n-1} \frac{n!}{(k)!\,(n-k-1)!} s^k - \sum_{k=1}^{n} \frac{n!}{(k-1)!\,(n-k)!} s^k$$

$$= n(1+s)^n - n - \sum_{k=1}^{n-1} \frac{n!}{(k-1)!\,(n-k-1)!}(\frac{1}{k} + \frac{1}{n-k})s^k - ns^n$$

$$= n[\,(1+s)^n - 1 - \sum_{k=1}^{n-1} \frac{n!}{(k)!\,(n-k)!} s^k - s^n]$$

$$= n[\,(1+s)^n - 1 - \sum_{k=1}^{n} \frac{n!}{(k)!\,(n-k)!} s^k]$$

$$= n\emptyset(s) \qquad 1.4$$

Which together with the observation that $\emptyset(0) = 0$ shows that $\emptyset(s)$ satisfies equation 1.2, by uniqueness of the solution, we have $\emptyset(s) = 0$, as desired [2].

2. An Alternate Proof of the Binomial Theorem using the Laplace transforms.

Theorem 2: Let n be any nonnegative integer and a, b two real numbers. Then from equation (1). [4]

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$$

where 0^0 is interpreted as unity whenever x = 0 or y = 0.

Proof. For x = 0 or y = 0, (1) clearly holds. Let t > 0 be any positive real number, then the Laplace transform of $(1 + t)^n$ for s > 0 is given by

$$L[(1+t)^n]s = \int_0^\infty e^{-st}(1+t)^n dt$$

$$= \frac{1}{s} + \frac{n}{s} L[(1+t)^{n-1}]s$$

$$= \frac{1}{s} + \frac{n}{s^2} + \frac{n(n-1)}{s^2} L[(1+t)^{n-2}]s$$

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$$L[(1+t)^n]s \quad = \sum_{k=0}^{n} \frac{n!}{(n-k)!} \frac{1}{s^{k+1}} \qquad 2.1$$

Taking the inverse Laplace transform, we obtain

$$[(1+t)^n] = L^{-1}\left\{\sum_{k=0}^{n} \frac{n!}{(n-k)!} \frac{1}{s^{k+1}}\right\}(t)$$

$$= \left\{\sum_{k=0}^{n} \frac{n!}{(n-k)!} L^{-1}\left(\frac{1}{s^{k+1}}\right)\right\}(t)$$

$$[(1+t)^n] = \left\{\sum_{k=0}^{n} \frac{n!}{k!(n-k)!} t^k\right\} \qquad 2.2$$

Similarly,

$$[(1-t)^n] = \left\{\sum_{k=0}^{n} \frac{n!}{k!(n-k)!} (-t)^k\right\} \qquad 2.3$$

From the above equation it is clear that (2.2) holds for all real t. The proof is now complete on substituting t = a/b in (2.2).

3. A Simple and Probabilistic Proof of the Binomial Theorem

Theorem 3: Let n be a positive integer and a, b two nonzero real numbers. Then from eq^n (1) [3]

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$$

Proof: By a straight forward induction argument, for all n ≥ 1, there exist positive integers C(n, 0), ... , C(n, n) such that for all a, b ∈ R,

$$(a+b)^n = \sum_{k=0}^{n} C(n,k)\ a^k b^{n-k} \qquad 3.1$$

Hence to establish (1) (i.e., to establish the general case described in equation 3, we need to show

$$C(n,k) = \binom{n}{k}, \quad k = 0 \ldots \ldots n, \qquad 3.2$$

To this end, let 0<a<1 and let $S_n = \sum_{i=0}^{n} X_i$, where $X_1,\ldots, X_n$ are independent and identically distributed random variables with $P(X_1 = 1) = a = 1 - P(X_1 = 0)$.

The sum S_n has the binomial (n, a) distribution which by an elementary probability and counting argument (which does not make use of the binomial theorem) is given by:

$$P(X_n = k) = \binom{n}{k} a^k (1-a)^{n-k}, \quad k = 0 \ldots \ldots \ldots n$$

Thus,

$$1 = \sum_{k=0}^{n} P(X_n = k) = \sum_{k=0}^{n} \binom{n}{k} a^k (1-a)^{n-k} \qquad 3.3$$

Thereby establishing the theorem in the special case 0 <a < 1 and b = 1 − a described in (1).
Next, by equation 3.1:

$$1 = (a+1-a)^n = \sum_{k=0}^{n} C(n,k)\ a^k (1-a)^{n-k} \qquad 3.4$$

then from equations 3.4 and 3.3,

$$\sum_{k=0}^{n} [C(n,k) - \binom{n}{k}] a^k (1-a)^{n-k} = 0$$

And consequently

$$\sum_{k=0}^{n} [C(n,k) - \binom{n}{k}] \left(\frac{a}{1-a}\right)^k = 0 \qquad 3.5$$

Since 0<a<1 is arbitrary

$$\sum_{k=0}^{n}[C(n,k)-\binom{n}{k}](X)^k = 0,\ 0 < X < \infty \qquad 3.6$$

The assertion equation 3.1 now follows immediately since the left-hand side of equation 3.5 is the zero polynomial and so all of its coefficients are zero. [3]

4. An Alternate Proof of the Binomial Theorem

While going through the proof of the binomial theorem using the Laplace transform in [1], this Proof arise

Theorem 4: Let n be a positive integer and a, b two nonzero real numbers. Then from equation 1 . [5]

$$(a+b)^n = \sum_{k=0}^{n}\binom{n}{k} a^k b^{n-k}$$

Proof: Let s = −1 be any nonzero real number. For each k, using the quotient rule for derivatives, we have $\frac{d}{ds}\left(\frac{s^k}{(1+s)^n}\right) = \frac{ks^{k-1}-(n-k)s^k}{(1+s)^{n+1}}$, which on multiplying throughout by $\frac{n!}{k!(n-k)!}$ and summing over k = 0 to n gives:

$$\frac{d}{ds}\sum_{k=0}^{n}\frac{n!}{k!(n-k)!}\left(\frac{s^k}{(1+s)^n}\right) = \sum_{k=0}^{n}\frac{n!}{k!(n-k)!}\left\{\frac{ks^{k-1}-(n-k)s^k}{(1+s)^{n+1}}\right\} \qquad 4.1$$

$$= \sum_{k=0}^{n}\frac{kn!}{k!\,(n-k)!}\frac{s^{k-1}}{(1+s)^{n+1}} - \sum_{k=0}^{n}\frac{(n-k)n!}{k!\,(n-k)!}\frac{s^k}{(1+s)^{n+1}}\}$$

$$= \sum_{k=0}^{n}\frac{n!}{(k-1)!\,(n-k)!}\frac{s^{k-1}}{(1+s)^{n+1}} - \sum_{k=0}^{n-1}\frac{n!}{k!\,(n-k-1)!}\frac{s^k}{(1+s)^{n+1}}\}$$

$$= \sum_{k=0}^{n}\frac{n!}{(k-1)!\,(n-k)!}\frac{s^{k-1}}{(1+s)^{n+1}} - \sum_{k=0}^{n}\frac{n!}{(k-1)!\,(n-k)!}\frac{s^{k-1}}{(1+s)^{n+1}}\}$$

$$= 0$$

So, by an application of either the fundamental theorem of calculus or the mean value theorem, we have $\sum_{k=0}^{n}\frac{n!}{k!(n-k)!}\left(\frac{s^k}{(1+s)^n}\right) = c$, for some fixed real number c, which on taking the limit as s → 0 gives c = 1. Therefore,

$$\sum_{k=0}^{n}\frac{n!}{k!(n-k)!}\left(\frac{s^k}{(1+s)^n}\right) = 1 \text{ or } \sum_{k=0}^{n}\frac{n!}{k!(n-k)!}s^k = (1+s)^n \qquad 4.2$$

Taking the limit as s → −1 in the preceding equation gives:

$$\sum_{k=0}^{n}\frac{n!}{k!(n-k)!}(-1)^k = (1-1)^n = 0$$

Thus, we have:

$$\sum_{k=0}^{n}\frac{n!}{k!(n-k)!}s^k = (1+s)^n \qquad 4.3$$

for all non zero real s. To complete the proof, we put s = a/b in equation 4.3

Conclusion:

We have seen the multiple techniques of proving Binomial Theorem like

1. Uniqueness theorem of initial values problem of Linear Differential Equation.
2. Laplace Transformation.
3. Simple probability and binomial distribution.
4. Properties of Laplace Transformation.

These proofs open the way for applications of analytic techniques and useful for proving further identities by differentiation or integration or by some other mathematical techniques.

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