

# On Strong Product of Two Fuzzy Graphs

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**Abstract-** In this paper, the strong product of two fuzzy graphs is defined. It is proved that when two fuzzy graphs are effective then their strong product is always effective and it is proved that the strong product of two complete fuzzy graphs is complete. Also it is proved that the strong product of two connected fuzzy graphs is always connected. The lower and upper truncations of the strong product of two fuzzy graphs are obtained. The degree of a vertex in the strong product of two fuzzy graphs is obtained. A relationship between the direct sum and the strong product of two fuzzy graphs is obtained.

**Index Terms-** Fuzzy Graph, Direct Sum, Strong Product, Effective Fuzzy Graph, Connectedness, Upper and Lower Truncations.

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## I. INTRODUCTION

Fuzzy graph theory was introduced by Azriel Rosenfeld in 1975. The properties of fuzzy graphs have been studied by Azriel Rosenfeld[9]. Later on, Bhattacharya[7] gave some remarks on fuzzy graphs, and some operations on fuzzy graphs were introduced by Mordeson.J.N. and Peng.C.S.[3]. The conjunction of two fuzzy graphs was defined by Nagoor Gani.A and Radha.K.[4]. We defined the direct sum of two fuzzy graphs and studied the effectiveness, connectedness and regular properties of the direct sum of two fuzzy graphs [8].

In this paper, the strong product of two fuzzy graphs is defined. It is proved that when two fuzzy graphs are effective then their strong product is always effective and it is proved that the strong product of two complete fuzzy graphs is complete. Also it is proved that the strong product of two connected fuzzy graphs is always connected. The lower and upper truncations of the strong product of two fuzzy graphs are obtained. The degree of a vertex in the strong product of two fuzzy graphs is obtained. A relationship between the direct sum and the strong product of two fuzzy graphs is obtained.

First let us recall some preliminary definitions that can be found in [1]-[9].

A fuzzy graph  $G$  is a pair of functions  $(\sigma, \mu)$  where  $\sigma$  is a fuzzy subset of a non empty set  $V$  and  $\mu$  is a symmetric fuzzy relation on  $\sigma$ . The underlying crisp graph of  $G:(\sigma, \mu)$  is denoted by  $G^*(V, E)$  where  $E \subseteq V \times V$ .

Let  $G:(\sigma, \mu)$  be a fuzzy graph. The underlying crisp graph of  $G:(\sigma, \mu)$  is denoted by  $G^*:(V, E)$  where  $E \subseteq V \times V$ . A fuzzy graph  $G$  is an effective fuzzy graph if  $\mu(u,v) = \sigma(u) \wedge \sigma(v)$  for all  $(u,v) \in E$  and  $G$  is a complete fuzzy graph if  $\mu(u,v) = \sigma(u) \wedge \sigma(v)$  for all  $u,v \in V$ . Therefore  $G$  is a complete fuzzy graph if and only if  $G$  is an effective fuzzy graph and  $G^*$  is complete.  $(\sigma', \mu')$  is a spanning fuzzy subgraph of  $(\sigma, \mu)$  if  $\sigma = \sigma'$  and  $\mu' \subseteq \mu$ , that is, if  $\sigma'(u) = \sigma(u)$  for every  $u \in V$  and  $\mu'(e) \leq \mu(e)$  for every  $e \in E$ .

The degree of a vertex  $u$  of a fuzzy graph  $G$  is defined as  $d_G(u) = \sum_{u \neq v} \mu(uv) = \sum_{uv \in E} \mu(uv)$ .

The Cartesian product of two fuzzy graphs  $G_1:(\sigma_1, \mu_1)$  and  $G_2:(\sigma_2, \mu_2)$  is defined as a fuzzy graph  $G = G_1 \times G_2: (\sigma_1 \times \sigma_2, \mu_1 \times \mu_2)$  on  $G^*:(V, E)$  where  $V = V_1 \times V_2$  and  $E = \{((u_1, v_1)(u_2, v_2)) / u_1 = u_2, v_1 v_2 \in E_2 \text{ or } v_1 = v_2, u_1 u_2 \in E_1\}$  with  $(\sigma_1 \times \sigma_2)(u,v) = \sigma_1(u) \wedge \sigma_2(v)$ , for all  $(u, v) \in V_1 \times V_2$  and

$$(\mu_1 \times \mu_2)((u_1, v_1)(u_2, v_2)) = \begin{cases} \sigma_1(u_1) \wedge \mu_2(v_1 v_2) & , \text{if } u_1 = u_2, v_1 v_2 \in E_2 \\ \sigma_2(v_1) \wedge \mu_1(u_1 u_2) & , \text{if } v_1 = v_2, u_1 u_2 \in E_1 \end{cases}$$

The conjunction or the tensor product of two fuzzy graphs  $G_1:(\sigma_1, \mu_1)$  and  $G_2:(\sigma_2, \mu_2)$  is defined as a fuzzy graph  $G = G_1 \wedge G_2: (\sigma, \mu)$  on  $G^*:(V, E)$  where  $V = V_1 \times V_2$  and  $E = \{((u_1, v_1)(u_2, v_2)) / u_1 u_2 \in E_1, v_1 v_2 \in E_2\}$  with  $\sigma(u_1, v_1) = \sigma_1(u_1) \wedge \sigma_2(v_1)$ , for all  $(u_1, v_1) \in V_1 \times V_2$  and  $\mu((u_1, v_1)(u_2, v_2)) = \mu_1(u_1 u_2) \wedge \mu_2(v_1 v_2)$ , for all  $((u_1, v_1)(u_2, v_2)) \in E$ .

If  $G_1:(\sigma_1, \mu_1)$  and  $G_2:(\sigma_2, \mu_2)$  are two fuzzy graphs such that  $\sigma_1 \leq \mu_2$  then  $\sigma_2 \geq \mu_1$ [6].

The lower and upper truncations of  $\sigma$  at level  $t$ ,  $0 < t \leq 1$ , are the fuzzy subsets  $\sigma_{(t)}$  and  $\sigma^{(t)}$  defined respectively by ,

$$\sigma_{(t)}(u) = \begin{cases} \sigma(u), & \text{if } u \in \sigma^t \\ 0 & , \text{if } u \notin \sigma^t \end{cases} \text{ and } \sigma^{(t)}(u) = \begin{cases} t & , \text{if } u \in \sigma^t \\ \sigma(u), & \text{if } u \notin \sigma^t \end{cases}$$

Let  $G:(\sigma,\mu)$  be a fuzzy graph with underlying crisp graph  $G^*:(V,E)$ . Take  $V_{(t)} = \sigma^t$ ,  $E_{(t)} = \mu^t$ . Then  $G_{(t)}:(\sigma_{(t)},\mu_{(t)})$  is a fuzzy graph with underlying crisp graph  $G_{(t)}^*:(V_{(t)}, E_{(t)})$ . This is called the lower truncation of the fuzzy graph  $G$  at level  $t$ . Here  $V_{(t)}$  and  $E_{(t)}$  may be proper subsets of  $V$  and  $E$  respectively. Take  $V^{(t)} = V$ ,  $E^{(t)} = E$ . Then  $G^{(t)}:(\sigma^{(t)}, \mu^{(t)})$  is a fuzzy graph with underlying crisp graph  $G^{(t)*}:(V^{(t)}, E^{(t)})$ . This is called the upper truncation of the fuzzy graph  $G$  at level  $t$  [5].

Let  $G_1:(\sigma_1,\mu_1)$  and  $G_2:(\sigma_2,\mu_2)$  denote two fuzzy graphs with underlying crisp graphs  $G_1^*:(V_1,E_1)$  and  $G_2^*:(V_2,E_2)$  respectively. Let  $V = V_1 \cup V_2$  and let  $E = \{uv / u,v \in V; uv \in E_1 \text{ or } uv \in E_2 \text{ but not both } \}$ .

Define  $G:(\sigma, \mu)$  by

$$\sigma(u) = \begin{cases} \sigma_1(u) & , \text{if } u \in V_1 - V_2 \\ \sigma_2(u) & , \text{if } u \in V_2 - V_1 \\ \sigma_1(u) \vee \sigma_2(u) & , \text{if } u \in V_1 \cap V_2 \end{cases} \text{ and } \mu(uv) = \begin{cases} \mu_1(uv) & , \text{if } uv \in E_1 \\ \mu_2(uv) & , \text{if } uv \in E_2 \end{cases}$$

Then if  $uv \in E_1$ ,  $\mu(uv) = \mu_1(uv) \leq \sigma_1(u) \wedge \sigma_1(v) \leq \sigma(u) \wedge \sigma(v)$ , if  $uv \in E_2$ ,  $\mu(uv) = \mu_2(uv) \leq \sigma_2(u) \wedge \sigma_2(v) \leq \sigma(u) \wedge \sigma(v)$ . Therefore  $(\sigma, \mu)$  defines a fuzzy graph. This is called the direct sum of two fuzzy graphs.

## II STRONG PRODUCT

### Definition 2.1

Let  $G_1:(\sigma_1,\mu_1)$  and  $G_2:(\sigma_2,\mu_2)$  denote two fuzzy graphs with underlying crisp graphs  $G_1^*:(V_1,E_1)$  and  $G_2^*:(V_2,E_2)$  respectively. The normal product of  $G_1^*$  and  $G_2^*$  is  $G^* = G_1^* \circ G_2^* : (V, E)$  where  $V = V_1 \times V_2$  and  $E = \{(u_1, v_1)(u_2, v_2) / u_1 = u_2, v_1 v_2 \in E_2 \text{ or } v_1 = v_2, u_1 u_2 \in E_1 \text{ or } u_1 u_2 \in E_1 \text{ and } v_1 v_2 \in E_2\}$ .

Define  $G:(\sigma, \mu)$ , where  $\sigma = \sigma_1 \circ \sigma_2$  and  $\mu = \mu_1 \circ \mu_2$  by

$\sigma(u_1, v_1) = \sigma_1(u_1) \wedge \sigma_2(v_1)$ , for all  $(u_1, v_1) \in V_1 \times V_2$  and

$$\mu((u_1, v_1)(u_2, v_2)) = \begin{cases} \sigma_1(u_1) \wedge \mu_2(v_1 v_2) & , \text{if } u_1 = u_2, v_1 v_2 \in E_2 \\ \sigma_2(v_1) \wedge \mu_1(u_1 u_2) & , \text{if } v_1 = v_2, u_1 u_2 \in E_1 \\ \mu_1(u_1 u_2) \wedge \mu_2(v_1 v_2) & , \text{if } u_1 u_2 \in E_1, v_1 v_2 \in E_2 \end{cases}$$

If  $u_1 = u_2, v_1 v_2 \in E_2$ ,  $\sigma_1(u_1) \wedge \mu_2(v_1 v_2) = \sigma_1(u_1) \wedge \sigma_1(u_2) \wedge \mu_2(v_1 v_2) \leq \sigma_1(u_1) \wedge \sigma_1(u_2) \wedge \sigma_2(v_1) \wedge \sigma_2(v_2) = \sigma(u_1, v_1) \wedge \sigma(u_2, v_2)$

Similarly if  $v_1 = v_2, u_1 u_2 \in E_1$ ,  $\sigma_2(v_1) \wedge \mu_1(u_1 u_2) \leq \sigma(u_1, v_1) \wedge \sigma(u_2, v_2)$

If  $u_1 u_2 \in E_1$  and  $v_1 v_2 \in E_2$ ,  $\mu_1(u_1 u_2) \wedge \mu_2(v_1 v_2) \leq \sigma_1(u_1) \wedge \sigma_1(u_2) \wedge \sigma_2(v_1) \wedge \sigma_2(v_2) = \sigma(u_1, v_1) \wedge \sigma(u_2, v_2)$

Hence  $\mu((u_1, v_1)(u_2, v_2)) \leq \sigma(u_1, v_1) \wedge \sigma(u_2, v_2)$ . Therefore  $G:(\sigma, \mu)$  is a fuzzy graph. This is called the normal product or the strong product of the fuzzy graphs  $G_1$  and  $G_2$  and is denoted by  $G_1 \circ G_2$ .

### Example 2.2

The following Figure1 gives an example of the strong product of two fuzzy graphs.

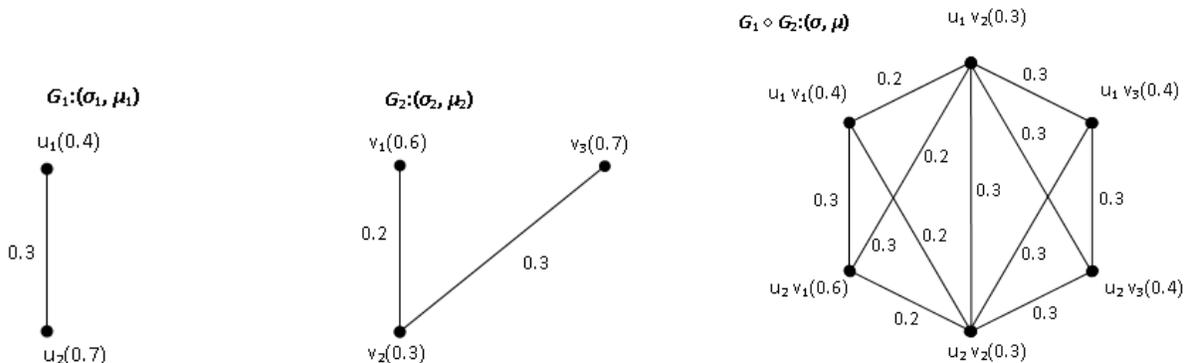


Figure 1: The strong product  $G_1 \circ G_2$  of  $G_1$  and  $G_2$

### Theorem 2.3:

If  $G_1$  and  $G_2$  are two effective fuzzy graphs, then  $G_1 \circ G_2$  is an effective fuzzy graph.

### Proof:

Let  $G_1$  and  $G_2$  be effective fuzzy graphs.

Then  $\mu_1(u_1 u_2) = \sigma_1(u_1) \wedge \sigma_1(u_2)$  for any  $u_1 u_2 \in E_1$  and  $\mu_2(v_1 v_2) = \sigma_2(v_1) \wedge \sigma_2(v_2)$  for any  $v_1 v_2 \in E_2$ .

Therefore proceeding as in the definition,

If  $u_1 = u_2, v_1 v_2 \in E_2$ ,  $\mu((u_1, v_1)(u_2, v_2)) = \sigma_1(u_1) \wedge \mu_2(v_1 v_2) = \sigma_1(u_1) \wedge \sigma_1(u_2) \wedge \sigma_2(v_1) \wedge \sigma_2(v_2) = (\sigma_1(u_1) \wedge \sigma_1(u_2)) \wedge (\sigma_2(v_1) \wedge \sigma_2(v_2)) = \sigma(u_1, v_1) \wedge \sigma(u_2, v_2)$ . Similarly,

If  $v_1 = v_2, u_1 u_2 \in E_1$ ,  $\mu((u_1, v_1)(u_2, v_2)) = \sigma(u_1, v_1) \wedge \sigma(u_2, v_2)$

If  $u_1 u_2 \in E_1$  and  $v_1 v_2 \in E_2$ ,  $\mu((u_1, v_1)(u_2, v_2)) = \sigma(u_1, v_1) \wedge \sigma(u_2, v_2)$ .

Hence  $G_1 \circ G_2$  is an effective fuzzy graph.

**Theorem 2.4:**

If  $G_1$  and  $G_2$  are two complete fuzzy graphs, then  $G_1 \circ G_2$  is a complete fuzzy graph.

**Proof:**

Let  $G_1$  and  $G_2$  be complete fuzzy graphs. Then  $G_1$  and  $G_2$  are effective fuzzy graphs and  $G_1^*$  and  $G_2^*$  are complete graphs. Therefore  $G_1 \circ G_2$  is an effective fuzzy graph by Theorem 2.2 and  $G_1^* \circ G_2^*$  is a complete graph. Hence  $G_1 \circ G_2$  is a complete fuzzy graph.

**Example 2.5:**

The following Figure 2 gives an example of the strong product of two effective fuzzy graphs.

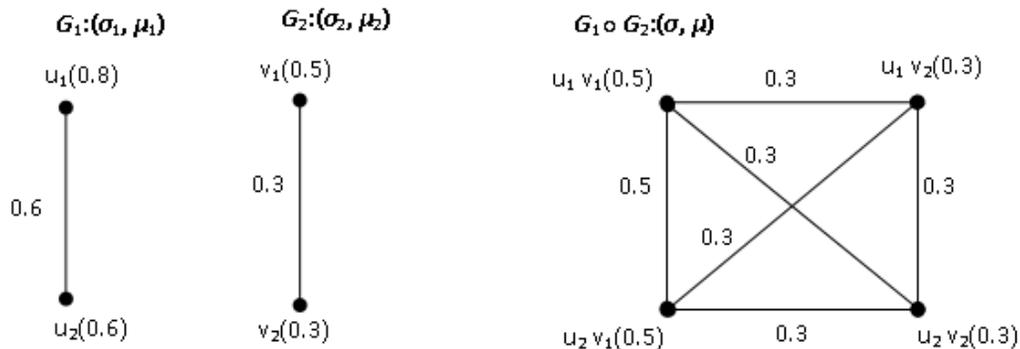


Figure 2: The strong product  $G_1 \circ G_2$  of two effective fuzzy graphs  $G_1$  and  $G_2$

**Example 2.6:**

The following Figure 3 gives an example of the strong product of two complete fuzzy graphs.

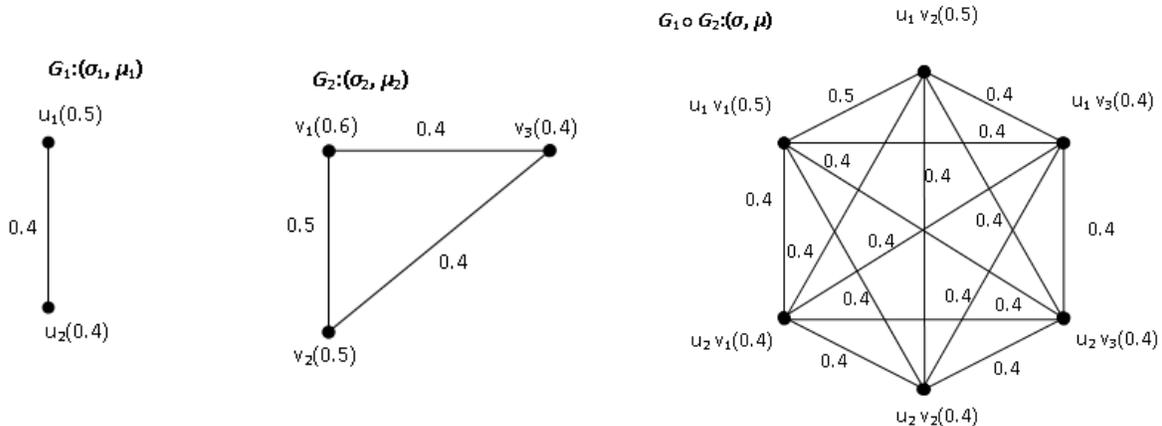


Figure 3: The strong product  $G_1 \circ G_2$  of two complete fuzzy graphs  $G_1$  and  $G_2$

**Theorem 2.7:**

The strong product of two connected fuzzy graphs is always a connected fuzzy graph.

**Proof:**

Let  $G_1:(\sigma_1, \mu_1)$  and  $G_2:(\sigma_2, \mu_2)$  be two connected fuzzy graphs with underlying crisp graphs  $G_1^*:(V_1, E_1)$  and  $G_2^*:(V_2, E_2)$  respectively.

Let  $V_1 = \{u_1, u_2, \dots, u_m\}$  and  $V_2 = \{v_1, v_2, \dots, v_n\}$ .

The strong product of two connected fuzzy graphs  $G_1:(\sigma_1, \mu_1)$  and  $G_2:(\sigma_2, \mu_2)$  can be taken as  $G:(\sigma, \mu)$  where  $\sigma = \sigma_1 \circ \sigma_2$  and  $\mu = \mu_1 \circ \mu_2$ .

Now consider the 'm' sub graphs of  $G$  with the vertex sets  $\{u_i v_1, u_i v_2, \dots, u_i v_n\}$  for  $i=1, 2, \dots, m$ .

Each of these sub graphs of  $G$  is connected since the  $u_i$ 's are the same and since  $G_2$  is connected, each  $v_i$  is adjacent to at least one of the vertices in  $V_2$ .

Also since  $G_1$  is connected, each  $u_i$  is adjacent to at least one of the vertices in  $V_1$ .

Therefore there exists at least one edge between any pair of the above 'm' sub graphs. Hence  $G$  is a connected fuzzy graph.

### III TRUNCATIONS OF THE STRONG PRODUCT OF TWO FUZZY GRAPHS

**Theorem 3.1:**  $(G_1 \circ G_2)_{(t)} = G_{1(t)} \circ G_{2(t)}$  and  $(G_1 \circ G_2)^{(t)} = G_1^{(t)} \circ G_2^{(t)}$ .

**Proof:**

$$\text{We have } (\sigma_1 \circ \sigma_2)_{(t)}(u, v) = \begin{cases} (\sigma_1 \circ \sigma_2)(u, v) & , \text{if } (\sigma_1 \circ \sigma_2)(u, v) \geq t \\ 0 & , \text{if } (\sigma_1 \circ \sigma_2)(u, v) < t \end{cases}$$

Now  $(\sigma_{1(t)} \circ \sigma_{2(t)})(u, v) = \sigma_{1(t)}(u) \wedge \sigma_{2(t)}(v)$

If  $(\sigma_1 \circ \sigma_2)(u, v) \geq t$ , then  $\sigma_1(u) \wedge \sigma_2(v) \geq t \Rightarrow \sigma_1(u) \geq t$  and  $\sigma_2(v) \geq t \Rightarrow \sigma_{1(t)}(u) = \sigma_1(u)$ ,  $\sigma_{2(t)}(v) = \sigma_2(v)$

$$\Rightarrow \sigma_{1(t)}(u) \wedge \sigma_{2(t)}(v) = \sigma_1(u) \wedge \sigma_2(v) = (\sigma_1 \circ \sigma_2)(u, v).$$

If  $(\sigma_1 \circ \sigma_2)(u, v) < t$ , then  $\sigma_1(u) \wedge \sigma_2(v) < t \Rightarrow$  either  $\sigma_1(u) < t$ ,  $\sigma_2(v) \geq t$  or  $\sigma_1(u) \geq t$ ,  $\sigma_2(v) < t$  or  $\sigma_1(u) < t$ ,  $\sigma_2(v) < t$

$$\Rightarrow \sigma_{1(t)}(u) = 0, \sigma_{2(t)}(v) = \sigma_2(v) \text{ or } \sigma_{1(t)}(u) = \sigma_1(u), \sigma_{2(t)}(v) = 0 \text{ or } \sigma_{1(t)}(u) = 0, \sigma_{2(t)}(v) = 0$$

$$\Rightarrow \sigma_{1(t)}(u) \wedge \sigma_{2(t)}(v) = 0.$$

$$\text{Therefore } (\sigma_{1(t)} \circ \sigma_{2(t)})(u, v) = \sigma_{1(t)}(u) \wedge \sigma_{2(t)}(v) = \begin{cases} (\sigma_1 \circ \sigma_2)(u, v) & , \text{if } (\sigma_1 \circ \sigma_2)(u, v) \geq t \\ 0 & , \text{if } (\sigma_1 \circ \sigma_2)(u, v) < t \end{cases}$$

Hence  $(\sigma_1 \circ \sigma_2)_{(t)}(u, v) = (\sigma_{1(t)} \circ \sigma_{2(t)})(u, v)$  for every  $(u, v) \in V_1 \times V_2$ .

$$\text{Now } (\mu_1 \circ \mu_2)_{(t)}((u_1, v_1)(u_2, v_2)) = \begin{cases} (\mu_1 \circ \mu_2)((u_1, v_1)(u_2, v_2)) & , \text{if } (\mu_1 \circ \mu_2)((u_1, v_1)(u_2, v_2)) \geq t \\ 0 & , \text{if } (\mu_1 \circ \mu_2)((u_1, v_1)(u_2, v_2)) < t \end{cases}$$

If  $(\sigma_1 \circ \sigma_2)(u, v) \geq t$ , then  $\sigma_1(u_1) \wedge \mu_2(v_1 v_2) \geq t$  or  $\sigma_2(v_1) \wedge \mu_1(u_1 u_2) \geq t$  or  $\mu_1(u_1 u_2) \wedge \mu_2(v_1 v_2) \geq t$

Proceeding as above, we can show that

$$\begin{aligned} (\mu_{1(t)} \circ \mu_{2(t)})((u_1, v_1)(u_2, v_2)) &= \begin{cases} \sigma_1(u_1) \circ \mu_2(v_1, v_2) & , \text{if } \sigma_1(u_1) \circ \mu_2(v_1, v_2) \geq t \\ \sigma_2(v_1) \circ \mu_1(u_1, u_2) & , \text{if } \sigma_2(v_1) \circ \mu_1(u_1, u_2) \geq t \\ \mu_1(u_1, u_2) \wedge \mu_2(v_1, v_2) & , \text{if } \mu_1(u_1, u_2) \wedge \mu_2(v_1, v_2) \geq t \\ 0 & , \text{if } \mu_1(u_1, u_2) \wedge \mu_2(v_1, v_2) < t \end{cases} \\ &= \begin{cases} (\mu_1 \circ \mu_2)((u_1, v_1)(u_2, v_2)) & , \text{if } (\mu_1 \circ \mu_2)((u_1, v_1)(u_2, v_2)) \geq t \\ 0 & , \text{if } (\mu_1 \circ \mu_2)((u_1, v_1)(u_2, v_2)) < t \end{cases} \end{aligned}$$

Therefore  $(\mu_1 \circ \mu_2)_{(t)}((u_1, v_1)(u_2, v_2)) = (\mu_{1(t)} \circ \mu_{2(t)})((u_1, v_1)(u_2, v_2))$  for every edge  $(u_1, v_1)(u_2, v_2)$  in  $G_1 \circ G_2$ .

Hence  $(G_1 \circ G_2)_{(t)} = G_{1(t)} \circ G_{2(t)}$ .

Proceeding in the same way, we can show that  $(G_1 \circ G_2)^{(t)} = G_1^{(t)} \circ G_2^{(t)}$ .

### IV. DEGREE OF A VERTEX IN THE STRONG PRODUCT OF TWO FUZZY GRAPHS

The degree of any vertex in the strong product  $G_1 \circ G_2$  of two fuzzy graphs  $G_1: (\sigma_1, \mu_1)$  and  $G_2: (\sigma_2, \mu_2)$  is given by,

$$d_{G_1 \circ G_2}(u_i, v_j) = \sum_{u_i = u_k, v_j v_\ell \in E_2} \sigma_1(u_i) \wedge \mu_2(v_j v_\ell) + \sum_{u_i u_k \in E_1, v_j = v_\ell} \mu_1(u_i u_k) \wedge \sigma_2(v_j) + \sum_{u_i u_k \in E_1, v_j v_\ell \in E_2} \mu_1(u_i u_k) \wedge \mu_2(v_j v_\ell).$$

This expression can be simplified using the terms of the degrees of vertices in  $G_1$  and  $G_2$  with some constraints.

**Theorem 4.1:**

If  $G_1: (\sigma_1, \mu_1)$  and  $G_2: (\sigma_2, \mu_2)$  are two fuzzy graphs such that  $\sigma_1 \geq \mu_2$  and  $\sigma_2 \geq \mu_1$  and  $\mu_1 \wedge \mu_2 = c$  (a constant), then the degree of a vertex in the strong product of the two fuzzy graphs  $G_1: (\sigma_1, \mu_1)$  and  $G_2: (\sigma_2, \mu_2)$  is given by,

$$d_{G_1 \circ G_2}(u_i, v_j) = d_{G_2}(v_j) + d_{G_1}(u_i) + [d_{G_1^*}(u_i) d_{G_2^*}(v_j)]c.$$

**Proof:**

Let  $G_1: (\sigma_1, \mu_1)$  and  $G_2: (\sigma_2, \mu_2)$  be two fuzzy graphs with underlying crisp graphs  $G_1^*: (V_1, E_1)$  and  $G_2^*: (V_2, E_2)$  respectively.

Suppose that  $\sigma_1 \geq \mu_2$  and  $\sigma_2 \geq \mu_1$  and  $\mu_1 \wedge \mu_2 = c$  (a constant), then

$$\sigma_1 \geq \mu_2 \Rightarrow \sigma_1 \wedge \mu_2 = \mu_2 \text{ and } \sigma_2 \geq \mu_1 \Rightarrow \sigma_2 \wedge \mu_1 = \mu_1$$

Now,

$$d_{G_1 \circ G_2}(u_i, v_j) = \sum_{u_i = u_k, v_j v_\ell \in E_2} \sigma_1(u_i) \wedge \mu_2(v_j v_\ell) + \sum_{u_i u_k \in E_1, v_j = v_\ell} \mu_1(u_i u_k) \wedge \sigma_2(v_j) + \sum_{u_i u_k \in E_1, v_j v_\ell \in E_2} \mu_1(u_i u_k) \wedge \mu_2(v_j v_\ell).$$

$$d_{G_1 \circ G_2}(u_i, v_j) = \sum_{u_i = u_k, v_j, v_\ell \in E_2} \mu_2(v_j, v_\ell) + \sum_{u_i, u_k \in E_1, v_j = v_\ell} \mu_1(u_i, u_k) + \sum_{u_i, u_k \in E_1, v_j, v_\ell \in E_2} c$$

$$= d_{G_2}(v_j) + d_{G_1}(u_i) + [d_{G_1^*}(u_i) d_{G_2^*}(v_j)]c.$$

**Theorem 4.2:**

If  $G_1:(\sigma_1, \mu_1)$  and  $G_2:(\sigma_2, \mu_2)$  are two fuzzy graphs such that  $\sigma_1 \geq \mu_2$  and  $\sigma_2 \geq \mu_1$  and  $\mu_1 \vee \mu_2 = C$  (a constant), then the degree of a vertex in the strong product of the two fuzzy graphs  $G_1:(\sigma_1, \mu_1)$  and  $G_2:(\sigma_2, \mu_2)$  is given by,  $d_{G_1 \circ G_2}(u_i, v_j) = [1 + d_{G_2^*}(v_j)]d_{G_1}(u_i) + [1 + d_{G_1^*}(u_i)]d_{G_2}(v_j) - [d_{G_1^*}(u_i) d_{G_2^*}(v_j)]C$ .

**Proof:**

Let  $G_1:(\sigma_1, \mu_1)$  and  $G_2:(\sigma_2, \mu_2)$  be two fuzzy graphs with underlying crisp graphs  $G_1^*:(V_1, E_1)$  and  $G_2^*:(V_2, E_2)$  respectively. Suppose that  $\sigma_1 \geq \mu_2$  and  $\sigma_2 \geq \mu_1$  and  $\mu_1 \vee \mu_2 = C$  (a constant), then  $\sigma_1 \geq \mu_2 \Rightarrow \sigma_1 \wedge \mu_2 = \mu_2$  and  $\sigma_2 \geq \mu_1 \Rightarrow \sigma_2 \wedge \mu_1 = \mu_1$

Now,

$$d_{G_1 \circ G_2}(u_i, v_j) = \sum_{u_i = u_k, v_j, v_\ell \in E_2} \sigma_1(u_i) \wedge \mu_2(v_j, v_\ell) + \sum_{u_i, u_k \in E_1, v_j = v_\ell} \mu_1(u_i, u_k) \wedge \sigma_2(v_j) + \sum_{u_i, u_k \in E_1, v_j, v_\ell \in E_2} \mu_1(u_i, u_k) \wedge \mu_2(v_j, v_\ell)$$

$$= \sum_{u_i = u_k, v_j, v_\ell \in E_2} \mu_2(v_j, v_\ell) + \sum_{u_i, u_k \in E_1, v_j = v_\ell} \mu_1(u_i, u_k) + \sum_{u_i, u_k \in E_1, v_j, v_\ell \in E_2} [\mu_1(u_i, u_k) + \mu_2(v_j, v_\ell) - \mu_1(u_i, u_k) \vee \mu_2(v_j, v_\ell)]$$

$$= d_{G_2}(v_j) + d_{G_1}(u_i) + \sum_{u_i, u_k \in E_1, v_j, v_\ell \in E_2} \mu_1(u_i, u_k) + \sum_{u_i, u_k \in E_1, v_j, v_\ell \in E_2} \mu_2(v_j, v_\ell) - \sum_{u_i, u_k \in E_1, v_j, v_\ell \in E_2} [\mu_1(u_i, u_k) \vee \mu_2(v_j, v_\ell)]$$

$$= d_{G_2}(v_j) + d_{G_1}(u_i) + d_{G_2^*}(v_j) d_{G_1}(u_i) + d_{G_1^*}(u_i) d_{G_2}(v_j) - \sum_{u_i, u_k \in E_1, v_j, v_\ell \in E_2} C$$

$$= [1 + d_{G_2^*}(v_j)]d_{G_1}(u_i) + [1 + d_{G_1^*}(u_i)]d_{G_2}(v_j) - [d_{G_1^*}(u_i) d_{G_2^*}(v_j)]C.$$

**Theorem 4.3:**

If  $G_1:(\sigma_1, \mu_1)$  and  $G_2:(\sigma_2, \mu_2)$  are two fuzzy graphs such that  $\sigma_1 \leq \mu_2$  and  $\mu_1 \wedge \mu_2 = c$  (a constant), then the degree of a vertex in the strong product is given by,  $d_{G_1 \circ G_2}(u_i, v_j) = d_{G_2^*}(v_j)\sigma_1(u_i) + d_{G_1}(u_i) + [d_{G_1^*}(u_i) d_{G_2^*}(v_j)]c$ .

**Proof:**

Let  $G_1:(\sigma_1, \mu_1)$  and  $G_2:(\sigma_2, \mu_2)$  be two fuzzy graphs with underlying crisp graphs  $G_1^*:(V_1, E_1)$  and  $G_2^*:(V_2, E_2)$  respectively. Suppose that  $\sigma_1 \leq \mu_2$ . Then  $\sigma_2 \geq \mu_1$ . This implies that  $\sigma_1 \leq \sigma_2$ . Also  $\mu_1 \wedge \mu_2 = c$  (a constant).

Now,

$$d_{G_1 \circ G_2}(u_i, v_j) = \sum_{u_i = u_k, v_j, v_\ell \in E_2} \sigma_1(u_i) \wedge \mu_2(v_j, v_\ell) + \sum_{u_i, u_k \in E_1, v_j = v_\ell} \mu_1(u_i, u_k) \wedge \sigma_2(v_j) + \sum_{u_i, u_k \in E_1, v_j, v_\ell \in E_2} \mu_1(u_i, u_k) \wedge \mu_2(v_j, v_\ell)$$

$$= \sum_{u_i = u_k, v_j, v_\ell \in E_2} \sigma_1(u_i) + \sum_{u_i, u_k \in E_1, v_j = v_\ell} \mu_1(u_i, u_k) + \sum_{u_i, u_k \in E_1, v_j, v_\ell \in E_2} \mu_1(u_i, u_k) \wedge \mu_2(v_j, v_\ell)$$

$$= d_{G_2^*}(v_j)\sigma_1(u_i) + d_{G_1}(u_i) + [d_{G_1^*}(u_i) d_{G_2^*}(v_j)]c.$$

**Example 4.4:**

If  $G_1:(\sigma_1, \mu_1)$  and  $G_2:(\sigma_2, \mu_2)$  are two fuzzy graphs such that  $\sigma_1 \leq \mu_2$  and  $\mu_1 \wedge \mu_2 = c$  (a constant), then their strong product  $G_1 \circ G_2 : (\sigma, \mu)$  is given in the following example.

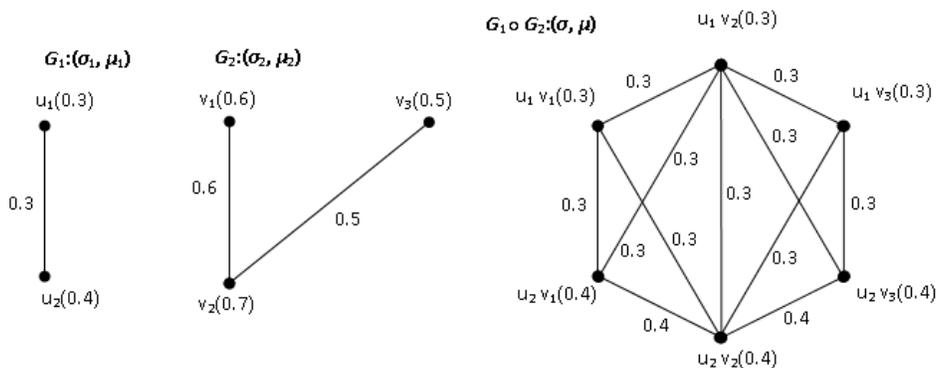


Figure 4: The Strong Product of two fuzzy graphs such that  $\sigma_1 \leq \mu_2$  and  $\mu_1 \wedge \mu_2 = 0.3$ .

Now,

$$d_{G_1 \circ G_2}(u_1, v_1) = d_{G_2^*}(v_1)\sigma_1(u_1) + d_{G_1}(u_1) + [d_{G_1^*}(u_1) d_{G_2^*}(v_1)]c = 1 \times 0.3 + 0.3 + 1 \times 1 \times 0.3 = 0.9$$

$$d_{G_1 \circ G_2}(u_2, v_2) = d_{G_2^*}(v_2)\sigma_1(u_2) + d_{G_1}(u_2) + [d_{G_1^*}(u_2)d_{G_2}(v_2)]c = 2 \times 0.4 + 0.3 + 1 \times 2 \times 0.3 = 1.7$$

## V. RELATIONSHIP BETWEEN THE DIRECT SUM AND THE STRONG PRODUCT

### Theorem 5.1:

The strong product of two fuzzy graphs  $G_1$  and  $G_2$  is the direct sum of the Cartesian product of  $G_1$  and  $G_2$  and the conjunction of  $G_1$  and  $G_2$ .

### Proof:

From the definitions,  $(\sigma_1 \circ \sigma_2)(u, v) = (\sigma_1 \times \sigma_2)(u, v) = (\sigma_1 \wedge \sigma_2)(u, v) = \sigma_1(u) \wedge \sigma_2(v)$  for every  $(u, v) \in V_1 \times V_2$ .

So  $((\sigma_1 \circ \sigma_2) \oplus (\sigma_1 \wedge \sigma_2))(u, v) = \sigma_1(u) \wedge \sigma_2(v)$  for every  $(u, v) \in V_1 \times V_2$ .

Hence  $(\sigma_1 \circ \sigma_2)(u, v) = ((\sigma_1 \times \sigma_2) \oplus (\sigma_1 \wedge \sigma_2))(u, v)$  for every  $(u, v) \in V_1 \times V_2$ .

From the definitions of Cartesian product and the conjunction,

$$(\mu_1 \times \mu_2) \oplus (\mu_1 \wedge \mu_2)((u_1, v_1)(u_2, v_2)) = \begin{cases} \sigma_1(u_1) \circ \mu_2(v_1, v_2) & , \text{if } u_1 = u_2, v_1 v_2 \in E_2 \\ \sigma_2(u_1) \circ \mu_1(u_1, u_2) & , \text{if } v_1 = v_2, u_1 u_2 \in E_1 \\ \mu_1(u_1, u_2) \wedge \mu_2(v_1, v_2) & , \text{if } u_1 u_2 \in E_1 \text{ and } v_1 v_2 \in E_2 \end{cases}$$

$$= (\mu_1 \circ \mu_2)((u_1, v_1)(u_2, v_2))$$

Hence  $G_1 \circ G_2 = (G_1 \times G_2) \oplus (G_1 \wedge G_2)$ .

## VI. CONCLUSION

In this paper, the strong product of two fuzzy graphs is defined. It is proved that when two fuzzy graphs are effective then their strong product is always effective and it is proved that the strong product of two complete fuzzy graphs is complete. Also it is proved that the strong product of two connected fuzzy graphs is always connected. The lower and upper truncations of the strong product of two fuzzy graphs are obtained. The degree of a vertex in the strong product of two fuzzy graphs is obtained. A relationship between the direct sum and the strong product of two fuzzy graphs is obtained. Operation on fuzzy graph is a great tool to consider large fuzzy graph as a combination of small fuzzy graphs and to derive its properties from those of the small ones. Through this paper, a step in that direction is made.

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