Nazarov Uncertainty Principle for Continuous Modulated Shearlet Transform on Euclidean Motion Group

Piyush Bansal[1] and Ashish Bansal[2]

[1]Department of Mathematics, St. Stephen's College (University of Delhi), University Enclave, North Campus, Delhi, 110007, India
(E-mail: piyush@ststephens.edu)
ORCID: 0009-0001-5738-6755

[2]Department of Mathematics, Keshav Mahavidyalaya (University of Delhi), H-4-5 Zone, Pitampura, Delhi, 110034, India
(Corresponding Author, E-mail: abansal@keshav.du.ac.in)
ORCID: 0000-0001-6507-0868

Abstract

Nazarov uncertainty principle has been proved for the Fourier transform and the continuous modulated shearlet transform on the Euclidean motion group. These principles can be used to deduce the Nazarov uncertainty principle for the well-known transforms such as the Gabor transform, the wavelet transform and the shearlet transform on Euclidean motion group.

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1 Introduction

The Nazarov uncertainty principle is an important concept from harmonic analysis which is closely related to the classical Heisenberg uncertainty principle in quantum mechanics, though formulated in a more general mathematical setting. Introduced by Fedor Nazarov, this principle places limits on the simultaneous localisation of a function and its Fourier transform [5]. In simpler terms, this principle provides a quantitative constraint that a function and its Fourier transform cannot both be sharply localized (concentrated) on the small sets of finite measure. Jaming [3] established the Nazarov uncertainty inequality for the Fourier transform on $\mathbb{R}^n$, which can be stated as follows.

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Theorem 1.1. *There exists a constant $C = C(n)$ such that, for every $\Lambda_1 \subset \mathbb{R}^n$ and $\Lambda_2 \subset \widehat{\mathbb{R}^n}$ of finite Lebesgue measures $|\Lambda_1|$ and $|\Lambda_2|$ respectively, and for every $f \in L^2(\mathbb{R}^n)$,*

$$\int_{\mathbb{R}^n} |f(x)|^2 \, dx \leq J \int_{\mathbb{R}^n \setminus \Lambda_1} |f(x)|^2 \, dx + J \int_{\widehat{\mathbb{R}^n} \setminus \Lambda_2} |\widehat{f}(\zeta)|^2 \, d\zeta,$$

where $J = Ce^{C \min\left\{|\Lambda_1|\,|\Lambda_2|, |\Lambda_1|^{1/n} w(\Lambda_2), w(\Lambda_1)|\Lambda_2|^{1/n}\right\}}$ and $w(\Lambda_1)$, $w(\Lambda_2)$ are mean widths of Λ_1, Λ_2 respectively.

Nazarov uncertainty principle was proved for quadratic-phase Fourier transform [7], for shearlet transform on $\mathbb{R}^n$ [1], for non-separable linear canonical wavelet transform [8] and for linear canonical transform [9].

In Section 2, we shall discuss some useful notations and results related to the Fourier transform. The continuous modulated shearlet transform (CMST) that was introduced in [2] shall also be reviewed. In the last section, we establish Nazarov uncertainty principle for the Fourier transform and the CMST on the Euclidean motion group.

2 Preliminaries and notations

In this section, we briefly review some standard results that shall be used throughout the paper. For a subset $\Lambda \subset \mathbb{R}^n$, let $|\Lambda|$ denote the Lebesgue measure of Λ which is assumed to be finite and let $\omega(\Lambda)$ denote the mean width of Λ (see [3, Pg. 36]).

2.1 Fourier transform

Assume G to be a unimodular, separable, locally compact group of type I. The left Haar measure on G is denoted by μ_G. Let $\widehat{G}$ denote the unitary dual of G, where $\widehat{G}$ is the set of all the equivalence classes of irreducible unitary representations of G. The space $\widehat{G}$ is equipped with the Mackey-Borel structure and the Plancherel measure $\mu_{\widehat{G}}$ on $\widehat{G}$ can be determined uniquely by the fixed Haar measure on G.

For each $\pi \in \widehat{G}$, let the Hilbert space $\mathcal{H}_\pi$ denote the representation space of π and $\mathrm{HS}(\mathcal{H}_\pi)$ denote the space of all Hilbert-Schmidt operators on $\mathcal{H}_\pi$. The space $\mathrm{HS}(\mathcal{H}_\pi)$ forms a Hilbert space with respect to the inner product defined by $\langle T_1, T_2 \rangle = \mathrm{tr}(T_2^* T_1)$. The family $\{\mathrm{HS}(\mathcal{H}_\pi)\}_{\pi \in \widehat{G}}$ of Hilbert spaces indexed by $\widehat{G}$, is a field of Hilbert spaces over $\widehat{G}$. The direct integral $\mathfrak{H}^2(\widehat{G})$ of the family $\{\mathrm{HS}(\mathcal{H}_\pi)\}_{\pi \in \widehat{G}}$ with respect to $\mu_{\widehat{G}}$, is the space of all measurable vector fields K on $\widehat{G}$ such that

$$\|K\|^2_{\mathfrak{H}^2(\widehat{G})} = \int_{\widehat{G}} \|K(\pi)\|^2_\pi \, d\mu_{\widehat{G}}(\pi) < \infty,$$

where $\|\cdot\|_\pi$ is the norm on the Hilbert space $\mathrm{HS}(\mathcal{H}_\pi)$. Also, $\mathfrak{H}^2(\widehat{G})$ forms a Hilbert space with the inner product given by

$$\langle K_1, K_2 \rangle_{\mathfrak{H}^2(\widehat{G})} = \int_{\widehat{G}} \mathrm{tr}\left[K_1(\pi)^* \, K_2(\pi)\right] \, d\mu_{\widehat{G}}(\pi).$$

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For $f \in L^1(G)$, the Fourier transform of f is defined by

$$\mathscr{F} f(\pi) = \pi(f) = \int_G f(x)\, \pi(x)^* \, d\mu_G(x).$$

The Plancherel formula states that for all $f \in L^1(G) \cap L^2(G)$, we have $\|\mathscr{F} f\|_{\mathfrak{H}^2(\widehat{G})} = \|f\|_2$.

2.2 Continuous modulated shearlet transform

In [2], Bansal, Bansal and Kumar defined the CMST that generalises the Gabor transform, the shearlet transform and the wavelet transform. In this section, we briefly discuss the CMST for the convenience of the reader. Let $\mathcal{L}$ be a locally compact group equipped with left Haar measure $d\mu_{\mathcal{L}}(l)$ and $\mathrm{Aut}(\mathcal{H})$ denote the automorphism group of $\mathcal{H}$, where $\mathcal{H}$ is a locally compact abelian group which is second countable and has Haar measure $d\mu_{\mathcal{H}}(h)$.

Consider the homomorphism $\alpha : \mathcal{L} \to \mathrm{Aut}(\mathcal{H})$ by $l \mapsto \alpha_l$, ensuring the continuity of the map $(l, h) \mapsto \alpha_l(h)$ from the product space $\mathcal{L} \times \mathcal{H}$ onto $\mathcal{H}$. The set $\mathcal{L} \times \mathcal{H}$ endowed with the product topology and the operations

$$(l, h)(l', h') = (l\, l', h\, \alpha_l(h'))$$
$$(l, h)^{-1} = (l^{-1}, \alpha_{l^{-1}}(h^{-1}))$$

is a locally compact group called the semi-direct product of $\mathcal{L}$ and $\mathcal{H}$ and is denoted by $\mathcal{L} \times_\alpha \mathcal{H}$. The set $\mathfrak{S} = (\mathcal{L} \times_\alpha \mathcal{H}) \times G$ forms a locally compact group. Let $1_{\mathcal{L}}$, $1_{\mathcal{H}}$ and 1_G denote the identity elements of $\mathcal{L}$, $\mathcal{H}$ and G respectively. For $u = (l, h, x, \pi) \in \mathfrak{S} \times \widehat{G}$, assume that

$$\mathfrak{H}_u = \pi(x)\mathrm{HS}(\mathfrak{H}_\pi),$$

where $\pi(x)\mathrm{HS}(\mathfrak{H}_\pi) = \{\pi(x)T : T \in \mathrm{HS}(\mathfrak{H}_\pi)\}$. Then, $\mathfrak{H}_u$ is a Hilbert space with the inner product given by

$$\langle \pi(x)T_1, \pi(x)T_2 \rangle_{\mathfrak{H}_u} = \mathrm{tr}\,(T_2^* T_1) = \langle T_1, T_2 \rangle_{\mathrm{HS}(\mathfrak{H}_\pi)}.$$

Suppose that the inner product on $\mathfrak{H}_u$ induces the norm $\|\cdot\|_u$. It may be easily verified that $\mathfrak{H}_u = \mathrm{HS}(\mathfrak{H}_\pi)$ for all $u \in \mathfrak{S} \times \widehat{G}$. The family $\{\mathfrak{H}_u : u \in \mathfrak{S} \times \widehat{G}\}$ is a field of Hilbert spaces over $\mathfrak{S} \times \widehat{G}$. The direct integral of $\{\mathfrak{H}_u : u \in \mathfrak{S} \times \widehat{G}\}$, denoted by $\mathfrak{H}^2(\mathfrak{S} \times \widehat{G})$, is the space of all vector fields K on $\mathfrak{S} \times \widehat{G}$ which are measurable and satisfy

$$\|K\|^2_{\mathfrak{H}^2(\mathfrak{S}\times\widehat{G})} = \int_{\mathfrak{S}\times\widehat{G}} \|K(u)\|_u^2 \, d\sigma(u) < \infty.$$

The space $\mathfrak{H}^2(\mathfrak{S} \times \widehat{G})$ is a Hilbert space having the inner product

$$\langle K_1, K_2 \rangle_{\mathfrak{H}^2(\mathfrak{S}\times\widehat{G})} = \int_{\mathfrak{S}\times\widehat{G}} \mathrm{tr}\,[K_2(u)^* \; K_1(u)] \, d\sigma(u)$$

and is equipped with the product measure

$$d\sigma(u) = d\mu_{\mathfrak{S}}(l, h, x)\; d\mu_{\widehat{G}}(\pi).$$

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For each $(l, h, x) \in \mathfrak{S}$ and $f, \psi \in L^2(\mathcal{H} \times G)$, define $\mathfrak{T}^{\psi}_{(l,h,x)} : \mathcal{H} \times G \to \mathbb{C}$ by

$$\mathfrak{T}^{\psi}_{(l,h,x)}(k, y) = \delta_{\alpha}^{1/2}(l)\ \psi(\alpha_{l^{-1}}(h^{-1}k), x^{-1}y)$$

and for all $(k, y) \in \mathcal{H} \times G$, define $\mathfrak{J}^{\psi}_{(l,h,x)} f : \mathcal{H} \times G \to \mathbb{C}$ by

$$\mathfrak{J}^{\psi}_{(l,h,x)} f(k, y) = f(k, y)\ \overline{\mathfrak{T}^{\psi}_{(l,h,x)}(k, y)}.$$

We call a function $\psi \in L^2(\mathcal{H} \times G)$ *admissible* if

$$C_{\psi} := \int_{\mathcal{L}\times G} \left| \mathscr{F}_{\mathcal{H}}\, \widetilde{\psi}(\eta \circ \lambda_l, x) \right|^2 d\mu_{\mathcal{L}}(l)\ d\mu_G(x) < \infty$$

which is independent of a.e. $\eta \in \widehat{\mathcal{H}}$.
For $f \in C_{00}(\mathcal{H} \times G)$, the set of all continuous complex-valued functions on $\mathcal{H} \times G$ with compact supports and admissible function $\psi \in L^2(\mathcal{H} \times G)$, the CMST of f with respect to ψ is a measurable field of operators on $\mathfrak{S} \times \widehat{G}$ defined by

$$\mathcal{MS}_{\psi} f(l, h, x, \pi) = \int_{\mathcal{H}} \int_{G} f(k, y)\ \overline{\mathfrak{T}^{\psi}_{(l,h,x)}(k, y)}\ \pi(y)^*\ d\mu_{\mathcal{H}}(k)\ d\mu_G(y).$$

$\mathcal{MS}_{\psi} f(l, h, x, \pi)$ is a bounded linear operator on $\mathfrak{H}_{\pi}$ such that

$$\|\mathcal{MS}_{\psi} f(l, h, x, \pi)\| \leq \|f\|_{L^2(\mathcal{H}\times G)}\ \|\psi\|_{L^2(\mathcal{H}\times G)}.$$

The operator $\mathcal{MS}_{\psi} : C_{00}(\mathcal{H} \times G) \to \mathfrak{H}^2(\mathfrak{S} \times \widehat{G})$ defined by $f \mapsto \mathcal{MS}_{\psi} f$ satisfies

$$\|\mathcal{MS}_{\psi} f\|_{\mathfrak{H}^2(\mathfrak{S}\times\widehat{G})} = C_{\psi}^{1/2}\ \|f\|_{L^2(\mathcal{H}\times G)} \tag{1}$$

and thus can be extended uniquely to a bounded linear operator from $L^2(\mathcal{H} \times G)$ into $\mathfrak{H}^2(\mathfrak{S} \times \widehat{G})$. This extension, which we still denote by $\mathcal{MS}_{\psi}$, satisfies (1) for each $f \in L^2(\mathcal{H} \times G)$.

3 Euclidean Motion Group $M(n)$

Consider $M(n) = \mathbb{R}^n \rtimes K$ with $K = SO(n)$ under the group operation defined by $(z, k)(w, s) = (z + k \cdot w, ks)$ for $z, w \in \mathbb{R}^n$ and $k, s \in K$. The set $M(n)$ is referred to as the *Euclidean Motion Group*. The set $M = SO(n-1)$ can be regarded as a subgroup of K that fixes the point $e_1 = (1, 0, 0, \ldots, 0)$. We can parameterize (upto unitary equivalence) all the irreducible unitary representations of $M(n)$ that are relevant for the Plancherel formula. This parameterization consists of pairs of the type (η, τ), where $\eta > 0$ and $\tau \in \widehat{M}$. Here, $\widehat{M}$ denotes the unitary dual of M and $\tau \in \widehat{M}$ is realized on a Hilbert space $\mathcal{H}_{\tau}$ of dimension d_{τ}. The motion group $M(n)$ is equipped with the Haar measure $dg = dz\ dk$, where dk denotes the normalized Haar measure on $SO(n)$ and dz denotes the Lebesgue measure on $\mathbb{R}^n$. For more details, one may refer to [6]. In this section, we shall establish Nazarov uncertainty principle for the Fourier transform and for the CMST on $M(n)$.

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Theorem 3.1. *For every $f \in L^2(M(n))$ and $\Lambda_1, \Lambda_2 \subset \mathbb{R}^n$ of finite Lebesgue measures, there exists a constant $C = C(n)$ such that*

$$\|f\|_2^2 \leq J \int_{\mathbb{R}^n \setminus \Lambda_1} \int_K |f(z,k)|^2 \, dz \, dk + J c_n \int_0^\infty \sum_{\tau \in \widehat{M}} d_\tau \, \|\widehat{f}(\eta,\tau)\|_{HS}^2 \, \eta^{n-1} \, d\eta$$
$$- J \int_{\Lambda_2} \int_K |\mathscr{F}_1 f(w,k)|^2 \, dw \, dk,$$

where $J = Ce^{C \min\left\{|\Lambda_1|\,|\Lambda_2|, |\Lambda_1|^{1/n} w(\Lambda_2), w(\Lambda_1)|\Lambda_2|^{1/n}\right\}}$ *and* $c_n = \dfrac{2}{2^{n/2}\,\Gamma\left(\frac{n}{2}\right)}$.

Proof. Let $f \in L^2(M(n))$, then for almost all $k \in K$,

$$\int_{\mathbb{R}^n} |f(z,k)|^2 \, dz < \infty.$$

For each $k \in K$, define $f_k(z) = f(z,k)$, for all $z \in \mathbb{R}^n$. Then, $f_k \in L^2(\mathbb{R}^n)$ for almost all $k \in K$. Let $\Lambda_1, \Lambda_2 \subset \mathbb{R}^n$ be of finite Lebesgue measures, then using Theorem 1.1, there exists a constant $C = C(n)$ such that

$$\begin{aligned}\int_{\mathbb{R}^n} |f_k(z)|^2 \, dz &\leq J \int_{\mathbb{R}^n \setminus \Lambda_1} |f_k(z)|^2 \, dz + J \int_{\mathbb{R}^n \setminus \Lambda_2} |\widehat{f_k}(w)|^2 \, dw \\ &= J \int_{\mathbb{R}^n \setminus \Lambda_1} |f(z,k)|^2 \, dz + J \int_{\mathbb{R}^n \setminus \Lambda_2} |\mathscr{F}_1 f(w,k)|^2 \, dw,\end{aligned}$$

where $J = Ce^{C \min\left\{|\Lambda_1|\,|\Lambda_2|, |\Lambda_1|^{1/n} w(\Lambda_2), w(\Lambda_1)|\Lambda_2|^{1/n}\right\}}$. Integrating both sides w.r.t. dk, we get

$$\|f\|_2^2 \leq J \int_{\mathbb{R}^n \setminus \Lambda_1} \int_K |f(z,k)|^2 \, dz \, dk + J \int_{\mathbb{R}^n \setminus \Lambda_2} \int_K |\mathscr{F}_1 f(w,k)|^2 \, dw \, dk, \tag{2}$$

where $\mathscr{F}_1$ denotes the Fourier transform in the first variable. Now, using Plancherel formula on $\mathbb{R}^n$ and $M(n)$ (see [4]), we have

$$\begin{aligned}&\int_{\mathbb{R}^n \setminus \Lambda_2} \int_K |\mathscr{F}_1 f(w,k)|^2 \, dw \, dk \\ &= \int_{\mathbb{R}^n} \int_K |\mathscr{F}_1 f(w,k)|^2 \, dw \, dk - \int_{\Lambda_2} \int_K |\mathscr{F}_1 f(w,k)|^2 \, dw \, dk \\ &= \int_{\mathbb{R}^n} \int_K |f(z,k)|^2 \, dz \, dk - \int_{\Lambda_2} \int_K |\mathscr{F}_1 f(w,k)|^2 \, dw \, dk \\ &= c_n \int_0^\infty \sum_{\tau \in \widehat{M}} d_\tau \, \|\widehat{f}(\eta,\tau)\|_{\mathrm{HS}}^2 \, \eta^{n-1} \, d\eta - \int_{\Lambda_2} \int_K |\mathscr{F}_1 f(w,k)|^2 \, dw \, dk,\end{aligned}$$

where $c_n = \dfrac{2}{2^{n/2}\,\Gamma\left(\frac{n}{2}\right)}$. Using above equality in (2), we obtain the result. ☐

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Theorem 3.2. *Let $G = M(n)$. For $f, \psi \in L^2(\mathcal{H} \times G)$ such that ψ is an admissible function and $\Lambda_1, \Lambda_2 \subset \mathbb{R}^n$ of finite Lebesgue measures, there exists a constant $C = C(n)$ such that*

$$\begin{aligned} C_\psi \|f\|^2_{L^2(\mathcal{H}\times G)} \leq & \, JC_\psi \int_{\mathcal{H}} \int_{\mathbb{R}^n \setminus \Lambda_1} \int_K |f(h,z,k)|^2 \, d\mu_{\mathcal{H}}(h) \, dz \, dk \\ & + Jc_n \int_{\mathfrak{S}} \int_0^\infty \sum_{\tau \in \widehat{M}} d_\tau \, \|\mathcal{MS}_\psi f(l,h,y,u,\eta,\tau)\|^2_{HS} \, d\sigma(l,h,y,u,\eta) \\ & - J \int_{\mathfrak{S}} \int_{\Lambda_2} \int_K \left| \mathscr{F}_2 \mathscr{F}_{\mathcal{H}} \left(\mathfrak{I}^\psi_{(l,h,y,u)} f \right)(I,w,k) \right|^2 \delta_\alpha(l) \, d\mu_{\mathcal{L}}(l) \, d\mu_{\mathcal{H}}(h) \, dy \, du \, dw \, dk, \end{aligned}$$

where $J = Ce^{C \min\left\{|\Lambda_1|\,|\Lambda_2|, |\Lambda_1|^{1/n} w(\Lambda_2), w(\Lambda_1)|\Lambda_2|^{1/n}\right\}}$ *and* $c_n = \dfrac{2}{2^{n/2}\,\Gamma\left(\frac{n}{2}\right)}$.

Proof. Let $f, \psi \in L^2(\mathcal{H} \times G)$ such that ψ is an admissible function and $\Lambda_1, \Lambda_2 \subset \mathbb{R}^n$ be of finite Lebesgue measures. By [2, Eq. (6.1)], $\mathscr{F}_{\mathcal{H}} \left(\mathfrak{I}^\psi_{(l,h,y,u)} f \right)(I, \cdot) \in L^2(G)$ for almost every $(l,h,y,u) \in \mathfrak{S}$. Using Theorem 3.1, we have for almost every $(l,h,y,u) \in \mathfrak{S}$,

$$\begin{aligned} & \int_{\mathbb{R}^n} \int_K \left| \mathscr{F}_{\mathcal{H}} \left(\mathfrak{I}^\psi_{(l,h,y,u)} f \right)(I,z,k) \right|^2 dz \, dk \\ & \leq J \int_{\mathbb{R}^n \setminus \Lambda_1} \int_K \left| \mathscr{F}_{\mathcal{H}} \left(\mathfrak{I}^\psi_{(l,h,y,u)} f \right)(I,z,k) \right|^2 dz \, dk \\ & \quad + Jc_n \int_0^\infty \sum_{\tau \in \widehat{M}} d_\tau \left\| \mathscr{F}_G \mathscr{F}_{\mathcal{H}} \left(\mathfrak{I}^\psi_{(l,h,y,u)} f \right)(I,\eta,\tau) \right\|^2_{\mathrm{HS}} \eta^{n-1} \, d\eta \\ & \quad - J \int_{\Lambda_2} \int_K \left| \mathscr{F}_2 \mathscr{F}_{\mathcal{H}} \left(\mathfrak{I}^\psi_{(l,h,y,u)} f \right)(I,w,k) \right|^2 dw \, dk \\ & = J \int_{\mathbb{R}^n \setminus \Lambda_1} \int_K \left| \mathscr{F}_{\mathcal{H}} \left(\mathfrak{I}^\psi_{(l,h,y,u)} f \right)(I,z,k) \right|^2 dz \, dk \\ & \quad + Jc_n \int_0^\infty \sum_{\tau \in \widehat{M}} d_\tau \, \|\mathcal{MS}_\psi f(l,h,y,u,\eta,\tau)\|^2_{\mathrm{HS}} \, \eta^{n-1} \, d\eta \\ & \quad - J \int_{\Lambda_2} \int_K \left| \mathscr{F}_2 \mathscr{F}_{\mathcal{H}} \left(\mathfrak{I}^\psi_{(l,h,y,u)} f \right)(I,w,k) \right|^2 dw \, dk \end{aligned}$$

Integrating both sides with respect to $\delta_\alpha(l) \, d\mu_{\mathcal{L}}(l) \, d\mu_{\mathcal{H}}(h) \, dy \, du$, we get

$$\begin{aligned} & \int_{\mathfrak{S}} \int_{\mathbb{R}^n} \int_K \left| \mathscr{F}_{\mathcal{H}} \left(\mathfrak{I}^\psi_{(l,h,y,u)} f \right)(I,z,k) \right|^2 \delta_\alpha(l) \, d\mu_{\mathcal{L}}(l) \, d\mu_{\mathcal{H}}(h) \, dy \, du \, dz \, dk \\ & \leq J \int_{\mathfrak{S}} \int_{\mathbb{R}^n \setminus \Lambda_1} \int_K \left| \mathscr{F}_{\mathcal{H}} \left(\mathfrak{I}^\psi_{(l,h,y,u)} f \right)(I,z,k) \right|^2 \delta_\alpha(l) \, d\mu_{\mathcal{L}}(l) \, d\mu_{\mathcal{H}}(h) \, dy \, du \, dz \, dk \\ & \quad + Jc_n \int_{\mathfrak{S}} \int_0^\infty \sum_{\tau \in \widehat{M}} d_\tau \, \|\mathcal{MS}_\psi f(l,h,y,u,\eta,\tau)\|^2_{\mathrm{HS}} \, d\sigma(l,h,y,u,\eta) \\ & \quad - J \int_{\mathfrak{S}} \int_{\Lambda_2} \int_K \left| \mathscr{F}_2 \mathscr{F}_{\mathcal{H}} \left(\mathfrak{I}^\psi_{(l,h,y,u)} f \right)(I,w,k) \right|^2 \delta_\alpha(l) \, d\mu_{\mathcal{L}}(l) \, d\mu_{\mathcal{H}}(h) \, dy \, du \, dw \, dk. \end{aligned}$$

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Using again [2, Eq. (6.1)], we obtain

$$C_\psi \|f\|^2_{L^2(\mathcal{H}\times G)} \leq JC_\psi \int_{\mathcal{H}} \int_{\mathbb{R}^n \setminus \Lambda_1} \int_K |f(h,z,k)|^2 \, d\mu_{\mathcal{H}}(h)\, dz\, dk$$
$$+ Jc_n \int_{\mathfrak{S}} \int_0^\infty \sum_{\tau \in \widehat{M}} d_\tau \, \|\mathcal{MS}_\psi f(l,h,y,u,\eta,\tau)\|^2_{\mathrm{HS}} \, d\sigma(l,h,y,u,\eta)$$
$$- J \int_{\mathfrak{S}} \int_{\Lambda_2} \int_K \left| \mathscr{F}_2 \mathscr{F}_{\mathcal{H}} \left(\mathfrak{J}^{\psi}_{(l,h,y,u)} f \right)(I,w,k) \right|^2 \delta_\alpha(l)\, d\mu_{\mathcal{L}}(l)\, d\mu_{\mathcal{H}}(h)\, dy\, du\, dw\, dk. \qquad \square$$

Remark 3.1. Using above result, one may deduce Nazarov uncertainty principle for Gabor transform on $M(n)$.

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