Treatment of the cyclic homology through topological spaces

S. Z. Rida [1] &Alaa Hassan Noreldeen [2] & Faten. R. Karar [3]
[1] Mathematics Department, Facultyof Science, South Valley University, Qena, Egypt
[2].[3] Mathematics Department, Facultyof Science, Aswan University, Aswan, Egypt
E-mail: szagloul@yahoo.com, E-mail: ala2222000@yahoo.com, E-mail: fatenragab2020@yahoo.com

DOI: 10.29322/IJSRP.12.09.2022.p12937
http://dx.doi.org/10.29322/IJSRP.12.09.2022.p12937

Paper Received Date: 14th August 2022
Paper Acceptance Date: 14th September 2022
Paper Publication Date: 26th September 2022

Abstract- The purpose of this paper is proving the analysis of vanishing and non-vanishing cyclic and reflexive (co)homology groups of algebras. Additionally, we present and prove some of the results and examples which related cyclic and reflexive (co)homolgy theory.

Index Terms*- Dihedral (co)homology –Reflexive homology - Operator algebras*- exact sequence.

2010 MSC. 18F25,19D99.

I. Introduction

The cohomology theory is discussed as the following: Triangular Banach algebras' second-order cohomology groups are researched in [10]. Factorization and bounded Hochschild cohomology of matrix-like Banach algebras are investigated in [14], while the group cohomology of triangular Banach algebra is addressed in [5]. In [7], projective modules in quantized homology are introduced. [6] Studies projective Hilbert modules over operator algebras. C^*-algebra projective homological classification is investigated. Using aspects of quantum theory, biprojective algebras, homological dimensions, and current advances in Banach homology theory are addressed in [2]. In [8], the studies the disappearance of the third simplicial cohomology group for specific classes of C^*-algebra. In [95], an approximate identity for ideals of Segal algebras on a compact group and homology theory are examined.

Murry and von Neumann published a number of articles in the 1930s and 1940s that served as the foundation for the theory of von Neumann algebras. Connes demonstrated in [1] that for any dual normal A-bimodule N, the cohomology of injective von Neumann algebra $H^*(A, N)$ disappears. The Kadison theorem [15], which states that all derivations of any von Neumann algebra A into itself are inner, is the earliest result in von Neumann algebra cohomology. It states that if the linear map $d: A \to A$ satisfies $d(x, y) = xd(y) + d(x)y$, then there exists v in A such that $d(x) = xv - vx$. According to this, $H^1\ (A, A) = 0$ for all A. in [9], the main finding in Hochschild cohomology of von Neumann algebras is that for every whose type II_2, $H^*(A, N)$ vanishes. The fully bounded cohomology of von Neumann algebra disappears; $H_{cb}\ (A, A) = 0$, according to Sinclair and Smith's presentation and extension of results from [9] in [11]. In [8], the norm continuous cohomology $H_{cb}\ (A, A) = 0$ is investigated. A few finite von Neumann algebras' cohomology groups were explored by Christensen, Smith, and in [3]. The Hochschild and cyclic cohomology groups vanish if $\mathcal{B}$ is C^*-algebra without a nuclear C^*-algebra or bounded traces as in [2]. The cyclic and dihedral cohomology of a nuclear C^*-algebra and the even dimensional dihedral cohomology of a biflat algebra do not vanish as shown in [6] and [1].

The contents of this paper are as follows. After recalling some definitions and background notations in the category of operator algebras, we study the vanishing and nonvanishing of the dihedral and reflexive cohomology groups of operator algebras. We study examples of the vanishing of the reflexive and dihedral cohomology topological algebras. Firstly, we review a few definitions and relations from [1], [3] and [4] on the category and dihedral (Reflexive) cohomology of topological algebra.

II. Simplicial object

This part, introduce notations and l some definitions of simplicial object. The main references in this material are [3], [8] and [11].

The simplicial category Δ has objects of Δ are a bounded sets $[n] := \{0,1, \dots, n\}, n \geq 0$, and morphisms are the functions $[n] \to [m]$. A contravariant functor from Δ to a category $\mathcal{C}$ is simplicial object ($\Delta^{op} \to \mathcal{C}$). The simplicial objects in $\mathcal{C}$ form a category, with morphisms being the natural transformations of functors.

We denote a category of objects with morphisms by a natural transformations of functors in $\mathcal{C}$ by $s\mathcal{C}$. If a topology $X \in \text{Ob}(s\mathcal{C})$, we denote $X_n := X([n])$.

The category Δ is generated by two distinguished classes of morphisms $\{\delta^i\}_{0\le i\le n}^{n\ge 1}$ and $\{\sigma^j\}_{0\le j\le n}^{n\ge 0}$, whose images under $X \in s\mathcal{C}$. The face and degeneracy maps of X are $\{\delta^i\}_{0\le i\le n}^{n\ge 1}$ and $\{\sigma^j\}_{0\le j\le n}^{n\ge 0}$, with $X \in s\mathcal{C}$. The injection map is $\delta^i: [n-1] \to [n]$ and the face map is $d_i := X(\delta^i): X_n \to X_{n-1}$. Similarly, the surjection $\sigma^i: [n+1] \to [n]$ for $n \ge 0$, in Δ. The image of σ^i under X is $s_i := X(\sigma^i): X_n \to X_{n+1}$. The relations between face and degeneracy maps are:

$$\begin{aligned}
d_i d_j &= d_{j-1} d_i && \text{if } i < j \\
d_i s_j &= s_{j-1} d_i && \text{if } i < j \\
d_i s_j &= s_j d_{i-1} && \text{if } i > j+1 \\
s_i s_j &= s_{j+1} s_i && \text{if } i \le j \\
d_i s_j &= \text{Id} && \text{if } i = j, j+1.
\end{aligned}$$

A set sSet define a category of the simplicial sets. The reduced simplicial set will be indicated as the entire subcategory of $sSet0$. A simplicial set X is said to be pointed if it contains distinct simplices $x_n \in X_n$, one for each degree, such that $n \ge 1$, $x_n = s_0(x_{n-1})$. A basepoint of X is the sequence $(x_0, x_1, x_2, \ldots) \in \prod_{n\ge 0} X_n$. A canonical basepoint exists in the category of pointed simplicial sets.

Let $X \in sSet$ denote the set of nondegenerate n-simplices of X, which is defined as
$\bar{X}_n := X_n \setminus \bigcup_{i=0}^{n-1} s_i(X_{n-1})$.

Definition (1):

For each object $A \in \mathcal{C}$, a simplicial object $A_* \in s\mathcal{C}$ can be associated, with $A_n = A$ and d_i, s_j becoming the identity map of A for any n, i, j. This results in a fully embedding $\mathcal{C} \hookrightarrow s\mathcal{C}$. The objects of $s\mathcal{C}$ that emerge in this manner are known as discrete simplicial objects.

Definition (2)

The topological space

$$\Delta^n := \left\{ (x_0, \ldots, x_n) \in \mathbb{R}^{n+1} \mid \sum_{i=0}^{n} x_i = 1, x_i \ge 0 \right\}$$

is the n-dimensional geometric simplex

Let e_i represent the vertex of Δ^n with the i^{th} one coordinate. There is a (unique) linear map $\mathbb{R}^{m+1} \to \mathbb{R}^{n+1}$ conveying e_i to $ee_{f(i)}$, that limits to a map of topological spaces $f^*: \Delta^m \to \Delta^n$ for each morphism $f: [m] \to [n]$ in. $\Delta^* := \{\Delta^n\}_{n\ge 0}$ forms a cosimplicial space, i.e. a (covariant) functor $\Delta \to$ Top, where Top denotes the category of topological spaces. This functor is faithful: it provides a topological realisation of the simplicial category, which was historically the original definition of Δ.

Definition (3)

The Yoneda embedding is $Y: \Delta \hookrightarrow$ sSet. The standard n-simplex is a functor Y assigns to $[n]$ a simplicial set $\Delta[n]_*$ which $\Delta[n]_*$ is writing by

$$\Delta[n]_k := \text{Hom}_\Delta([k], [n]) \cong \{(n_0, \ldots, n_k) \mid 0 \le n_0 \le \cdots \le n_k \le n\},$$

a map $f: [k] \to [n]$ is given with the sequence of its values $(f(0), \ldots, f(k))$. The degeneracy and face maps in $\Delta[n]_*$ are getting by

$$d_i(n_0, \ldots, n_k) = (n_0, \ldots, \hat{n}_i, \ldots, n_k),\ s_j(n_0, \ldots, n_k) = (n_0, \ldots, n_j, n_j, \ldots, n_k).$$

The natural bijection $\text{Hom}_{\text{sSet}}(\Delta[n]_*, X) \cong X_n$ is exist for a simplicial set X, The $\Delta[n]_*$ is represent the functor: $sSet \to Set$, $X \mapsto X_n$.

The functor $|-| : sSet \to Top$ defiend to each simplicial set X, a topological space $|X|$ denoted by:

$$|X| := \bigsqcup_{n\ge 0} (X_n \times \Delta^n)/\sim,$$

Every X_n corresponding with discrete topology and given by;

$$(d_i x, p) \sim (x, d^i p) \text{ for } (x, p) \in X_n \times \Delta^{n-1}$$

$$(s_j x, p) \sim (x, s^j p) \text{ for } (x, p) \in X_{n-1} \times \Delta^n.$$

Let $(X, *)$ be a pointed topological space. The singular complex of the topology X is a simplicial set $S_*(X)$ denoteby the form;

$$S_n(X) := \text{Hom}_{\text{Top}}(\Delta^n, X).$$

The Eilenberg subcomplex of $S_*(X)$ is $\bar{S}_n(X) := \{f: \Delta^n \to X: f(v_i) = * \text{ for all vertices} v_i \in \Delta^n\}$.

If X is connected, the natural inclusion $\bar{S}_*(X) \hookrightarrow S_*(X)$ is a weak equivalence of simplicial sets.

The induce inverse equivalences of a homotopy categories is : $\text{Ho}(\text{sSet}_0) \simeq \text{Ho}(\text{Top}_{0,*})$.

III. Representation Cohomology

Now it is the time to articulate the research work with ideas gathered in above steps by adopting any of below suitable approaches:

We give the basic theorem of the homotopy types of simplicial groups with those of pointed connected spaces. And study the cohomology theory of the topological spaces. The important theorem in the cyclic and dihredral cohomology of algebra is given with proving in this part.

Definition (4)
For a topological space $X \in \mathrm{Top}_{0,*}$, we give the derived representation scheme $\mathrm{DRep}_G(X)$ to be $\mathrm{DRep}_G(\Gamma X)$, such that ΓX is a simplicial group model of X. The representation homology of X in G is denote by the form:

$$\mathrm{HR}_*(X, G) := \pi_* L(\Gamma X)_G.$$

By definition, $\mathrm{HR}_*(X, G)$ is a graded commutative algebra, with $\mathrm{HR}_0(X, G)$ naturally isomorphic to $[\pi_1(X)]_G = \mathcal{O}[\mathrm{Rep}_G(\pi_1(X))]$, the coordinate ring of the representation scheme $\mathrm{Rep}_G[\pi_1(X)]$. The last isomorphism with the natural isomorphism $\pi_0(\Gamma X) \cong \pi_1(X)$.
For a simplicial set $X \in$ s Set $_0$, a loop group $\mathbb{G}X$ provides a functorial simplicial group model for $|X|$. Since this simplicial group is semi-free as, $(\mathrm{HR}_*(X, G) \cong \pi_*(\mathbb{G}X)_G$
This form used to calculate the representation homology in simple cases.

IV. THE CATEGORY OF ALGEBRAS

We recall that $\mathcal{X} = \{\mathcal{X}_n\}_{n\in\mathbb{Z}}$ is a topological algebra. The n^{th}-dimensional homology of $\mathcal{X}$ is the quotient spaces $\mathcal{H}_n(\mathcal{X}) = Z_n(\mathcal{X})/B_n(\mathcal{X})$. $Z_n(\mathcal{X}) = Ker\{d_n: \mathcal{X}_n \to \mathcal{X}_{n-1}\}$ is the n^{th}-dimensional cycles and $B_n(\mathcal{X}) = Im\{d_{n+1}: \mathcal{X}_{n+1} \to \mathcal{X}_n\}$ is the n^{th}-dimensional boundaries since $B_n(\mathcal{X}) \subset Z_n(\mathcal{X})$. The family $\{\mathcal{H}_n(\mathcal{X})\}$ is known as the homology of $\mathcal{X}$.
The short exact sequence $0 \to B_n(\mathcal{X}) \overset{i_n}{\to} Z_n(\mathcal{X}) \overset{p_n}{\to} \mathcal{H}_n(\mathcal{X}) \to 0$ is splitting since there are continuous operators $q_n: \mathcal{H}_n(\mathcal{X}) \to Z_n(\mathcal{X})$ which satisfies:

$$p_n \circ q_n = Id: \mathcal{H}_n(\mathcal{X}) \to \mathcal{H}_n(\mathcal{X}),$$
$$Id - q_n \circ p_n = d_{n+1} \circ s_n: Z_n(\mathcal{X}) \to B_n(\mathcal{X})$$

Let $\overline{\mathcal{X}} = \{\overline{\mathcal{X}_n}\}$ be a dual complex of the space. The differentials $d_n: \mathcal{X}_n \to \mathcal{X}_{n-1}$ induce the differentials of $\overline{d}_n: \overline{\mathcal{X}}_{-n+1} \to \mathcal{X}_{-n}$. If $\mathcal{X}$ is an admissible Banach complex, then $\overline{\mathcal{X}}$ would also be an admissible Banach complex. The homology of the dual complex $\overline{\mathcal{X}}$ is the cohomology of $\mathcal{X}$ defined by the form:

$$\mathcal{H}^*(\mathcal{X}) = \{\mathcal{H}^n(\mathcal{X})\}, \ \mathcal{H}^n(\mathcal{X}) = \mathcal{H}_{-n}(\overline{\mathcal{X}}).$$

Let $\mathcal{X}$ and $\mathcal{Y}$ be two Banach complexes where the mapping is $f: \mathcal{X} \to \mathcal{Y}$ $s.h.$ $f = \{f_n\}$, $f_n: \mathcal{X}_n \to \mathcal{Y}_{n+m}$. We then define the differentials $d(f)_n$ which maps among most of the linear spaces $Hom_m(\mathcal{X}, \mathcal{Y})$ as $d: Hom_m(\mathcal{X}, \mathcal{Y}) \to Hom_{m-1}(\mathcal{X}, \mathcal{Y})$ and define them as:

$$d(f)_n = d_{n+m} \circ f_n + (-1)^m f_{n-1} \mathrm{o}\, d_n: \mathcal{X}_n \to \mathcal{Y}_{n+m-1}$$

Clearly $d \circ d = 0$, thus the family $Hom\,(\mathcal{X}, \mathcal{Y}) = \{Hom_m\,(\mathcal{X}, \mathcal{Y})\}$ forms the complex in the category of linear spaces.
The chain map of $(\mathcal{X}, \mathcal{Y})$ is $f: \mathcal{X} \to \mathcal{Y}$ induces the following homology map:

$$\mathcal{H}_*(f) = \{\mathcal{H}_n(f)\}: \mathcal{H}_*(\mathcal{Y}) \to H_*(\mathcal{X}),$$
$$\mathcal{H}_n(f): \mathcal{H}_n(\mathcal{Y}) \to \mathcal{H}_n(\mathcal{X})$$

and the cohomology map:

$$\mathcal{H}^*(f) = \{\mathcal{H}^n(f)\}: \mathcal{H}^*(\mathcal{X}) \to \mathcal{H}^*(\mathcal{Y}),$$
$$\mathcal{H}^n(f): \mathcal{H}^n(\mathcal{X}) \to \mathcal{H}^n(\mathcal{Y}).$$

For the two chain functions $f, g: \mathcal{X} \to \mathcal{Y}$, their homotopic is $f \cong g$ if there exists a homotopic map $k: \mathcal{X} \to \mathcal{Y}$ that satisfies that

$$d(h) = g - f \text{ or } d_{n+1} \circ h_n + h_{n-1} \circ d_n = g_n - f_n.$$

If for the topological complexes $\mathcal{X}, \mathcal{Y}$ there are chain mappings $f: \mathcal{X} \to \mathcal{Y}$ and $g: \mathcal{Y} \to \mathcal{X}$ with $g \circ f \cong Id_{\mathcal{X}}$, $f \circ g \cong Id_{\mathcal{Y}}$, then $\mathcal{X} \cong \mathcal{Y}$ are said to be homotopy equivalent. If $f \cong g: \mathcal{X} \to \mathcal{Y}$, $\overline{f} \cong \overline{g}: \overline{\mathcal{X}} \to \overline{\mathcal{Y}}$ and $\mathcal{X} \cong \mathcal{Y}$ then $\overline{\mathcal{X}} \cong \overline{\mathcal{Y}}$.
The tensor product of any two topological complexes $\mathcal{X}$ and $\mathcal{Y}$ is given by the formula (see [6]):

$$(\mathcal{X} \otimes \mathcal{Y})_n = \sum_{p+q=n} \mathcal{X}_p \otimes \mathcal{Y}_q, \tag{1}$$

The differential is:

$$d_n(x_p \otimes y_q) = d_p(x_p) \otimes y_q + (-1)^p x_p \otimes d_q(y_q), \quad p + q = n.$$

In the following part we introduce the cohomology theory of topological algebras. so, we explain some examples of topological algebras and we introduce some states of trivial and nontrivial cohomology theory of topological algebras.

1. (Co)homology of topological algebra

We begin by briefly recalling the basic definitions concerning (co)homology of algebras. For a gentler introduction see [1, 6 and 17].
Consider the algebra $\mathcal{B}$ which is a space with associative multiplication $\pi: \mathcal{B} \otimes \mathcal{B} \to \mathcal{B}$, and the B-construction $B\mathcal{B}$ since:

$$(B\mathcal{B})_n = \begin{cases} \mathcal{B}^{\otimes n}, & n \geq 1 \\ 0, & n \leq 0 \end{cases}$$

The elements in $\mathcal{B}^{\otimes n}$ are in the form $[a_1, \dots a_2]$, $\forall a_i \in \mathcal{B}$. The differential d is defined as:

$$d_n[a_1, \dots a_2] = \sum_{i=1}^{n-1} (-1)^{i+1} [a_1, \dots, \pi(a_i \otimes a_{i+1}), \dots, a_n]$$

If $B\mathcal{B}$ is admissible, then $\mathcal{B}$ is admissible. Every finite dimensional of algebras are admissible. For example, for the finite dimensional of admissible topological algebra, let $\mathcal{B} = l_1$ be the convergent series $\sum_{n=1}^{\infty} \mathcal{X}_n$ with multiplication:

$$\sum_{n=1}^{\infty} \mathcal{X}_n \cdot \sum_{n=1}^{\infty} \mathcal{Y}_n = \sum_{n=2}^{\infty} \left(\sum_{k=1}^{n-1} \mathcal{X}_k \cdot \mathcal{Y}_{n-k} \right)$$

The elements of the series are e_i which remaining zeros. The map $s: \mathcal{B}^{\otimes n} \to \mathcal{B}^{\otimes n+1}$ is defined as:

$$s(e_{i_1} \otimes \dots \otimes e_{i_n}) = \begin{cases} e_1 \otimes e_{i_1 - 1} \otimes \dots \otimes e_{i_n}, & i_1 > 1 \\ 0, & i_1 = 1 \end{cases} \tag{2}$$

Satisfying $d \circ s \circ d = d$, meaning that $\mathcal{B}$ is admissible algebra.

Let $\mathcal{B} = \mathcal{B}` \otimes \mathcal{B}``$ be algebra. The elements in $\mathcal{B}^{\otimes n}$ are $a_1 \otimes \dots \otimes a_n$ such that $a_i \in \mathcal{B}` \ or \ \mathcal{B}``$. If we rewrite these elements to the form $b_1 \otimes \dots \otimes b_n$ where $b_1, \dots, b_n$ are the tensor products of $\mathcal{B}` \ or \ \mathcal{B}``$, then we find that $b_i \in B\mathcal{B}` \ or \ b_i \in B\mathcal{B}``$.

For the map $s: \mathcal{B}^{\otimes n} \to \mathcal{B}^{\otimes n+1}$, we have:

$$\begin{aligned} s(b_1 \otimes \dots \otimes b_m) &= s(b_1) \otimes b_2 \otimes \dots \otimes b_m + (-1)^{n_1} \big(b_1 - d \circ s(b_1)\big) \otimes s(b_2) \otimes \dots \otimes b_m + \cdots \\ &+ (-1)^{n_1 + \cdots + n_{m-1}} \big(b_1 - d \circ s(b_1)\big) \otimes \dots \otimes \big(b_m - d \circ s(b_{m-1})\big) \otimes s(b_m) \end{aligned}$$

Since, n_i are dimensions of b_i [10].

The (co)homology of $\mathcal{B}$ as the (co)homology of B-construction $B\mathcal{B}$ over $\mathcal{B}$ is denoted by $\mathcal{H}_*(\mathcal{B})\big(\mathcal{H}^*(\mathcal{B})\big)$, then $\mathcal{H}_*(\mathcal{B}) = \mathcal{H}_*(B\mathcal{B})\big(\mathcal{H}^*(\mathcal{B}) = \mathcal{H}^*(B\mathcal{B})\big)$ [17].

Consider algebra $\mathcal{B}$ with unity e, then $B\mathcal{B}$ is contractible and complex with contracting homotopy $h: B\mathcal{B} \to B\mathcal{B}$ which is defined as $h[a_1, \dots, a_n] = [e, a_1, \dots, a_n]$. This means that the (co)homology of $\mathcal{B}$ is trivial.

Let $\mathcal{B}$ be a unity algebra where the factorization algebra $\mathcal{B}/\mathbb{C}$. The (co)homology of $\mathcal{B}$ is the (co)homology of $\mathcal{B}/\mathbb{C}$.

An example of a homologically trivial Banach algebra is the algebra $L_1(G)$ in locally compact amenable group G. The algebra $\mathcal{K}(H)$ of compact operators is in separable since H is Hilbert space [5].

We now go on to introduce some examples of the topology spaces which have non-trivial homology.

Example (5):

Consider $\mathcal{B} = \mathbb{C}$ has trivial multiplication since $(B\mathcal{B})_n \cong \mathbb{C}$ and the identity differential is equal to zero. From [6], the homology of $\mathbb{C}$ for all $n \geq 1$ are isomorphic to $\mathbb{C}$.

Example (6):

If $\mathcal{B} = l_1$ is the convergent series Banach algebra, then the one-dimensional homology of $\mathcal{B}$ is isomorphic to $\mathbb{C}$.

Example (7):

If $\mathcal{B} = l_1^m = l_1/l_1^{(m)}$. By [17], the one-dimensional homology of $\mathcal{B}$ is amorphic to $\mathbb{C}$.

Example (8):

If $\mathcal{B} = \mathcal{C}_{z_0}^{(m)}(\mathfrak{D})$ is the polynomial algebra of the analytical functions with domain $\mathfrak{D} \subset \mathbb{C}$ that vanishes at the z_0, then we get the one-dimensional homology $\mathcal{B}$ that is isomorphic to $\mathbb{C}^{\times m+1}$.

Note that space $Z^n(\mathcal{B}, \mathcal{B}^*)$ is a closed subspace of $\mathcal{C}^n(\mathcal{B}, \mathcal{M})$ but the subspace of $B^n(\mathcal{B}, \mathcal{B}^*)$ is not closed see, [12].

$\mathcal{C}^n(\mathcal{B}, \mathcal{B}^*) \approx \mathcal{C}^{n+1}(\mathcal{B}, \mathbb{C})$, where $\mathcal{C}^{n+1}(\mathcal{B}, \mathbb{C})$ is the space of the bounded $(n+1)$-linear forms in $\mathcal{B}$. The relation between $\mathcal{C}^n(\mathcal{B}, \mathcal{B}^*)$ and $\mathcal{C}^{n+1}(\mathcal{B}, C)$ is given by:

$$\begin{gathered} < a_0, \phi(a_1, \dots, a_n) > = \omega_\phi(a_0, a_1, \dots, a_n), \quad \phi \in \mathcal{C}^n(\mathcal{B}, \mathcal{B}^*), \quad \omega_\phi \in \mathcal{C}^{n+1}(\mathcal{B}, C), \\ < , >: \mathcal{B} \times \mathcal{B}^* \to \mathrm{C}, \quad < a, f > = f(a). \end{gathered} \tag{3}$$

Theorem (9):[13]

If $\mathcal{B}$ is a properly infinite von Neumann algebra and $\mathcal{B}^*$ is the dual of $\mathcal{B}$, then:

$$\mathcal{H}^n(\mathcal{B}, \mathcal{B}^*) = 0 \qquad \forall n \in \mathbb{N}.$$

Corollary (10):

Let $\mathcal{B}$ be the algebra without bounded traces or nuclear algebra, and $\mathcal{H}^n(\mathcal{B}, \mathcal{B}^*)$ is the Hochschild cohomology of $\mathcal{B}$ with coefficients in $\mathcal{B}^*$. Then:

$$\mathcal{H}^n(\mathcal{B}, \mathcal{B}^*) = 0 \qquad \forall n \in \mathbb{N}.$$

Theorem (11):

The cyclic cohomology of the norm continuous algebra $\mathcal{B}$ is vanishing since $\mathcal{B}$ is the algebra without bounded traces.

Proof:

Let $\mathcal{B}$ be the C^*-algebra without bounded traces, then $\mathcal{HC}^0(\mathcal{B}) = 0$. From theorem (9), for $n \in \mathbb{N}$, we get $\mathcal{H}^n(\mathcal{B}, \mathcal{B}^*) = 0$. From [17], if we need to get a long exact sequence, we will find that $\mathcal{HC}^1(\mathcal{B}) = 0$, $\mathcal{HC}^{n-1}(\mathcal{B}) = \mathcal{HC}^{n+1}(\mathcal{B})$. By this way we get the required result.

Corollary (12):

For the nuclear algebra $\mathcal{B}$, $\mathcal{HC}^n(\mathcal{B})$ are vanished if n is odd. If n is even, then $\mathcal{HC}^n(\mathcal{B})$ are isomorphic to the spaces of all bounded traces in $\mathcal{B}$.

Proof:

Since $\mathcal{B}$ is a nuclear C^*-algebra, then $\mathcal{H}^n(\mathcal{B}) = 0$. Then we have $\mathcal{HC}^n(\mathcal{B}) = 0$ if n odd and for even groups, $\mathcal{HC}^n(\mathcal{B}) \cong \mathcal{HC}^0(\mathcal{B})$ such that $\mathcal{HC}^0(\mathcal{B})$ is the space with bounded traces, see [6].

Now we give some results of this paper, which is a study of cases in which reflexive cohomology of algebras is vanish. We begin by recalling the cyclic cohomology of algebras and how it relates to reflexive cohomology of algebras by the long exact sequence.

2-**The reflexive and dihedral cohomology of algebras**

Let $\mathcal{B}$ be an involution C^*-algebra over a field $\mathbb{C}$. The map $\omega \in \mathcal{C}^{n+1}(\mathcal{B}, \mathbb{C})$ is known to be α -dihedral, fulfilling the following accompanying statements:

(i) $\omega(a_0, \ldots, a_{n-1}, a_n) = (-1)^n \omega(a_n, a_0, \ldots, a_{n-1})$, is a cyclic operator.

(ii) $\omega(a_0, \ldots, a_{n-1}, a_n) = (-1)^{\frac{n(n+1)}{2}} \omega(a_0^*, a_n^*, \ldots, a_1^*)$, is a reflexive.

Define a_i^* as the image of $a_i \in \mathcal{B}$, $i = 0,1, \ldots n$ under the involution operator $*: \mathcal{B} \to \mathcal{B}$. The map $f \in \mathcal{C}^n(\mathcal{B})$ is α-cochain dihedral if it coincides with its α -dihedral $\omega \in \mathcal{C}^{n+1}(\mathcal{B}, \mathbb{C})$. The space $\mathcal{CD}^n(\mathcal{B})$ is made up of dihedral n-cochains and is invariant with map $\delta^{n+1}: \mathcal{C}^{n+1}(\mathcal{B}, \mathbb{C}) \to \mathcal{C}^{n+2}(\mathcal{B})$.

Definition (13):

The n-dimensional dihedral cohomology of a C^*-algebra $\mathcal{B}$ is defined by ${}_{\alpha}\mathcal{HD}^n(\mathcal{B})$ where

$$ {}_{\alpha}\mathcal{HD}^n(\mathcal{B}) = {ZD^n(\mathcal{B})}\big/{BD^n(\mathcal{B})}, \qquad \alpha = \pm 1, \tag{4} $$

$ZD^n(\mathcal{B})$ is the dihedral n-cocycles and $BD^n(\mathcal{B})$ is the dihedral n-coboundaries, which are defined by the below formula :

$$ ZD^n(\mathcal{B}) = CD^n(\mathcal{B}) \cap Z^{n+1}(\mathcal{B}, \mathbb{C}), \tag{5} $$

$$ BD^n(\mathcal{B}) = CD^n(\mathcal{B}) \cap B^{n+1}(\mathcal{B}, \mathbb{C}) \tag{6} $$

The functional $\omega \in \mathcal{C}^{n+1}(\mathcal{B}, \mathbb{C})$ is called α-reflexive if it satisfies the condition (ii). We can similarly get the reflexive cohomology ${}_{\alpha}\mathcal{HR}^n(\mathcal{B})$, $\alpha = \pm 1$.

$$ {}_{\alpha}\mathcal{HR}^n(\mathcal{B}) = {Z\mathcal{R}^n(\mathcal{B})}\big/{B\mathcal{R}^n(\mathcal{B})}, \qquad \alpha = \pm 1, \text{ where} $$

$Z\mathcal{R}^n(\mathcal{B}) = C\mathcal{R}^n(\mathcal{B}) \cap Z^{n+1}(\mathcal{B}, \mathbb{C})$,

$B\mathcal{R}^n(\mathcal{B}) = C\mathcal{R}^n(\mathcal{B}) \cap B^{n+1}(\mathcal{B}, \mathbb{C})$.

Lemma (14):

For a topological algebra $\mathcal{B}$ where $\mathcal{X}$ is the $\mathcal{B}$-bimodule topological algebra with $\mathbb{K} \in B^n(\mathcal{B}, \mathcal{X})$, we have

$$ \sum (-1)^{\sigma} \mathbb{K}(a_{\sigma(1)}, \ldots, a_{\sigma(n)}) = 0 \tag{7} $$

Where $a_1, \ldots, a_n \in \mathcal{B}$, σ is the permutation with n ordered.

Proof:

Let $f \in \mathcal{C}^{n+1}(\mathcal{B}, \mathcal{X})$ be a cochain, that is $\mathbb{K} = \delta^{n-1} f$. From [15] and [16], we can show that $\sum (-1)^{\sigma} \delta^{n-1} f(a_{\sigma(1)}, \ldots, a_{\sigma(n)}) = 0$.

For a permutation σ and τ where

$$ \sigma(i+1) = \begin{cases} \tau(i), & i = 1,2,\ldots,n\text{-}1 \\ \tau(n) & i = 0 \end{cases} $$

the terms $a_{\sigma(1)} \cdot f(a_{\sigma(2)}, \ldots, a_{\sigma(n)})$ and $f\left(a_{\tau(1)}, \ldots, a_{\tau(n-1)}\right) \cdot a_{\tau(n)}$ are the approach and left-hand side of relation (7). This reality holds for the accompanying:

$$ f\left(x_{\sigma(1)}, \ldots, x_{\sigma(k)} x_{\sigma(k+1)}, \ldots x_{\sigma(n)}\right) = f(x_{\tau(1)}, \ldots, x_{\tau(k)} x_{\tau(k+1)}, \ldots x_{\tau(n)}) $$

For every $\mathbb{K} < n$. For more information, see [10] and [12].

Theorem (15):

For an algebra $\mathcal{B}$, the sequence which is related to the cyclic $\mathcal{HC}^n(\mathcal{B})$ and dihedral $\mathcal{HD}^n(\mathcal{B})$ cohomology of $\mathcal{B}$ is the long exact sequence:

$$ \cdots \to {}^{-}\mathcal{HD}^n(\mathcal{B}) \to \mathcal{HC}^n(\mathcal{B}) \to \mathcal{HD}^n(\mathcal{B}) \to {}^{-}\mathcal{HD}^n(\mathcal{B}) \to \cdots \tag{8} $$

Proof:

We can get the required from the short exact sequence:

$$ 0 \to Tot\, \mathcal{C}(\mathcal{B}) \to Tot\mathcal{D}(\mathcal{B}) \to Tot\, {}^{-}\mathcal{D}(\mathcal{B}) \to 0. $$

Theorem (16):

Let $\mathcal{B}$ be the commutative unital algebras with involution. If $\mathcal{B}$ has a condition $codim\, \mathcal{B}^n \geq n$, $n > 1$, then ${}_{\alpha}\mathcal{HD}^n(\mathcal{B}) \neq 0$, $\alpha = \pm 1$.

Proof:

Since $n \leq codim\, \mathcal{B}^2$, there exists a linear independent element $e_1, \ldots, e_n \in \frac{\mathcal{B}}{\mathcal{B}^2}$ defines the functional,

$$ \phi_i \in \mathcal{B}^* = Hom_A(\mathcal{B}, C), 1 \leq i \leq n+1 $$

Such that:

$$ \phi_i/\mathcal{B}^2 = 0, \phi_i(e_j) = \delta_{ij} = \begin{cases} 1, & \mathrm{i} \neq j \\ 0, & \mathrm{i} = j \end{cases}. $$

Consider the cochain $\oint \in \mathcal{C}^n(\mathcal{B},\mathcal{B}^*)$ as follows,

$$\oint(a_1,\dots,a_n) = j_1(a_1)\dots j_n(a_n)j_1, \qquad a_1,\dots a_n \in \mathcal{B}.$$

Clearly, $\delta \oint(a_1,\dots,a_n) = 0$ hence $\oint \in Z^n(\mathcal{B},\mathcal{B}^*)$ and from (7), it is equal ϕ_1, therefore the cocycles do not equal the coboundaries. Thus $\mathcal{H}^n(\mathcal{B},\mathcal{B}^*) \neq 0$.

Now consider the co-chain $\oint \in \mathcal{C}^n(\mathcal{B},\mathcal{B}^*)$ such that:

$$\oint(a_1^*,\dots,a_n^*) = j_2(a_1^*)\dots j_3(a_2^*)\dots j_{n+1}(a_n^*)j_1 + (-1)^n j_1(a_1^*)\dots j_n(a_1^*)j_{n+1} +$$
$$+\sum_{1=2}^{n}(-1)^{n_i}\; j_1(a_1^*)\dots j_2(a_{i+1}^*)\dots j_{n-i+1}(a_n^*)j_{n-i+3} + (a_1^*)\dots j_{n-1}(a_{1-1}^*)\dots j_{n-i+2}$$

An immediate count shows that the map is $\oint \in Z\mathcal{D}^n(\mathcal{B})$. Using (15) and relation (7), we get:

$$_{\alpha}\mathcal{H}\mathcal{D}^n(\mathcal{B}) \neq 0, \quad \alpha = \pm 1.$$

Corollary (17):

One of the reflexive cohomology is $_{\alpha}\mathcal{H}\mathcal{R}^n(\mathcal{B}) \neq 0,\ \alpha = \pm 1$.

Example (18):

Let $\mathcal{A}$ be a nuclear algebra, then $_{\alpha}\mathcal{H}\mathcal{D}\mathcal{A}^{2K}(A) \cong {}_{\alpha}\mathcal{A}^{tr}$. Where $_{\alpha}\mathcal{A}^{tr}$- is all bounded traces in $\mathcal{A}$,

$$\mathcal{A}^{tr} = \alpha\mathcal{A}^{tr}, \qquad \alpha \in \mathcal{A}, \alpha = (-1)^k, k > 0.$$

Theorem (19):

Let $\mathcal{X} = L(\mathbb{H})$ be the algebra of the bounded operators in Hilbert algebra $\mathbb{H}$. We then have the vanishing state of the reflexive cohomology in the Hilbert space as the form:

$$^{\alpha}\mathcal{H}\mathcal{R}^n(\mathcal{X}) = 0, \qquad n \geq 0,\ \alpha = \pm 1$$

Proof:

Let $\mathcal{X}$ be a C^*-algebra (that has no bounded traces). This means that $\mathcal{H}^0(\mathcal{X}) = \mathcal{H}\mathcal{C}^n(\mathcal{X}) = 0$. From [5] and [6], if we relate among the sequence:

$$\cdots \to {}^{-\alpha}\mathcal{H}\mathcal{D}^0(\mathcal{X}) \to {}^{-\alpha}\mathcal{H}\mathcal{R}^0(\mathcal{X}) \to \cdots \to {}^{\alpha}\mathcal{H}\mathcal{D}^{n-1}(\mathcal{X}) \to \qquad {}^{-\alpha}\mathcal{H}\mathcal{D}^{n+1}(\mathcal{X}) \to \cdots \tag{9}$$

This is in addition to two short exact sequences:

$$0 \to {}^{-\alpha}\mathcal{H}\mathcal{R}^n(\mathcal{X}) \to 0 \to {}^{\alpha}\mathcal{H}\mathcal{R}^n(\mathcal{X}) \to 0$$

Then we find that, $^{\alpha}\mathcal{H}\mathcal{D}^n(\mathcal{X}) = {}^{\alpha}\mathcal{H}\mathcal{R}^n(\mathcal{X}) = 0,\ \ n \geq 0, \ \ \alpha = \pm 1$. (see [9]).

Theorem (20):

Let $\mathcal{X}$ be an arbitrary stable or nuclear algebra. Then $^{\alpha}\mathcal{H}\mathcal{D}^n(\mathcal{X}) = {}^{\alpha}\mathcal{H}\mathcal{R}^n(\mathcal{X}) = 0,\ n \geq 0,\ \alpha = \pm 1$, n is odd.

Proof:

For the nuclear C^*-algebra $\mathcal{X}$, from theorem (11), we have $\mathcal{H}\mathcal{C}^n(\mathcal{X}) = 0$ if n is odd and it is isomorphic to $\mathcal{X}^{tr}$ if n is even. Using theorem (15) we have the reflexive cohomology of $\mathcal{X}$ as vanishing for all odd groups and the dihedral cohomology is $^{\alpha}\mathcal{H}R^n(\mathcal{X}) = {}_{\alpha}\mathcal{X}^{tr}, \quad \alpha = \pm 1$ as n is even.

Theorem (21):

Let $\mathcal{P}$ be the commutative unital algebra algebras with an involution and $codim\ \mathcal{P}^n \geq n,\ n > 1$, then $_{\alpha}\mathcal{H}R^n(\mathcal{P}) \neq 0,\ \alpha = \pm 1$.

Proof:

Same manner as in theorem (16) and using [13] and [15] obtain our proof.

V. CONCLUSION

We study some result regarding the vanishing cohomology theory of the topological spaces. The reflexive and dihedral cohomology groups of algebra are vanishing in some states as we proven. So, we have proved that there are instances of nontrivial (co)homology groups in the operator algebras.

ACKNOWLEDGMENT

The authors thank the referees for their assistance and suggestions with the main draught of the current work.

REFERENCES

[1] Connes A., Non-commutative differential geometry, Publ. Math. I.H.E.S., 62 (1985), 41-144. Zbl0592.46056MR87i:58162.
[2] Engel A., Wrong way maps in uniformly finite homology and homology of groups, J. Homotopy Relate Struct., 13(2), 423–441(2018).
[3] J. P. May, Simplicial Objects in Algebraic Topology, Chicago Lectures in Mathematics, University of Chicago Press,Chicago, IL, 1992.
[4] Hatcher A., Algebraic Topology, Cambridge University Press, Cambridge (2002).
[5] Alaa H. N. Treatment of Operator Algebras through Cohomology Theory, Inf. Sci. Lett. 9, No. 1, 1-5 (2020).http://dx.doi.org/10.18576/isl/090101.
[6] Helemskii A. Ya., The homology in Banach and topological algebras, Kluwer Acd. Press (1989).
[7] J.-L. Loday, Cyclic homology, Fundamental Principals of Mathematical Sciences, vol. 301, Springer-Verlag, Berlin, 1992.
[8] P. Goerss and R. Jardine, Simplicial Homotopy Theory, Modern Birkh¨auser Classics. Birkh¨auser Verlag, Basel, 2009.

[9] Tsygan B. L., Homology of matrix algebras over rings and Hochschild homology, Uspehi Mat. Nauk, 38 :2 (1983), 217-218; Russian Math. Surveys, 38 (1983), 198-199.
[10] Kassel C., L'homologie cyclique des algèbre senveloppantes, Invent. Math., 91 (1988), 221-251. Zbl0653.17007MR89e:17015.
[11] C. A. Weibel, An Introduction to Homological Algebra, Cambridge Studies in Advanced Mathematics **38**, Cambridge University Press, 1995.
[12] Quillen D., Super connection character forms and the Cayley transform, Topology, 27 (1988), 211-238. Zbl0671.57013MR89j:58134.
[13] Christensen E. and Sinclair A. M., "On the vanishing of $H^n(A, A^*)$ for certain c^*- algebras". Pasific J. of Math. 137, No. 1. (1989), 55-63.
[14] C. A. Weibel, An introduction to homological algebra, Cambridge Studies in Advanced Mathematics, Vol.38. Cambridge UniversityPress, 1994.
[15] Frederic G. and White M. C., Vanishing of the third simplicial Cohomology group $l^1(z_+)$, Trans. A.M.s., 353, No 5(2001),2003-2017.
[16] - J.-L. Loday, Cyclic homology, Fundamental Principals of Mathematical Sciences, vol. 301, Springer-Verlag, Berlin, 1992.
[17] Gouda Y. Gh., Reflexive and dihedral (co)homology of pre-additive category, Int. J. of Math, & Math. Sci 24:7 (2001), 429-438.

AUTHORS

First Author – S.Z. Rida, Mathematics Department, Facultyof Science, South Valley University, Qena, Egypt, E-mail: szagloul@yahoo.com
Second Author – Alaa Hassan Noreldeen, Mathematics Department, Facultyof Science, Aswan University, Aswan, Egypt, E-mail: ala2222000@yahoo.com.
Third Author – Faten. R. Karar, Mathematics Department, Facultyof Science, Aswan University, Aswan, Egypt, E-mail: fatenragab2020@yahoo.com.

Correspondence Author – Faten. R. Karar, Mathematics Department, Facultyof Science, Aswan University, Aswan, Egypt, E-mail: fatenragab2020@yahoo.com.