On the Domain of Four Dimensional Pascal Matrix in the Space l_q^2

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Abstract- We introduce the double sequence space p_q^2=P(l_q^2) as the domain of four dimensional Pascal matrix P in the space l_q^2 , for 1≤q<∞. Furthermore, we show that p_q^2 is a BK-space, Banach space, establish its Schauder basis, topological properties, isomorphism and some inclusions.

***Index Terms*-** 4-dimensional Pascal Matrix, Isomorphism and inclusions

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I. Basic Notations and Background

Let $\iota: \mathbb{N} \times \mathbb{N} \to \tau$ be a function, where τ may stand for any nonempty set and $\mathbb{N}$ a set of counting numbers. Then $(j,k) \to \iota(j,k) = x_{jk}$ can be termed to be a double sequence.

In the paper by Basar & Sever [2] the Banach space l_q^2 of double sequences corresponding to the well-known l_q of single sequences was introduced, its properties were studied, and its $\beta(v)$-dual determined and the established that the α- and δ-duals of the space coincide with the $\beta(v)$-dual; where $1 \leq q < \infty$ and $v \in \{p, bp, r\}$. If P denotes the Pascal mean (a four dimensional matrix), then $p_\infty^2, p_c^2, p_{bc}^2$ and p_0^2 are collections of all double sequences whose P-transforms are in the spaces l_∞^2, c^2, c_b^2 and c_0^2 respectively; where l_∞^2, c^2, c_b^2 and c_0^2 are the double spaces of bounded, convergent, both bounded and convergent and null sequences respectively in the Pringsheim's sense, see Moricz [3]. We introduce a new double sequence space p_q^2 of Pascal as the set of all double sequences whose P-transforms are in the space l_q^2. Next, fixing some notations is necessary.

Let ω^2 be a vector space of all real or complex valued double sequences for which coordinatewise addition and scalar multiplications are defined, see Moricz [3]. Further, a vector subspace of ω^2 is termed as a double sequence space, see Moricz [3]. The space l_∞^2 denotes the space of all bounded sequences with norm$\|x\|_\infty = \sup_{j,k\in\mathbb{N}} |x_{jk}| < \infty, N = \{1,2,3,\ldots\}$. If $x = x_{jk} \in \mathbb{C}$, then x is convergent to a number l in Pringsheim's sense if for every $\varepsilon > 0$, there exists a number $n_0 = n_0(\varepsilon) \in \mathbb{N}$ and $l \in \mathbb{C}$ such that $|x_{jk} - l| < \varepsilon \; \forall \, j,k \geq n_0$, and we write $P - \lim_{j,k\to\infty} x_{jk} = l$, $\mathbb{C}$ being the complex field, see Pringsheim [4]. c^2 is used to denote the space of all convergent double sequences in Pringsheim's sense, see Moricz [3]. In Moricz [3], it is pointed out that there are some double sequences in c^2 that are not in l_∞^2; for example, the double sequence $x = (x_{jk})$ defined by

$$x_{jk} := \begin{cases} n, & j = 0, k \in \mathbb{N}, \\ 0, & j \geq 1, k \in \mathbb{N}. \end{cases}$$

In Moricz [3], it can be seen that x is convergent in Pringsheim's sense but not bounded, since $p - \lim_{j,k\to\infty} x_{jk} = 0$ but $\|x\|_\infty = \infty$. Hence, we consider the space c_b^2 of all double sequences which are both convergent in Pringsheim's sense and bounded; that is, $c_b^2 = c^2 \cap l_\infty^2$, Moricz [3]. So, c_0^2 is the space of all double sequences converging to zero in Pringsheim's sense, and c_{0b}^2 is the space of all double sequences that are bounded and converging to zero in Pringsheim's sense, i.e., $c_{0b}^2 = c_0^2 \cap l_\infty^2$, Moricz [3]. Also, Basar and Sever [2] defined the space l_q^2 by

$$l_q^2 := \left\{ x = (x_{jk}) \in \omega^2 : \sum_{j,k=0,0}^{\infty,\infty} |x_{jk}|^q < \infty \right\}, (1 \leq q < \infty).$$

For $1 \leq q < \infty$, Basar and Sever [2] proved that the double sequence space l_q^2, corresponds to l_q of single sequences, and is a Banach space with the norm

$$\|x\|_{l_q^2} := \left(\sum_{j,k=0,0}^{\infty,\infty} |x_{jk}|^q \right)^{\frac{1}{q}},$$

$$1 \leq q < \infty \qquad (1)$$

In Basar and Sever [2], in case $q \in (0,1)$ in (1), then the space l_q^2 is a q-norm with the q-norm

$$|||x||| := \|x\|_{l_q^2}^q = \sum_{j,k=0,0}^{\infty,\infty} |x_{jk}|^q.$$

Let X and Y be two double sequence spaces and $A = (a_{mnjk})$ be any four-dimensional infinite matrix of complex numbers as in Robison [16]. In Robison [16], it was also indicated that A is said defines a matrix mapping from X into Y and write $A: X \to Y$ for every $x = (x_{jk}) \in X$, so that the $A-$transform of $x = (x_{jk})$ is $Ax = \{(Ax)_{mn}\}_{mn}$, where

$$(Ax)_{mn} = P - \sum_{j,k}^{m,n} a_{mnjk} x_{jk} \qquad (2)$$

for each $m, n \in \mathbb{N}$, exists. For matrix domains, Yesilkayagil & Basar [10], the $v-$matrix domain $\chi_A^{(v)}$ of A in X is defined by

$$\chi_A^{(v)} = \left\{ x = (x_{jk}) \in \omega^2 : P - \sum_{j,k}^{m,n} a_{mnjk} x_{jk} \text{ exists and is in } Y \right\}.$$

Clearly, (2) suggests that A maps X into Y if $X \subset Y_A^{(v)}$; and $(X:Y)$ can denote the set of all four-dimensional matrices transforming X into Y, see Yesilkayagil & Basar [10]. $A = (a_{mnjk}) \in (X:Y)$ if, and only if the double series on the right of (1) converges in Pringsheim's sense for each $m, n \in \mathbb{N}$, that is, $A_{mn} \in \chi^{\beta(v)}$ for all $j, k \in \mathbb{N}$ and any $x \in X$ have $Ax \in Y$, see Yesilkayagil & Basar [10]. It is well known, for example in Cooke [15], that $A = (a_{mnjk})$ is a triangular matrix if $a_{mnjk} = 0$ for $j > m, k > m$ or both, and $a_{mnjk} \neq 0$ for all $m, n \in \mathbb{N}$ and every triangular matrix has a unique inverse which also happens to be a triangular matrix too.

II. The Pascal Double Sequence Space p_q^2

Pascal matrix of finite order existed for a very long time as pointed out by Aggarwala & Lamoureux [5], where the authors declared that there was no reason, whatsoever, to stop at a finite matrix of this type for, one can extend the Pascal matrix of finite order to an infinite lower triangular matrix. We felt that this extension aroused Polat [1] to introduce some Pascal sequence spaces, each which is a matrix domain via infinite Pascal matrix as follows:

$$(l_\infty)_P = p_\infty = \left\{ x = (x_k) \in \omega : \sup_n \left| \sum_k^n \binom{n}{n-k} x_k \right| < \infty \right\}$$

$$(c)_P = p_c = \left\{ x = (x_k) \in \omega : \lim_{n\to\infty} \sum_k^n \binom{n}{n-k} x_k \text{ exists} \right\}$$

$$(c_0)_P = p_0 = \left\{ x = (x_k) \in \omega : \lim_{n\to\infty} \sum_k^n \binom{n}{n-k} x_k = 0 \right\}.$$

Recently, Kiltho, et al. [17] introduced Pascal double sequence spaces, $p_\infty^2, p_c^2, p_{bc}^2$ and p_0^2 as matrix domains of four-dimensional Pascal matrix, as an extension of the work of Polat [1]. This paper will therefore wish to introduce the Pascal double sequence space p_q^2, the set of all double sequences whose P-transforms are in the space l_q^2. We define the four-dimensional Pascal matrix $P = (p_{mn}^{jk})$ as follows:

$$p_{mn}^{jk} = \begin{cases} \binom{m}{m-j}\binom{n}{n-k}, & 0 \le j \le m, 0 \le k \le n \\ 0 & , j > m \text{ and } k > n. \end{cases} \quad (4)$$

with inverse $P^{-1} = Q = (q_{mn}^{jk})$ defined by

$$q_{mn}^{jk} = \begin{cases} (-1)^{(m-j)+(n-k)} \binom{m}{m-j}\binom{n}{n-k}, & 0 \le j \le m, 0 \le k \le n \\ 0 & , j > m \text{ and } k > n. \end{cases} \quad (5)$$

Now, we introduce the space p_q^2 as the collection of all double sequences such that its P-transform is in the space l_q^2, as follows

$$p_q^2 = \left\{ x = (x_{jk}) \in \omega^2 : \sum_{m,n} \left| \sum_{j,k=0}^{m,n} \binom{m}{m-j}\binom{n}{n-k} x_{jk} \right|^q < \infty \right\} \quad (6)$$

Following Yesilkayagil & Basar [10] the space p_q^2 is linear with coordinatewise addition and scalar multiplication, where we are going to show that it is a complete q-normed space with the q-norm:

$$\|x\|_{p_q^2} = \left(\sum_{m,n} \left| \sum_{j,k=0}^{m,n} \binom{m}{m-j}\binom{n}{n-k} x_{jk} \right|^q \right)^{\frac{1}{q}}. \quad (7)$$

Let $\chi_A = \{x = (x_{jk}) \in \omega^2 : Ax \in X\}$ be a matrix domain of a four-dimensional matrix A, then the Pascal sequence space in (6) is also a matrix domain, as $p_q^2 = (l_q^2)_P = P(l_q^2)$; while P-transform of a double sequence space $x = (x_{jk})$ in (6) can be defined as

$$y_{mn} = (Px)_{mn} = \sum_{j,k=0}^{m,n} \binom{m}{m-j}\binom{n}{n-k} x_{jk} \quad (8)$$

for all $m, n \in \mathbb{N}$. The terms of the double series $x = (x_{mn})$ and $y = (y_{mn})$ are assumed to be connected with the relation (8).

Definition 1: A Banach space is called a BK-space provided each of the maps $p_{jk}: X \to \mathbb{C}$ defined by $p_{jk} = x_{jk}$ is continuous for all $j, k \in \mathbb{N}$, see Choudhary & Nanda [12]

Lemma 1: If A is a triangle and X is a BK-space, then X_A is a BK-space with the norm given by $\|x\|_{X_A} = \|x\|_X$ for all $x \in X_A$, see Boos [11].

By considering the notion of BK-space, one can say that the sequence space l_q^2 is a BK-space according to its l_q^2-normed defined by $\|x\|_{l_q^2} = \left(\sum_{m,n=0,0}^{\infty,\infty} |x_{jk}|^q\right)^{\frac{1}{q}}$, where $1 \le q < \infty$.

Next, we present our results viz:

Theorem 1: The Pascal sequence p_q^2 is a BK-space according to the norm defined by

$$\|x\|_{p_q^2} = \|Px\|_{l_q^2} = \left(\sum_{m,n} \left| \sum_{j,k=0}^{m,n} \binom{m}{m-j}\binom{n}{n-k} x_{jk} \right|^q \right)^{\frac{1}{q}},$$ where $1 \le q < \infty$ as in (7).

Proof: Since l_q^2 is a BK space, we define a map $A_P: p_q^2 \to l_q^2$ by $A_P(x) = P(x) \; \forall \, x \in p_q^2$. Since P is a triangular matrix, then A_P is linear, 0ne-to-one and onto. If A_P^{-1} is the inverse of A_P, it is also linear, one-to-one and onto, so that $p_q^2 = A_P^{-1}(l_q^2)$ is a Banach space. It remains to show the coordinates are continuous in l_q^2. To do this, let $x_{jk} \to x$ in p_q^2. Then $y_{jk}^{[r,s]} = P(x^{[r,s]}) \Longrightarrow y_{jk} = P(x)$, since l_q^2 is a BK space. Let $P^{-1} = Q$ be the inverse of P, which is also a triangle. Then $x_{mn}^{[r,s]} = \sum_{j,k=0}^{m,n} Q y_{mn}^{[r,s]} \to$

$\sum_{j,k=0}^{m,n} Qy_{mn} = x$. This shows that the coordinates are continuous on p_q^2. Hence p_q^2 is a BK space.

Theorem 2: The set p_q^2 becomes a linear space with the coordinatewise addition and scalar multiplication and the following statements hold:

i) If $q \in (0,1)$, then p_q^2 is a complete q-normed space with

$$\|x\|_{p_q^2} = \sum_{m,n} \left| \sum_{j,k=0}^{m,n} \binom{m}{m-j}\binom{n}{n-k} x_{jk} \right|^q$$

which is q-norm isomorphic to the space l_q^2.

ii) If $q \in [1,\infty)$q∈1, ∞ then p_q^2 is a Banach space with

$$\|x\|_{p_q^2} = \left(\sum_{m,n} \left| \sum_{j,k=0}^{m,n} \binom{m}{m-j}\binom{n}{n-k} x_{jk} \right|^q \right)^{\frac{1}{q}}$$

which is isomorphic to the space l_q^2.

Proof: We are going to give the proof of the second part (ii), since the first part (i) can be proved in a similar way.

The first part of the theorem is a routine verification, where it can be easily seen that (a) p_q^2 is not empty; (b) the sum of any two elements in p_q^2 is also in p_q^2; and (c) the scalar multiplication $\alpha x \in p_q^2 \ \forall \alpha \in \mathbb{C}$ and $x \in p_q^2$. Thus, p_q^2 is a linear space with coordinatewise addition and scalar multiplication. Now, we can show that p_q^2 is a Banach space with the norm defined by (7). Let $(x^\alpha)_{\alpha\in\mathbb{N}}$ be any Cauchy sequence in the space p_q^2, where $x^\alpha = \{x_{jk}{}^{(\alpha)}\}_{j,k=0}^{\infty}$ for every fixed $\alpha \in \mathbb{N}$. Then for a given $\varepsilon > 0$, there exists a positive integer $N = N(\varepsilon)$ such that

$$\|x^\alpha - x^\beta\|_{p_q^2} = \left(\sum_{m,n} \left| \sum_{j,k=0}^{m,n} \binom{m}{m-j}\binom{n}{n-k} (x_{jk}^\alpha - x_{jk}^\beta) \right|^q \right)^{\frac{1}{q}} < \varepsilon \quad \forall \alpha, \beta > N$$

which yields for each $m, n \in \mathbb{N}$ and applying Minkowki's inequality, that

$$\|x^\alpha - x^\beta\|_{p_q^2} = \left(\sum_{m,n} \left| \sum_{j,k=0}^{m,n} \binom{m}{m-j}\binom{n}{n-k} x_{jk}^\alpha - \sum_{j,k=0}^{m,n} \binom{m}{m-j}\binom{n}{n-k} x_{jk}^\beta \right|^q \right)^{\frac{1}{q}}$$

$$\leq \left| \left(\sum_{m,n} \left| \sum_{j,k=0}^{m,n} \binom{m}{m-j}\binom{n}{n-k} x_{jk}^\alpha \right|^q \right)^{\frac{1}{q}} - \left(\left| \sum_{j,k=0}^{m,n} \binom{m}{m-j}\binom{n}{n-k} x_{jk}^\beta \right|^q \right)^{\frac{1}{q}} \right| < \varepsilon$$

This means that $\left(\sum_{m,n} \left|\sum_{j,k=0}^{m,n} \binom{m}{m-j}\binom{n}{n-k} x_{jk}^\alpha\right|^q\right)^{\frac{1}{q}}_{\alpha\in\mathbb{N}}$ is a Cauchy sequence with complex terms for every fixed $m, n \in \mathbb{N}$n. Since $\mathbb{C}$ is complete, it converges, i.e.

$$\left(\sum_{m,n} \left| \sum_{j,k=0}^{m,n} \binom{m}{m-j}\binom{n}{n-k} x_{jk}^\alpha \right|^q \right)^{\frac{1}{q}} \to \left(\sum_{m,n} \left| \sum_{j,k=0}^{m,n} \binom{m}{m-j}\binom{n}{n-k} x_{jk} \right|^q \right)^{\frac{1}{q}} \quad \text{as } \alpha \to \infty, \tag{9}$$

such that

$$\lim_{\alpha\to\infty} \|x^\alpha - x^\beta\|_{p_q^2} = 0.$$

Since, $\left(\sum_{m,n} \left|\sum_{j,k=0}^{m,n} \binom{m}{m-j}\binom{n}{n-k} x_{jk}^\alpha\right|^q\right)^{\frac{1}{q}}_{m,n\in\mathbb{N}} \in p_q^2$ for each fixed $\alpha \in \mathbb{N}$, there exists a positive real number K_α such that

$$\left(\sum_{m,n} \left| \sum_{j,k=0}^{m,n} \binom{m}{m-j}\binom{n}{n-k} x_{jk}^\alpha \right|^q \right)^{\frac{1}{q}} \leq K_\alpha.$$

Therefore, taking summation over m, n in the following relation

$$\left| \sum_{j,k=0}^{m,n} \binom{m}{m-j}\binom{n}{n-k} x_{jk} \right|^{\frac{1}{q}}$$

$$= \left| \sum_{j,k=0}^{m,n} \binom{m}{m-j}\binom{n}{n-k} x_{jk} - \sum_{j,k=0}^{m,n} \binom{m}{m-j}\binom{n}{n-k} x_{jk}^\alpha + \sum_{j,k=0}^{m,n} \binom{m}{m-j}\binom{n}{n-k} x_{jk}^\alpha \right|^{\frac{1}{q}}$$

$$\leq \left| \sum_{j,k=0}^{m,n} \binom{m}{m-j}\binom{n}{n-k} x_{jk} - \sum_{j,k=0}^{m,n} \binom{m}{m-j}\binom{n}{n-k} x_{jk}^\alpha \right|^{\frac{1}{q}} + \left| \sum_{j,k=0}^{m,n} \binom{m}{m-j}\binom{n}{n-k} x_{jk}^\alpha \right|^{\frac{1}{q}}$$

$$\leq \varepsilon + K_\alpha.$$

This shows that $x = (x_{jk}) \in p_q^2$. Since $\{x^\alpha\}_{\alpha\in\mathbb{N}}$ is an arbitrary Cauchy sequence, then the space p_q^2 is complete. Thus, p_q^2 is a Banach space with the norm $\|x\|_{p_q^2} = \left(\sum_{m,n} |(Px)_{mn}|^q\right)^{\frac{1}{q}}$.

To prove the fact that p_q^2 is linearly isomorphic to l_q^2, we have to show the existence of a linear bijection between the spaces p_q^2 and l_q^2. Consider the transformation τ defined from p_q^2 to l_q^2 by $x \mapsto y = \tau x = \{(Px)_{mn}\}$y =τx. Clearly, τ is linear, $\tau(u) + \tau(v) = \tau(u + v)$τu+τv = τ(u + v) for all $u = (u_{jk}), v = (v_{jk}) \in p_q^2$; and $K \cdot \tau(x) = \tau(Kx)$ for all $K \in \mathbb{C}, x = (x_{jk}) \in p_q^2$. Further, we can see that $x = \theta$, whenever $\tau x = \theta$ which shows that τ is injective. Now, let $y = (y_{jk}) \in l_q^2$ and define a sequence $x = (x_{jk})$ via y by

$$x_{jk} = \sum_{u,v=0}^{j,k} (-1)^{(j-u)+(k-v)} \binom{j}{j-u}\binom{k}{k-v} y_{uv} \quad \forall u, v \in \mathbb{N}.$$

Hence, by taking into account the hypothesis $y \in l_q^2$, one can derive by taking summation over $m,n \in \mathbb{N}$ on the following equality

$$|(Px)_{mn}| = \left|\sum_{j,k=0}^{m,n}\binom{m}{m-j}\binom{n}{n-k}\sum_{u,v=0}^{j,k}(-1)^{(j-u)+(k-v)}\binom{j}{j-u}\binom{k}{k-v}y_{uv}\right| = |y_{mn}|.$$

That is, $\|Px\|_q = \|y\|_q$, which implies that $x \in p_q^2$. Therefore, τ is surjective. Hence, $p_q^2 \cong l_q^2$.

Definition 2: A barreled space is a topological vector space for which every barreled set in the space is a neighbourhood for the zero vector. A barreled set in a topological vector space is a set that is convex, balanced, absorbing, and closed, see [Narici & Beckenstein [13]

Lemma 2: If the sequence space X is a Banach space or a Frechet space, then it is a barreled space, see Schaefer [14].

Theorem 3: The double sequence space l_q^2 is a barreled space for $1 \le q < \infty$.

Proof: We have seen that l_q^2 is a Banach space in Theorem 2. Thus, the proof of Theorem 3 is obvious by Theorem 2 and Lemma 2.

Definition 3:The space X of double sequence spaces is monotone if $xu = (x_{jk}u_{jk}) \in X$ fore very $x = (x_{jk}) \in X$ and $u = (u_{jk}) \in \chi^2$, where $\chi^2 = \{0,1\}^{\mathbb{N}\times\mathbb{N}}$ denotes the double sequence space of 0s and 1s, see Yesilkayagil & Basar [6].

Theorem 4: The space l_q^2 is monotone for all $1 \le q < \infty$.

Proof: Let $1 \le q < \infty$, $x = (x_{jk}) \in l_q^2$ and $u = (u_{jk}) \in \chi^2$. Then we have $|x_{jk}u_{jk}|^q = |x_{jk}|^q|u_{jk}|^q \le |x_{jk}|^q$ for each $j,k \in \mathbb{N}$. This simply means that the inequality $\sum_{j,k}|x_{jk}u_{jk}|^q \le \sum_{j,k}|x_{jk}|^q$ holds. That is, $xu = (x_{jk}u_{jk}) \in l_q^2$. This shows that l_q^2 is monotone for all $1 \le q < \infty$.

Definition 4A double sequence $(x_{jk})_{j,k=0}^{\infty}$ in a double sequence space X is called a Schauder double basis if, for every $x \in X$, there exists a unique double sequence of scalars $(\lambda_{jk})_{j,k=0}^{\infty}$ such that $x = \sum_{j,k=0}^{\infty}\lambda_{jk}x_{jk}$, see Loganathan & Moorthy [8].

Theorem 5: Let $m,n,j,k \in \mathbb{N}$ and define $q^{(jk)} = \{q^{(jk)}\}_{mn}$ by

$$q_{mn}^{jk} = \begin{cases} 0 & ,j > m \text{ and } k > n \text{ or both,} \\ (-1)^{(m-j)+(n-k)}\binom{m}{m-j}\binom{n}{n-k} & ,0 \le k \le n \text{ and } 0 \le j \le m. \end{cases}$$

Then the set $\{q^{(jk)}\}$ is a double basis for the double sequence space p_q^2 such that any $x \in p_q^2$ has a unique representation of the form

$$x = \sum_{jk}\zeta_{jk}q^{(jk)}; \qquad (10)$$

where $\zeta_{jk} = (Px)_{jk}\ \forall\ j,k \in \mathbb{N}$.

Proof:

a) We want to show that $\{q^{(jk)}\} \subset p_q^2$. Since $Pq^{jk} = e^{jk} \in l_q^2$ for $j,k = 0,1,2,$ ...k=0,1,2,... e^{jk} is a double sequence whose non-zero term is 1 in the $(j,k)^{th}$ place for each j,kk Now, let $x \in p_q^2$. For every r and s, we write $x^{[r,s]} = \sum_{j,k=0}^{r,s}\zeta_{jk}Pq^{(jk)}$

P is continuous. So, we can apply P to (10) to have

$$x^{[r,s]} = \sum_{j,k=0}^{r,s}\zeta_{jk}Pq^{(jk)} = x^{[r,s]} = \sum_{j,k=0}^{r,s}(Px)_{jk}e^{jk} \qquad (12)$$

and $\{P(x - x^{[r,s]})\}_{it} = \begin{cases} 0, 0 \le i \le r \,\&\, 0 \le t \le s \\ (Px)_{it} \quad ,i > r \,\&\, t > s \end{cases}$. Let $\varepsilon > 0$ be given. Then there exist r_0 and s_0 such that $|(Px)_{r,s}| < \frac{\varepsilon}{2}\ \forall r > r_0$ and $s > s_0$. Therefore,

$$\|x - x^{[r,s]}\|_{p_q^2} = \left(\sum_{i,t>r,s}|(Px)_{jk}|^q\right)^{\frac{1}{q}} \le \left(\sum_{m,n>r_0,s_0}|(Px)_{jk}|^q\right)^{\frac{1}{q}} \le \frac{\varepsilon}{2} < \varepsilon$$

$\forall r > r_0$ and $s > s_0$. Clearly, this shows that $x \in p_q^2$ as in (10).

Next, we show the uniqueness of the representation of x in (10). For this, suppose on the contrary that there exists another representation of $x = \sum_{jk}\xi_{jk}q^{(jk)}$. The linear transformation $P: p_q^2 \to l_q^2$ is continuous. It implies that we can have

$$(P(x))_{mn} = \sum_{j,k}\xi_{jk}(Pq^{(jk)})_{mn} = \sum_{j,k}\xi_{jk}e^{jk} = \xi_{jk} \qquad (j,k \in \mathbb{N}),$$

which is a clear contradiction to the fact that $(P(x))_{mn} = \zeta_{jk}$. So, the representation (10) is unique. This completes the proof.

Theorem 6: The inclusion $l_q^2 \subset p_q^2$ holds for $1 \le q < \infty$.

Proof: Let us take arbitrary $x = (x_{jk}) \in l_q^2$. Then there exists a positive real number KKsuch that $\sum_{j,k}|x_{jk}|^q \le K$. Applying Holder's inequality to (8), we have

$$|y_{mn}|^q = \left|\sum_{j,k=0}^{m,n}\binom{m}{m-j}\binom{n}{n-k}x_{jk}\right|^q \le \left(\sum_{j,k=0}^{m,n}\binom{m}{m-j}\binom{n}{n-k}|x_{jk}|^q\right) \times \left(\sum_{j,k=0}^{m,n}\binom{m}{m-j}\binom{n}{n-k}\right)^{q-1} \le \sum_{j,k=0}^{m,n}\binom{m}{m-j}\binom{n}{n-k}|x_{jk}|^q$$

This implies that

$$\sum_{m,n=0,0}^{\infty,\infty}|y_{mn}|^q \le \sum_{m,n=0,0}^{\infty,\infty}\left(\sum_{j,k=0}^{m,n}\binom{m}{m-j}\binom{n}{n-k}|x_{jk}|^q\right)$$

$$= \sum_{j,k=0}^{m,n} |x_{jk}|^q \left(\sum_{m,n=0,0}^{\infty,\infty} \binom{m}{m-j} \binom{n}{n-k} \right)$$

$$\sum_{j,k=0}^{m,n} |x_{jk}|^q \left(\sum_{m=0}^{\infty} \binom{m}{m-j} \sum_{n=0}^{\infty} \binom{n}{n-k} \right) \leq K$$

That shows that $x = (x_{jk}) \in p_q^2$ for all $1 \leq q < \infty$<∞ Hence, $l_q^2 \subset p_q^2$ holds for $1 \leq q < \infty$.

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