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Uniformly Rotund in Every Direction (URED) Norm

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Abstract

The basic ideas of URED norm is given.We discuss questions of renorming spaces so as to be URED. We examine the relationship of this concept with other rotundities and normal structure and it is applied to fixed point property of a non expansive mapping.

Key words:Renorming,Normal structure,Fixed point Property,Rotundity

1 Introduction

The notion of uniformly rotund in every direction (URED) norm was first used by A. L. Garkavi in 1962 [2]. He proved for any closed convex set C of URED space there is at most one $x \in C$ such that

$$\sup_{y\in C} \|x-y\| = \inf_{z\in C} \sup_{y\in C} \|z-y\|,$$

i.e.for the purpose of characterizing those spaces for which every bounded set has at most one Cebysev center. This class goes by the name of spaces uniformly convex in every direction. It is also used by him in connection with the uniqueness of the solution of an approximation problem.Their structure has been studied by M.M Day ,R.C James and S. Swaminathan [5,7] and showed that for any uncountable set $\Gamma, C_0(\Gamma)$ does not have an equivalent URED. They also showed in [5] that the closed bounded convex subsets of URED spaces have normal structure. They also discussed renorming phenomena related to the concept. Every separable B-space admits an equivalent U.C.E.D. norm [5]. The concept of uniform convexity in a normed linear space is based on the geometric condition that if two members of the unit ball are far apart, then their midpoint is well inside the unit ball. We consider here a generalization of this concept whose geometric significance is that the collection of all chords of the unit ball that are parallel to a fixed direction and whose lengths are bounded below by a positive number has the property that the midpoints of the chords lie uniformly deep inside the unit ball[5]. D.N .Kutzarova and S.L.Troyanski[8] proved that for any probability space

$(\Omega, \mu), L_1(\mu)$ admits locally uniform rotund (LUR) and URED. M.A . Smith and B.Turret proved that for any $1 < p < \infty$ the normed space X is URED implies the space $L_p(\mu, X)$[9]. Smith gave an example of a Banach space that is URED but not uniformly rotund in weakly compact (URWC) sets and another Banach space that is rotund(R) but not URED both examples are formed by equivalently renorming of the l_2 space[3].

2 Some Definitions and Notations

Definition. 2.1. Let X be a normed space. Given $z \in S(X)$, it is said that X is uniformly rotund in the direction $z(UR_z)$ if for every $\epsilon > 0$ there exists $\delta > 0$ such that if $x, y \in S(X), x - y \in span(z)$ and $\|x - y\| \geq \epsilon$, then $\|\frac{x+y}{2}\| < 1 - \delta. X$ is uniformly rotund in every direction (URED) if it is URz for every $z \in S(X)$.

Definition. 2.2. The norm $\|\cdot\|$ of a Banach space X is said to be uniformly rotund in every direction or uniformly convex in every direction or directionally uniformly rotund if

$$\lim_n \|x_n - y_n\| = 0 \quad \text{whenever} \quad x_n, y_n \in S(X) \quad \text{are such that} \quad \lim_n \|x_n + y_n\| = 2$$

and there is a $z \in X$ and real numbers α_n such that

$$x_n - y_n = \alpha_n z \text{ for all} \quad n \in \mathbb{N}, \text{ then we have} \quad \lim_n \alpha_n = 0$$

Equivalently, The norm $\|\cdot\|$ of a Banach space X is said to be uniformly rotund in every direction (URED for short) if $\lim_n \|x_n - y_n\| = 0$ whenever $x_n, y_n \in S(X)$ are such that

$$\lim_n \|x_n + y_n\| = 2 \quad \text{and there is a } z \in X \text{such that} \quad x_n - y_n \in \text{span z for all} \quad n.$$

Remark. We also define a norm $\|\cdot\|$ a Banach space X is URED if, for any $x, x_n \in X$ with $\|x_n\| \to 1, \|x_n + x\| \to 1$ and $\|2x_n + x\| \to 2 \Rightarrow x = 0$.

If Γ is an uncountable set .Then $C_0(\Gamma)$ has no equivalent URED norm ([1] 1, p.74).

Definition. 2.3. Renorming of Banach space consists of replacing the given norm, which is usually provided by the very definition of the space, by another norm which may have better (or sometimes worse) geometric properties of Convexity or smoothness, or both[1].

Definition. 2.4. A Banach space X is said to have fixed point property if any non expansive mapping $T : C \to C i.e. \|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$ on a weakly compact convex subset of X there is $x \in C$ such that $Tx = x$.

Theorem 2.1. *[1]. The norm $\|\cdot\|$ on a Banach space X is uniformly rotund in every direction if and only if whenever*

$$x_n, y_n \in X, n = 1, 2, \cdots \text{ are such that } \lim_n (2\|x_n\|^2 + 2\|y_n\|^2 - \|x_n + y_n\|^2) = 0 \quad \text{and}$$

$\{x_n\}$ is bounded and there is a point $z \in X - \{0\}$ and real numbers $\alpha_n, n = 1, 2, \cdots$ which satisfy

$$x_n - y_n = \alpha_n z \text{ for all } n, \text{ then } \lim_n \alpha_n = 0.$$

To prove this theorem we need the following lemmas.

Lemma 2.2. *[10 , p.271,ex 8.44]. let* $x_n, y_n \in S(X)$, *and* $z_n \in [x_n, y_n], n = 1, 2, \cdots$ *be such that* $\lim_n \|z_n\| = 1$. *and*

$$\inf\{\min \|x_n - z_n\|, \|y_n - z_n\|\} > 0. \tag{1}$$

Proof. Let $f_n \in S(X^*)$ be such that $f_n(1/2(x_n + y_n)) = 1/2\|x_n + y_n\| \to 1$. We observe that $f(x_n) \to 1$ and $f(x_n) \to 1$.

$$\text{Indeed, if} \quad f(x_n) \leq 1 - \delta \quad \text{for all n ,then} f(x_n) > 1 + \delta/2 \quad \text{for large m .}$$

Contradicting $\|f_n\| = 1, \|y_m\| = 1$. So $f(x_n) \to 1, f(y_n) \to 1$. Then also $f(z_n) \to 1$ and hence

$$\begin{aligned}
f_n(z_n) &\leq \|f_n\|\|z_n\| \\
&\leq 1.\|z_n\| \\
&\Rightarrow 1 \geq \|z_n\| \geq f_n(z_n) \to 1 \\
&\Rightarrow 1 \geq \|z_n\| \geq 1 \quad \text{as} \quad n \to \infty \\
&\Rightarrow \lim \|z_n\| = 1 \\
\text{Proof of second part} \ &: \\
\lim \|z_n\| = 1 &\Rightarrow \ \forall\, \varepsilon > 0\ \exists\, N : \ \forall\, n \geq N, |\|z_n\| - 1| < \varepsilon \\
&\Rightarrow 1 - \varepsilon < \|z_n\| < 1 + \varepsilon \\
&\Rightarrow \|z_n\| > 1 - \varepsilon \quad \text{, using this value as below} \\
\inf\{\min \|x_n - z_n\|, \|y_n - z_n\|\} &\geq \inf\{\min(\|x_n\| - \|z_n\|, \|y_n\| - \|z_n\|)\}. \\
&> \inf\{\min(1 - 1 + \varepsilon, 1 - 1 + \varepsilon\}. \\
&= \inf\{\min(\varepsilon, \varepsilon\}. \\
&= \varepsilon > 0.
\end{aligned}$$

□

Lemma 2.3. *[1]. Let* $\|\cdot\|$ *be a norm on a Banach space* X *and let* $x_n, y_n \in S(X)$. *and* $z_n \in [x_n, y_n], n = 1, 2, \cdots$ *be such that* $\lim_n \|z_n\| = 1$ *and*

$$\inf\{\min \|x_n - z_n\|, \|y_n - z_n\|\} > 0 \tag{2}$$

Then

$$\lim_n (\min_{0 \leq t \leq 1} \|tx_n + (1 - t)y_n\|) = 1.$$

Proof. Let $z_n = t_n x_n + (1 - t_n)y_n, t_n \in [0, 1]$. By using (1), choose $\delta > 0$ such that for all $n\delta < t_n < 1 - \delta$. For every n, choose $f_n \in S(X^*)$ such that

$$f_n(z_n) = \|z_n\|.$$

Then $\forall\, n$,

$$\|z_n\| = f_n(z_n) = t_n f_n(x_n) + (1 - t_n) f_n(y_n) \tag{3}$$

Since $\lim_n \|z_n\| = 1, 0 \leq t_n \leq 1, f_n(x_n) \leq 1$ and $f_n(y_n) \leq 1$ for all n from (2) it follows that

$$\begin{aligned} 1 = \|z_n\| &= t_n f_n(x_n) + (1-t_n) f_n(y_n) \\ &\leq |t_n f_n(x_n) + (1-t_n) f_n(y_n)| \\ &\leq |t_n f_n(x_n)| + |(1-t_n) f_n(y_n)| \\ \Rightarrow 1 &\leq |f_n(x_n)| \quad \text{as} \quad \lim_n t_n = 1 \\ \therefore \lim_n f_n(x_n) \geq 1 \\ \text{Similarly}, \lim f_n(y_n) \geq 1 \quad \text{as} \quad \lim_n t_n = 0 \\ \therefore \lim_n f_n(x_n) = \lim_n f_n(y_n) = 1. \end{aligned}$$

Hence for every $\varepsilon > 0$ there is an n_0 such that

$$\begin{aligned} \forall\ n \geq n_0, &|f_n(x_n) - 1| < \varepsilon \\ &\Rightarrow -\varepsilon < f_n(x_n) - 1 < \varepsilon \\ &\Rightarrow 1 - \varepsilon < f_n(x_n) < 1 + \varepsilon \\ \Rightarrow f_n(x_n) &> 1 - \varepsilon \\ \text{Similarly}, f_n(y_n) &> 1 - \varepsilon \end{aligned}$$

Therefore if $n \geq n_0$ and $t \in [0, 1]$, then

$$\begin{aligned} \|tx_n + (1-t_n)y_n\| &\geq f_n(tx_n + (1-t)y_n) \\ . &= tf_n(x_n) + (1-t)f_n(y_n) \\ &> t(1-\varepsilon) + (1-t)(1-\varepsilon) \\ &= (t + 1 - t)(1-\varepsilon) \\ &= 1 - \varepsilon \end{aligned}$$

Thus by definition infimum ,we get

$$\lim_n (\min_{0 \leq t \leq 1} \|tx_n + (1-t)y_n\|) = 1.$$

$\square$

Lemma 2.4. *[1]. Assume that the norm* $\|\cdot\|$ *on* X *is URED. If* $x_n, y_n \in B(X), n = 1, 2, \cdots,$ *are such that* $\lim_n \|x_n + y_n\| = 2, x_n - y_n = \alpha_n z$ *where* $z \in S(X)$ *and* α_n *are real numbers, then* $\lim_n \alpha_n = 0$.

Proof. Assume that there are $x_n, y_n \in B(X)$ and $\varepsilon > 0$such that

$$\lim_n \|x_n + y_n\| = 2, x_n - y_n = \alpha_n z, z \in S(X)` \quad \text{and} |\alpha_n| \geq \varepsilon > 0 \quad \text{for all} \quad n.$$

Let $\widehat{x}_n, \widehat{y}_n \in S(X)$ be such that

$$[x_n, y_n] \subset [\widehat{x}_n, \widehat{y}_n] \quad \text{and put} \quad z_n = 1/2(x_n + y_n).$$

Then $\lim_n \|z_n\| = 1[\because \lim_n \|x_n + y_n\| = 2,]$ and

$$\inf_n\{min(\|z_n - \widehat{x}_n\|, \|z_n - \widehat{y}_n\|)\} \geq 1/2 \inf\{\|x_n - y_n\|\} \geq \varepsilon/2,$$

$$\begin{aligned} \because \|x_n - y_n\| &= \|\alpha_n z\| \\ &= |\alpha_n|\|z\| \\ &> \varepsilon \because \|z\| = 1 \quad \text{for all} \quad n. \end{aligned}$$

From lemma 2.2, used for $\widehat{x}_n$ and $\widehat{y}_n$, it follows that $\lim \|1/2(\widehat{x}_n + \widehat{y}_n\| = 1$. Since for each n, $\widehat{x}_n - \widehat{y}_n = \beta_n z$ for some$|\beta_n| \geq |\alpha_n| \geq \varepsilon$, it follows that $\|\cdot\|$ does not have the URED property so the conditions on the theorem must hold. □

Proof of theorem 2.1, let us assume that the norm $\|\cdot\|$ is URED but does not satisfy the condition in the theorem .Then there are $x_n, y_n \in X$ and an $\varepsilon > 0$ such that

$$\lim 2\|x_n\|^2 + 2\|y_n\|^2 - \|x_n + y_n\|^2 = 0, \{x_n\}\text{is bounded,}$$

$x_n - y_n = \alpha_n z$ for some $z \in X\ \{0\}$ and $|\alpha_n| \geq \varepsilon\ \forall\ n$. Since $f\{x_n\}$ is bounded and since $\lim_n(\|x_n\| - \|y_n\|) = 0$, assume without squeezing generality $\|x_n\| = \lim_n \|y_n\| = 1$. Put

$$\forall\ n \in \mathbb{N}, \delta_n = max\{\frac{1}{\|x_n\|}, \frac{1}{\|y_n\|}\}$$

and $x'_n = \delta_n x_n$ and $y'_n = \delta_n y_n$ Then

$$\forall\ n, x'_n, y'_n \in B(X), \lim_n \|x'_n + y'_n\| = 2, x'_n - y'_n = \delta_n \alpha_n z,$$

and $\delta_n \alpha_n| > \varepsilon/2$ for large n. Since we assumed that $\|\cdot\|$ is URED we obtained a contradiction with above theorem. Therefore the norm $\|\cdot\|$satisfies the condition in theorem 2.1 whenever $\|\cdot\|$ is URED. Conversely assume that for all $x_n, y_n \in X, n = 1, 2, \cdots$ are such that

$$\lim_n(2\|x_n\|^2 + 2\|y_n\|^2 - \|x_n + y_n\|^2) = 0 \quad \text{and}$$

$\{x_n\}$ is bounded and there is a point $z \in X - \{0\}$ and real numbers $\alpha_n, n = 1, 2, \cdots$ which satisfy

$$x_n - y_n = \alpha_n z \quad \text{for all} \quad n, \text{ then } \quad \lim_n \alpha_n = 0.$$

Then

$$\lim_n \|x_n - y_n\| = |\alpha_n|\|z\| = 0$$

So by definition $\|\cdot\|$ is URED.

3 Relation between URED ,UR and LUR norms

It is clear that uniformly rotund(UR) norm implies URED norm .But the converse is not always true. The notions LUR and URED are not related in general. The answers can be extracted from [1,3,4]. A norm that is URED does not have to be LUR. An example is l_∞

which admits an equivalent URED norm but admits no equivalent LUR norm. If there is a countable total set in X^* then X admits an equivalent URED norm [1,Corollary II.6.9 (iii)p.66]. This is the case for l_∞(just take the coordinate functionals.) On the other hand A norm that is LUR does not have to be URED. An example is $c_0(\Gamma)$, where $\Gamma$is an uncountable set, which admits an equivalent LUR norm (for example the Day norm) but according to Proposition II.7.9 p.74.in [1] if Γ is an uncountable set then $c_0(\Gamma)$, does not admit any equivalent URED norm. Similarly, URED implies R but not conversely.For example if Γ is an uncountable set then $c_0(\Gamma)$ is R but not URED. A separable Banach space admits a norm that is simultaneously LUR,URED and uniformly Gateaux (UG) which are seen in theorem 7.16 of [1] .Similarly, the space L_1 admits a norm that is LUR and URED.

Theorem 3.1. *[1,5]. Let X, Z are Banach spaces and let$T : X \to Z$ be a bounded linear one-to-one operator from X into Z. Suppose $|\cdot|$ is a norm on X and let $\|\cdot\|$ be a norm on Z which is URED. Define a norm $|||\cdot|||$ on X for $x \in X$ by*

$$|||x|||^2 = |x|^2 + \|Tx\|^2$$

Then $|||x|||$ is an equivalent URED norm.

Proof. Fist to show $|||\cdot|||$ is an equivalent norm . We write

$$\begin{aligned} |x|^2 \leq |||x|||^2 &= |x|^2 + \|Tx\|^2 \\ &\leq |x|^2 + \|T\|^2|x|^2 \\ &= |x|^2(1 + \|T\|^2) \end{aligned}$$

$\therefore |x|^2 \leq |||x||| \leq M|x|$, where $M^2 = (1 + \|T\|^2)$ To show $|||\cdot|||$ is URED , let $\{x_n\}$and $\{y_n\}$ be the sequences in X,and $\{x_n\}$ is bounded , that there are real numbers α_n and an element $z \in X - \{0\}$ such that $x_n - y_n = \alpha_n z$ and

$$\lim_n(2 \, ||| \, x_n \, |||^2 + 2 \, ||| \, y_n \, |||^2 - ||| \, x_n + y_n|||^2) = 0$$

We have to show $\lim_n \alpha_n = 0$. But from fact 2.3[in 1, p.45] we have

$$\lim_n(2\|Tx_n\|^2 + 2\|Ty_n\|^2 - \|Tx_n + Ty_n\|^2) = 0.$$

As $\{x_n\}$ is bounded $\{T(x_n)\}$ is also bounded. Since T is linear operator , $x_n - y_n = \alpha_n z \Rightarrow T(x_n - y_n) = T(x_n) - T(y_n) = \alpha_n T(z)$. Moreover $T(z) \neq 0$ Since T is one-to-one and $z \neq 0$ So by URED rotundity of $\|\cdot\| \Rightarrow \lim_n \alpha_n = 0$. Thus the norm $|||\cdot|||$ is URED . □

Definition. 2.5. A Banach space X is said to have normal structure if for any bounded closed convex subset C of X,not reduced to a point , there is $x \in C$such that$r_x(C) < d(C)$. where

$$r_x(C) = sup\{\|x - y\|, y \in C\} forall \in C$$

$$d(C) = sup\{\|x - y\| : x, y \in C\}$$

This property was introduced by Brodskii and Milman and has been significant in the development of fixed-point theory. Geometrically, X has normal structure if for every bounded and convex subset C of X, there exists a ball of radius less than the diameter of C centered at a point o f C and containing C.The concepts o f normal structure and complete normal structure have been of fundamental importance in some recent investigations concerned with determining fixed points of different mappings[1,11].

Theorem 3.2. *[1, p. 67, 6, p. 212]. If a norm of a Banach space X is URED , then X has normal structure*

Proof. Assume that a Banach space X has URED norm . Let C be a convex bounded subset of $X$and let C has two distinct point x, ysuch that $z = (x+y)/2$. We show that $r_x(C) < d(C)$. If possible suppose that this condition does not hold .Then there is a sequence of point $\{x_n\}$ of C such that

$$\lim \|z - x_n\| = d(C) \tag{4}$$

$$\Rightarrow \lim_n \|(x+y)/2 - x_n\| = d(C) \tag{5}$$

$$\Rightarrow \lim_n \|(x - x_n + y - x_n)\| = 2d(C) \tag{6}$$

$$\forall\ n \geq 1, \text{we have} \max\{\|x - x_n\|, \|y - x_n\|\} \leq d(C) \tag{7}$$

$$\tag{8}$$

From (4) and (7) we see that $\lim_n \|x - x_n\| = \lim_n \|y - x_n\| = d(C)$. By URED of the norm $\|\cdot\|$ we have

$$\|x - y\| = \lim \|(x - x_n + y - x_n\| = 0.$$

This shows that $x = y$ which is a contradiction of $x \neq y$. it is due to the assumption $r_x(C) < d(C)$ does not hold . So it forces $r_x(C) < d(C)$. Therefore Xhas normal structure . ∎

Theorem 3.3. *[1, p.74]. Let Γbe an uncountable set. Show that $c_0(T)$ has no equivalent URED norm.*

Proof. Let $\|\cdot\|$ be the supremum norm of $c_0(\Gamma)$, and the norm $|\cdot|$ is an equivalent norm on $c_0(\Gamma)$. Assume that$|\cdot|$ is URED. Put

$$M = sup\{|x| : x \in c_0(\Gamma), \|x\| \leq 1\}$$

Let $u_n \in c_0(\Gamma), \|u_n\| \leq 1, n = 1, 2, \cdots$ be such that

$$\lim_n |u_n| = M$$

Since each element of $c_0(\Gamma)$ is function on Γ with countable support and Γ is uncountable, there is $z \in c_0(\Gamma), \|z\| = 1$,and its support disjoint from all the supports of u_n. Then for $n = 1, 2, \cdots$ put

$$x_n = u_n + 1/2z \quad \text{and} \quad y_n = u_n - 1/2z$$

Considering the support of x_n , we note that that for each $n, \|x_n\| \leq 1, \|y_n\| \leq 1$ and thus

$$\|x_n\| \leq 1 \Rightarrow |x_n| \leq M \Rightarrow \lim_n |x_n| = M.$$

$$\|y_n\| \leq 1 \Rightarrow |y_n| \leq M \Rightarrow \lim_n |y_n| = M.$$

More over ,

$$\frac{x_n + y_n}{2} = u_n \Rightarrow \lim_n |\frac{x_n + y_n}{2}| = \lim_n |u_n| = M.$$

Combining above results we get

$$\lim_n |x_n| = \lim_n |y_n| = \lim_n |\frac{x_n + y_n}{2}| = M.$$

Also ,

$$\lim_n (2|x_n|^2 + 2|y_n|^2 - |x_n + y_n|^2) = 2M^2 + 2M^2 - 4M^2 = 0.$$

But $x_n - y_n = z, \|z\| = 1 \neq 0$. A contradiction of the norm $|\cdot|$ URED .Thus $|\cdot|$ is not URED. □

Theorem 3.4. *[1,p.67]. If there is a countable set in X^* which is total on X, then X admits a URED norm. In particular, every separable space admits an URED norm.*

Proof. Let $\{f_i\}_{(n=1)}^{\infty}$ be a countable set in X^* which is total on X, i.e. for every $x \in X, x \neq 0$, there is an $i \in \mathbb{N}$ such that $f_i(x) \neq 0$. Assume, without loss of generality, that $\{f_i\} \subset S(X^*)$. Define an operator T of X into $l_2(\mathbb{N}), x \in X$ by

$$T(x)(i) = 2^{-i} f_i(x), i = 1, 2, \cdots .$$

Then T is a bounded linear one -to-one operator of X into$l_2(\mathbb{N})$. Indeed,

$$\begin{aligned} \|T(x)\|_2^2 &= \sum_i^{\infty} 2^{-2i} |f_i(x)|^2 \\ &\leq \sum_i^{\infty} 1/2^{2i} \|x\|^2 \\ &= 1/3 \|x\|^2 \\ \therefore \|T(x)\| &\leq 1/\sqrt{3} \|x\| \end{aligned}$$

T is one-one, since $T(x) = 0$, for $x \neq 0$ then for $a = x/\|x\| \neq 0$. $f_i(a) = 2^i T(a)(i) = 0$, contradicting the fact that $\{f_i\}_{(n=1)}^{\infty}$ is total in X So $T(x) = 0 \Rightarrow x = 0, i.e. T$ is one-one function . Thus T is bounded ,one-to-one and linear operator from X into $l_2(\mathbb{N})$ but norm on $l_2(\mathbb{N})$ is URED , So by theorem 2.1 and [Theorem 6.8 (iii) ,1], X admits an equivalent URED in z. If $X$is separable,$\{x_i\}_{i=1}^{\infty} \subset S(X)$ is dense in $S(X)$ and $f_i \in S(X^*), i = 1, 2, \cdot$ are such that$f_i(x_i) = 1, i = 1, 2, \cdots$, then for $x \in S(X)$ and $\varepsilon > 0$ given, choose x_i such that $\|x - x_i\| < \varepsilon$. Then

$$f_i(x) = f_i(x_i) - (f_i(x_i) - f_i(x)) \geq f_i(x_i) - \|x - x_i\| > 1 - \varepsilon$$

Therefore $\|x\| = sup_i f_i(x) \ \forall \ x \in X$, and in particular, $\{f_i\}$is total on X. □

We discuss the application of norms that are uniformly rotund in every direction in the fixed point theory for non expansive mapping.

Theorem 3.5. *[1,cor.6.11,p.67 ,6,11]. Let X be a Banach space with a URED norm. Let C be a weakly compact convex subset of X and assume that $T : C \to C$ is a non expansive mapping,i.e. that $\|T(x) - T(y)\| \leq \|x - y\|$ for every $x, y \in C$. Then there exists $x_0 \in C$ such that $T(x_0) = x_0$.*

Proof. Let $\mathrm{K} = \{C' \subset C\ ,C'$ is closed convex subsets of $C\}$ such that $T(C') \subset C'$. Since C is closed convex subsets of C itself and $T(C) \subset C, C \in \mathrm{K}$ and with respect to the inclusion ,Thus K satisfies the assumptions of Zorn's lemma. so there exists a minimal element $C_0 \in \mathrm{K}$. If we let $C_1 = \overline{conv}[T(c_0)]$, we have $C_1 \subset C_0$ and thus

$$T(C_1) \subset T(C_0) \subset \overline{conv}[T(C_0)] = C_1.$$

Hence $C_1 \in \mathrm{K}$ and by minimality, we have $C_1 = C_0$. The function $\phi(x) = r_x(C_0) = \sup\{\|x - y\| : y \in C\}$ is weakly lsc and convex. since it is a supremum over $y \in C_0$ of the lower semi-continuous (lsc) convex functions $\|, -y\|$, $r_x(C_0)$ is a function of y only.

$$\text{Let} \quad C_2 = \{x \in C_0 : \phi(x) = \inf_C \phi\}.$$

The set $C_2$is a nonempty weakly closed set because $C_0 \subset C$is weakly compact . Since $C_1 = C_0$, we have for every $x \in C$,

$$\phi(Tx) = r_T(x)(C_0) = \sup\{\|T(x) - T(y)\| : y \in C_0\}. \tag{9}$$

Because T is non expansive, (9) shows that

$$\phi(Tx) = \sup \|T(x) - T(y)\| \leq \sup \|x - y\| = \phi(x)$$
$$\Rightarrow \phi(T(x)) \leq \phi(x) \quad \text{for every} \quad x \in C.$$
$$\Rightarrow \phi(T(x)) = \inf_C \phi \Rightarrow T(x) \in C_2$$

by definition of C_2 and thus if $T(x) \in T(C_2)$ then $T(C_2) \subset C_2$. So by the minimality, we have $C_2 = C_0$ and ϕ is constant of C_0. In fact we have $\phi \equiv d(C_0)$ on C_0(indeed, for any $\varepsilon > 0$ we find $x_\varepsilon, y_\varepsilon$ in C_0 with

$$\|x_\varepsilon - y_\varepsilon\| + \varepsilon > d(C_0)$$

and then $\phi(x_\varepsilon) \leq d(C_0) - \varepsilon)$. Since X has normal structure theorem 2.2 , this can happen only if $C_0 = \{x_0\}$. Thus $T(x_0) = x_0$.

$$[\because \phi(T(x_0)) = r_{T(x_0)}\{x_0\} = \sup_{x_0 \in \{x_0\}} \|T(x_0)\| = \sup \|x_0\| = \phi(x_0)]$$

$$\text{Thus} \quad \phi(T(x_0)) = \phi(x_0) \Rightarrow T(x_0) = x_0 \quad \text{because of the same domain of} \quad \phi.$$

□

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