On rational Diophantine Triples and Quadruples

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Abstract- This paper concerns with the study of constructing strong rational Diophantine triples and quadruples with suitable property.

Index Terms- Strong rational Diophantine triples and quadruples, Pythagorean equation
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I. Introduction

Let q be a non-zero rational number. A set $\{a_1, a_2, ...a_m\}$ of non-zero rational is called a rational $D(q)$ - mtuple, if $a_i a_j + q$ is a square of a rational number for all $1 \le i \le j \le m$. The mathematician Diophantus of Alexandria considered a variety of problems on indeterminant equations with rational or integers solutions. In particular, one of the problems was to find the sets of distinct positive rational numbers such that the product of any two numbers is one less than a rational square [20] and Diophantus found four positive rationals $\frac{1}{16}, \frac{33}{16}, \frac{17}{4}, \frac{105}{16}$ [4,5]. The first set of four positive integers with the same property, the set $\{1,3,8,120\}$ was found by Fermat. It was proved in 1969 by Baker and Davenport [3] that a fifth positive integer cannot be added to this set and one may refer [6, 7,11] for generalization. However, Euler discovered that a fifth rational number can be added to give the following rational Diophantine quintuple $\left\{1,3,8,120,\frac{777480}{8288641}\right\}$. Rational sextuples with two equal elements have been given in [2]. In this 1999, Gibs [13] found several examples of rational Diophantine sextuples, eg., $\left\{\frac{11}{192},\frac{35}{192},\frac{155}{27},\frac{512}{27},\frac{1235}{48},\frac{180873}{16}\right\}$, $\left\{\frac{17}{448},\frac{265}{448},\frac{2145}{448},252,\frac{23460}{7},\frac{2352}{7921}\right\}$.

All known Diophantine quadruples are regular and it has been conjectured that there are no irregular Diophantine quadruples [1,13] (this is known to be true for polynomials with integer co-efficients [8]). If so then there are no Diophantine quintuples. However there are infinitely many irregular rational Diophantine quadruples. The smallest is $\frac{1}{4}, 5, \frac{33}{4}, \frac{105}{4}$. Many of these irregular quadruples are examples of another common type for which two of the subtriples are regular i.e., $\{a,b,c,d\}$ is an irregular rational Diophantine quadruple, while $\{a,b,c\}$ and $\{a,b,\mathrm{d}\}$ are regular Diophantine triples. These are known as semi – regular rational Diophantine quadruples. These are only finitely many of these for any given common denominator l and they can readily found.

Moreover in [12], it has been proved that $D(\mp k^2)$ - triple $\{k^2, k^2 \pm 1, 4k^2 \pm 1\}$ cannot be extended to a $D(\mp k^2)$ - quintuple. In [10], it has been proved that $D(-k^2)$ - triple $\{1, k^2+1, k^2+4\}$ cannot be extended to a $D(-k^2)$ - quadruple if $k \ge 5$. Also, one may refer [14 -20].

These result motivated us to search for strong rational Diophantine triples and quadruples with suitable property.

II. Method of Analysis

Section A

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In this section we generate a sequence of strong rational Diophantine triples such that (A,B,C), (B,C,D), (C,D,E),…… with the property $D\left((2n+1)^2 k^2\right)$

Case 1:

Consider $A=\frac{a}{b}$ and $B=\frac{-(4n+1)k^2b}{a}$

Note that $AB+(2n+1)^2 k^2$ is a perfect square

Let C be any non-zero rational integer such that

$$AC+(2n+1)^2 k^2=\alpha^2 \tag{1}$$

$$BC+(2n+1)^2 k^2=\beta^2 \tag{2}$$

From (1), we have

$$C=\frac{\alpha^2-(2n+1)^2 k^2}{A} \tag{3}$$

Consider the linear transformations

$$\alpha=X+\frac{a}{b}T \tag{4}$$

$$\beta=X+\frac{-(4n+1)k^2b}{a} \tag{5}$$

On substituting (3) in (2) and by using (4) and (5), we get

$$C=\frac{(2nkb+a)^2-(2n+1)^2 k^2b^2}{ab}$$

Case 2:

Let $B=\frac{-(4n+1)k^2b}{a}$ and $C=\frac{(2nkb+a)^2-(2n+1)^2 k^2b^2}{ab}$

Let D be any nonzero rational integer such that

$$BD+(2n+1)^2 k^2=\alpha^2 \tag{6}$$

$$CD+(2n+1)^2 k^2=\beta^2 \tag{7}$$

From (6), we have

$$D=\frac{\alpha^2-(2n+1)^2 k^2}{B} \tag{8}$$

Using the linear transformations

$$\alpha=X+\frac{(-(4n+1)k^2b)}{a}T \tag{9}$$

$$\beta=X+\frac{(2nkb+a)^2-(2n+1)^2 k^2b^2}{ab}T \tag{10}$$

On substituting (8) in (7) and by using (9) and (10), we get

$$D=\frac{a}{b}$$

Case 3:

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Consider $C=\frac{(2nkb+a)^2-(2n+1)^2k^2b^2}{ab}$ and $D=\frac{a}{b}$

Let E be any non-zero rational integer such that

$$CE+(2n+1)^2k^2=\alpha^2 \tag{11}$$

$$DE+(2n+1)^2k^2=\beta^2 \tag{12}$$

From (11)

$$E=\frac{\alpha^2-(2n+1)^2k^2}{C} \tag{13}$$

Let us assume the linear transformations

$$\alpha=X+\frac{(2nkb+a)^2-(2n+1)^2k^2b^2}{ab}T \tag{14}$$

$$\beta=X+\frac{a}{b}T \tag{15}$$

On substituting (13) in (12) and by using (14) and (15), we get

$$E=\frac{(2nkb+2a)^2-(2n+1)^2k^2b^2}{ab}$$

Case 4:

Consider $D=\frac{a}{b}$ and $E=\frac{(2nkb+2a)^2-(2n+1)^2k^2b^2}{ab}$

Let F be any non-zero rational integer such that

$$DF+(2n+1)^2k^2=\alpha^2 \tag{16}$$

$$EF+(2n+1)^2k^2=\beta^2 \tag{17}$$

From (16)

$$F=\frac{\alpha^2-(2n+1)^2k^2}{D} \tag{18}$$

Consider the linear transformation

$$\alpha=X+\frac{a}{b}T \tag{19}$$

$$\beta=X+\frac{(2nkb+2a)^2-(2n+1)^2k^2b^2}{ab}T \tag{20}$$

On substituting the value of (18) in (17) and by using (19) and (20), we get

$$F=\frac{(2nkb+3a)^2-(2n+1)^2k^2b^2}{ab}$$

From all the above cases, (A,B,C), (B,C,D), (D,E,F),…. will form a sequence of strong rational diophantine triples

Section B

In this section, we search for distinct rational quadruple (A,B,C,D) such that product of any two of them added with 4 is a perfect square.

Assume $A=\frac{a}{b}$ (21)

and $B=\frac{b}{a}(\mathrm{n}^2\pm 2\,\mathrm{n}-3)$ (22)

where $\mathrm{a}=12r^2+s^2-4rs(n\pm 1)$

and $\mathrm{b}=2\,\mathrm{rs}$ $(r,s\neq 0)$

Let C be any non – zero rational integer such that

$$AC+4=\alpha^2 \quad (23)$$

$$BC+4=\beta^2 \quad (24)$$

From (23) we get,

$$C=\frac{\alpha^2-4}{A} \quad (25)$$

Assume

$$\alpha=X+\frac{a}{b}T \quad (26)$$

$$\beta=X+\frac{b}{a}(\mathrm{n}^2+2\,\mathrm{n}-3)T \quad (27)$$

On substituting (25) in (24) and by using (26) and (27), we get

$$C=\frac{\left(12r^2+s^2-4rs(n\pm 1)\right)^2+4r^2s^2(\mathrm{n}^2+2\,\mathrm{n}-3)+4\left(12r^2+s^2-4rs(n\pm 1)\right)(rs)(n\pm 1)}{\left(12r^2+s^2-4rs(n\pm 1)\right)(2rs)}$$

Let D be any non – zero rational integer such that

$$AD+4=\alpha^2 \quad (28)$$

$$BD+4=\beta^2 \quad (29)$$

$$CD+4=\gamma^2 \quad (30)$$

From (29) we get

$$D=\frac{\beta^2-4}{B} \quad (31)$$

Assume

$$\alpha=X+\frac{b}{a}(\mathrm{n}^2+2\,\mathrm{n}-3)T \quad (32)$$

$$\beta=X+\frac{(bn+b+a)^2-4b^2}{ab} \quad (33)$$

On substituting (30) in (28) and by using (32) and (33), we get

$$D=\frac{\left(12r^2+s^2-4rs(n\pm 1)\right)^2+16r^2s^2(\mathrm{n}^2\pm 2\,\mathrm{n}-3)+8\left(12r^2+s^2-4rs(n\pm 1)\right)(rs)(n\pm 1)}{\left(12r^2+s^2-4rs(n\pm 1)\right)(2rs)}$$

Hence (A, B, C, D) is a strong rational diophantine quadruple in which the product of any two when added with 4 is a perfect square.

Remark:

If we take B different from (22) we can generate different quadruples. Some of them are given below

1) If $B=\frac{b}{a}(n^2+4n)$ with the property $D(4)$, then the quadruple is

$$\left(\frac{12r^2+s^2-4rs(n+2)}{2rs},\frac{2rs}{12r^2+s^2-4rs(n+2)}(n^2+4n),\right.$$
$$\frac{\left(12r^2+s^2-4rs(n+2)\right)^2+4r^2s^2n(n+4)+4rs\left(12r^2+s^2-4rs(n+2)\right)(n+2)}{\left(12r^2+s^2-4rs(n+2)\right)(2rs)},$$
$$\left.\frac{\left(12r^2+s^2-4rs(n+2)\right)^2+16r^2s^2n(n+4)+8rs\left(12r^2+s^2-4rs(n+2)\right)(n+2)}{\left(12r^2+s^2-4rs(n+2)\right)(2rs)}\right)$$

2) If $B=\frac{b}{a}(2n+1)k^2$ with the property $D(n^2k^2)$, then the quadruple is

$$\left(\frac{a}{b},\frac{b}{a}(2n+1)k^2,\frac{a^2+k^2b^2(1+2n)+2kab(n+1)}{ab},\frac{\left(ka(n+1)+2k^2b(2n+1)\right)^2-n^2k^2a^2}{ab(2n+1)k^2}\right)$$

where $a=\frac{n\left(3r^2+s^2\right)-2rs(n+2)}{n}$ and $b=\frac{2rs}{nk}$

3) If $B=\frac{b}{a}\left(p^2-2npq\right)$ with the property $D(n^2q^2)$, then the quadruple is

$$\left(\frac{a}{b},\frac{b}{a}\left(p^2-2npq\right),\frac{\left(q(p-nb)+a\right)^2-n^2b^2q^2}{ab},\frac{\left(a(p-nq)+2b\left(p^2-2npq\right)\right)^2-n^2q^2a^2}{ab\left(p^2-2npq\right)}\right)$$

where $a=3r^2+s^2+4rs\left(1-\frac{p}{qn}\right)$ and $b=\frac{2rs}{nq}$

4) If $B=\frac{b}{a}\left(n^2q^2\right)$ with the property $D(p^2-2npq)$, then the quadruple is

$$\left(\frac{a}{b},\frac{b}{a}\left(n^2q^2\right),\frac{a^2+q^2n^2b^2+2ab(p-qn)}{ab},\frac{a^2+4q^2n^2b^2+4ab(p-qn)}{ab}\right)$$ provided $n=-\frac{a}{qb}$

5) If $B=\frac{b}{a}(-2npq)$ with the property $D(p^2+n^2q^2)$, then the quadruple is

$$\left(\frac{a}{b},\frac{b}{a}(-2npq),\frac{a^2-2b^2pqn+2b(p-qn)}{a},\frac{a^2-8npqb^2+4ab(p-qn)}{ab}\right)$$ provided $n=\frac{a^2+4abp}{6b^2pq+4abq}$ or $n=\frac{a^2+4abp}{10b^2pq+4abq}$

III. Conclusion

To conclude, one may search for other families of strong rational diophantine triples and quadruples .

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