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On the Initial, Final and Zero Objects and Zero Morphisms in the Category L-FCyc

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Abstract- In our earlier papers[14 &15],we have introduced the category **L-FCyc** whose objects are L-fuzzy subgroups of finite cyclic groups and morphisms are L-fuzzy homomorphisms, and studied some of the properties enjoyed by it. In this paper we discuss the existence of initial objects, final objects, zero objects and zero morphisms in this category.

Index Terms- Lattice, L-fuzzy group, category, L-FCyc, initial objects, final objects, zero objects, zero morphisms.

I. Introduction

Category Theory was introduced in 1945 by MacLane and Eilenberg [5].*Lattice Theory* was developed into a subject of its own right through the works of many mathematicians; Richard Dedekind and Garrett Birkhoff [2] being two among them. *Fuzzy Set Theory* was introduced by L.A.Zadeh in 1965[17]. J.A.Gougen[7] considered a complete and distributive lattice L as the membership set ,instead of the interval [0,1]used by Zadeh. In 1971, A.Rosenfeld[10]introduced fuzzy groups and later on it was generalized into L-fuzzy groups.

We have presented some results obtained in our studies on categories of L-fuzzy groups in [11]. In that paper, we have formed four categories of L-fuzzy groups and discussed some relations between them. In [12], we have discussed maximal lattices of cyclic groups and developed a method to construct it for finite cyclic groups. The method was further extended to the case of infinite cyclic groups in [13]. As a continuation of these works, in [14] we introduced L-fuzzy homomorphism of L-fuzzy subgroups of finite cyclic groups through an embedding of lattices. We then formed a category whose objects are L-fuzzy subgroups of finite cyclic groups and morphisms are L-fuzzy homomorphisms. We named this category as **L-FCyc** and discussed some categorical properties enjoyed by it. In [15] we discussed isomorphisms in this category. In the present paper, we discuss the existence of initial objects, final objects, zero objects and zero morphisms in this category.

Throughout this paper L denotes a complete and distributive lattice and L_i denotes sublattices of L. We represent the greatest element of L_i by I_i and the least element by O_i. Terms and notations in lattice theory used in this paper are taken from Bernard Kolman[1] Birkhoff G.[2], Davey B.A.[4] and Vijay K.Khanna[16];those in Algebra are from Fraleigh[8]; in fuzzy algebra are from George J.Klir [6], Zadeh [17],Goguen [7] ,Rosenfeld[10]and Mordeson[9] and those from category theory are from Bodo Pareigis [3].

II. Basic concepts

In this section, we give a summary of the relevant terms and results presented in papers[12 to 15], as they form the basis of the ideas developed in the present paper.

For a given group G and lattice L, a function $A : G \to L$ is called an L-fuzzy group on G (or L-fuzzy subgroup of G) if $\forall$ x, y ϵ G

(i) $A(x, y) \geq A(x) \wedge A(y)$ and (ii) $A(x^{-1}) \geq A(x)$.

We shall denote this by writing A ϵ (G, L).

2.1. *Definition[12]*. Let $L = (\{a_1,a_2,\ldots\ldots.a_n\},\leq)$ be a lattice. We say that L is a *finite lattice* containing n points and write |L|= n.

2.2. *Example[12]*. D_6= {1,2,3,6} is a lattice under divisibility. It is a finite lattice containing four points and so $|D_6|=4$.

2.3.*Definition[12]*. Let G be a group, L be a finite lattice and A: $G\to L$ be an L-fuzzy group. A is said to *saturate* L if Im(A)=L . If there is an L-fuzzy group A on G which saturates L, then we say that *G saturates **L***.

2.4. *Example[12]* .Consider G = <Z,+> and L=({0,1/3,1/2,1},$\leq$). Define A: $G\to L$ by A(0)=1, A(x)=1/2 if x$\in$ 4Z-{0}, A(x)=1/3 if x$\in$ 2Z-4Z and A(x)=0 if x$\in$ Z-2Z. Then A is an L-fuzzy group on G with Im (A) =L. Hence, A as well as G saturates L.

2.5. *Example[12]* .Let $G=<Z_4,+_4>$ and L=({0,1},$\leq$). Define A: $G\to L$ by A(0)=1;A(x)=0 ,if x$\neq$0.Then A is an L-fuzzy group on G which saturates L and so G also saturates L.

2.6 .*Example[12]*. Let $G=<Z_4,+_4>$ and L=({0,1/2,1},$\leq$). Define A: $G\to L$ by A(0)=1;A(x)=0 ,if x$\neq$0. Here A is an L-fuzzy group on G with Im(A) $\neq$L.Hence A does not saturate L. But if we define B: $G\to L$ by B(0)=1;B(2)=1/2 and B(1)=B(3)=0,then B is an L-fuzzy group on G which saturates L and hence G also saturates L.

2.7. *Definition[12]*. Let G be a group and L be a finite lattice. A sublattice L_1 of L is said to be a *maximal lattice* saturated by G if there is an A $\in$ (G,L) which saturates L_1 and there is no B$\in$ (G,L) which saturates a sublattice L_2 of L with $|L_2|>|L_1|$.

2.8.*Example[12]*. Consider the sublattices L_1={0,1} and L_2={0,1/2,1} of L ={0,1/3,1/2,1} and let $G=<Z_4,+_4>$.Define A: $G\to L$ by A(0)=1;A(x)=0 ,if x$\neq$0. Also define B: $G\to L$ by

B(0)=1;B(2)=1/2 and B(1)=B(3)=0. Then L_1 is not a maximal lattice of G, because there is L_2 with $|L_2|>|L_1|$ and B: G $\rightarrow$ L which saturates L_2. It can be shown that L_2 is a maximal lattice for G.
We may recall that a group G is said to be of prime power order if $|G|= p^n$, for some prime number p and positive integer n.

2.9.*Theorem[16]*.Let G be a cyclic group of prime power order. Then the lattice of all subgroups of G is a chain■

2.10.*Theorem[12]*. Let G be a cyclic group of prime power order. Then a maximal lattice L_G for G is a chain isomorphic to the chain of all subgroups of G■

It is well-known that every finite cyclic group of order n is isomorphic to Z_n. So, henceforth we represent cyclic groups of order n by Z_n.

2.11. *Proposition [8]*.The group Z_m X Z_n is isomorphic to Z_{mn} if and only if m and n are relatively prime■

2.12. *Proposition[8]*. The group $\prod_{i=1}^{n} Z_{m_i}$ is cyclic and isomorphic to $Z_{m_1 m_2 m_3 \dots\dots m_n}$ if and only if the numbers m_i ,for i=1,2…n are pairwise relatively prime■

2.13. *Theorem[12]*. Suppose n = $p_1^{n_1} p_2^{n_2} \dots \dots . . p_r^{n_r}$, where p_i's are distinct primes. Then the maximum lattice for Z_n $= Z_{p_1^{n_1}} X Z_{p_2^{n_2}} \dots \dots . X \; Z_{p_r^{n_r}}$ is the product lattice of the maximum chains for the factors $Z_{p_i^{n_i}}$.■

2.14. *Definition[14]*. Let A$\in$(G,L) . Then for a$\in$L, the set A^{-1} (a) ={x$\in$G/A(x) =a} is called the *A-pre image* of a.

2.15. *Definition[14]*. Let A $\in$(G, L). Then the collection of all A-pre images

Pr (L) = {A^{-1} (a)/a$\in$L}

is called the *A-pre image set* of L

2.16.*Definition[3]*. Let L and M be lattices. A mapping f: L$\rightarrow$ M is called a *lattice homomorphism* if for all a, b $\in$ L

f $(a \wedge b) = f(a) \wedge f(b)$ and

$f(a \vee b) = f(a) \vee f(b)$.

If, in addition, the mapping f is one-to-one and onto, we call f a *lattice isomorphism*. If f: L$\rightarrow$ M is an isomorphism, we say that L is *isomorphic* to M.

2.17 *Definition[3]*. Let L and M be lattices. If f: L$\rightarrow$M is a one-to-one homomorphism, then f is said to be an *embedding* of L into M. In this case, L is isomorphic to the sublattice f (L) of M and we say that L is *embeddable* on M.

2.18. *Definition.[14]* Let G_1 and G_2 be two finite cyclic groups with maximum lattices L_1 and L_2 respectively, where L_1 is embeddable on L_2. Let A $\in$(G_1, L_1) and B $\in$(G_2, L_2), each of them saturating its respective lattice. Let f: $L_1$$\rightarrow$ L_2 be a map such that

(1) $f(I_1)=I_2$,where I_1 and I_2 are maximum elements of L_1 and L_2 respectively, and
(2) f defines an embedding of L_1 on L_2.

Then a map F: Pr(L_1) $\rightarrow$ Pr(L_2) is said to be an *L-fuzzy homomorphism* from A to B through f if
F (A^{-1} (a)) =B^{-1}(f (a)) for every a in L_1.

We shall denote this by writing F_f*: A* $\rightarrow$ *B is a homomorphism*.

2.19. *Definition [3]*. A *category* $\mathcal{C}$ consists of:

(1) a *class of objects*, denoted as *Ob*$\mathcal{C}$ and whose members are denoted as A, B, C,…
(2) a family of mutually disjoint sets {Mor(A, B)} for all objects A,B in $\mathcal{C}$, whose elements f,g,h…$\in$ Mor(A,B) are called *morphisms* and
(3) a family of maps called *composition* { Mor(A,B) X Mor(B,C) $\rightarrow$ Mor(A,C)} in which
(f, g) $\mapsto$ gf for all A,B,C $\in$ Ob$\mathcal{C}$.

satisfying the following axioms:

(1) Associativity: For all A,B,C,D $\in$ Ob$\mathcal{C}$ and all f $\in$ Mor(A,B), g$\in$Mor(B,C) and h $\in$ Mor(C,D), we have, h(gf)=(hg)f
(2) Identity: For each A$\in$Ob $\mathcal{C}$ there is a morphism I_A $\in$Mor(A,A), called the *identity*, such that we have, fI_A=f and I_Ag=g for all B,C $\in$ Ob$\mathcal{C}$, and all f $\in$ Mor(A,B) and g $\in$ Mor(C,A).

2.20. *Example [3]*. All sets together with the set maps and their composition form a category. This category is denoted by **Set.**

2.21.*Theorem.[14]* L-fuzzy subgroups of finite cyclic groups which saturate their respective maximum lattices together with L-fuzzy homomorphism of L-fuzzy groups through an embedding form a category■

2.22.*Notation.[14]* We shall denote the above category by **L-FCyc.** Whenever we say that A is an object in **L-FCyc,** we shall mean that there is a finite cyclic group G_1 and a corresponding maximum lattice L_1 such that A$\in$ (G_1, L_1). Similarly, the statement F_f: A$\rightarrow$ B is a morphism in **L-FCyc** shall imply that A$\in$ (G_1, L_1), B$\in$ (G_2, L_2), F: Pr $L_1$$\rightarrow$ Pr L_2 and that f: $L_1$$\rightarrow$ L_2 is an embedding such that F (A^{-1} (a)) =B^{-1} (f (a)) for every point a in L_1.

2.23. Proposition[14]. Let F_f: A$\rightarrow$ B and F_g: A$\rightarrow$ B be two morphisms in **L-FCyc**. Then f = g■

2.24.Proposition [14] Let F_f: A$\rightarrow$ B and G_f: A$\rightarrow$ B be two morphisms in **L-FCyc.** Then F = G■

2.25.Definition [3]. Let $\mathcal{B}$ and $\mathcal{C}$ be categories. Let $\mathcal{F}$ consist of

(1) a map Ob$\mathcal{B} \ni A \mapsto \mathcal{F}(A) \in Ob\mathcal{C}$
(2) a family of maps { $Mor_{\mathcal{B}}(A,B) \ni f \mapsto \mathcal{F}(f) \in Mor_{\mathcal{C}}(\mathcal{F}(A), \mathcal{F}(B))$}

for all A,B $\in Ob\mathcal{B}$

Then $\mathcal{F}$ is called a *covariant functor* (or simply a *functor*) if $\mathcal{F}$ complies with the following axioms:

(i) $\mathcal{F}(1_A) = 1_{\mathcal{F}(A)}$ for all A $\in Ob\mathcal{B}$
(ii) $\mathcal{F}(fg) = \mathcal{F}(f)\mathcal{F}(g)$ for all $f \in Mor_{\mathcal{B}}(B,C), g \in Mor_{\mathcal{B}}(A,B)$ and for all A,B,C $\in Ob\mathcal{B}$

*2.26.Definition [3].*Let $\mathcal{B}$ and $\mathcal{C}$ be categories. Let $\mathcal{F}$ consist of
(1) a map $Ob\mathcal{B} \ni A \mapsto \mathcal{F}(A) \in Ob\mathcal{C}$
(2) a family of maps $\{Mor_{\mathcal{B}}(A,B) \ni f \mapsto \mathcal{F}(f) \in Mor_{\mathcal{C}}(\mathcal{F}(B),\mathcal{F}(A))\}$ for all A,B $\in Ob\mathcal{B}$
Then $\mathcal{F}$ is called a *contravariant functor* if $\mathcal{F}$ complies with the following axioms:
(i) $\mathcal{F}(1_A) = 1_{\mathcal{F}(A)}$ for all A $\in Ob\mathcal{B}$
(ii) $\mathcal{F}(fg) = \mathcal{F}(g)\mathcal{F}(f)$ for all $f \in Mor_{\mathcal{B}}(B,C), g \in Mor_{\mathcal{B}}(A,B)$ and for all A,B,C $\in Ob\mathcal{B}$.

2.27.Proposition[3]. Let $\mathcal{C}$ be a category and A $\in Ob\mathcal{C}$.Then Mor(A,-): $\mathcal{C} \to$ **Set** with
Ob $\mathcal{C} \ni B \longmapsto$ Mor(A,B) $\in$ Ob **Set**
Mor (B,C) $\ni f \longmapsto$ Mor(A,f) $\in$ Mor(Mor(A,B),Mor(A,C))
is a covariant functor. Furthermore, Mor(-,A): $\mathcal{C} \to$ **Set** with
Ob $\mathcal{C} \ni B \longmapsto$ Mor(B,A) $\in$ Ob **Set**
Mor (B,C) $\ni f \longmapsto$ Mor(f,A) $\in$ Mor(Mor(C,A),Mor(B,A))
is a contravariant functor■

2.28. Remark [3]. The above proposition says that corresponding to any object A in a category $\mathcal{C}$, one can form a covariant functor Mor(A,-) and a contravariant functor Mor(-,A). Of these, Mor(A,-) is called *covariant representable functor* and Mor(-,A) is called *contravariant representable functor.*

2.29..Definition [3]. Let $\mathcal{C}$ be a category and f a morphism in $\mathcal{C}$. f is called a *monomorphism* if the map Mor(B, f) is injective for all B $\in Ob\mathcal{C}$.

2.30.Definition[3]. Let $\mathcal{C}$ be a category and f a morphism in $\mathcal{C}$. f is called an *epimorphism* if the map Mor(f,B) is injective for all B $\in Ob\mathcal{C}$.

2.31. Lemma[3]. (a). f $\in$ Mor(A,B) is a monomorphism in $\mathcal{C}$ if and only if fg=fh implies g=h for all C $\in Ob\mathcal{C}$ and for all g,h $\in$ Mor(C,A).
(b). f $\in$ Mor(A,B) is an epimorphism in $\mathcal{C}$ if and only if gf=hf implies g=h for all C $\in Ob\mathcal{C}$ and for all g,h $\in$ Mor(B,C)■

The above lemma enables us to roughly define a *monomorphism* as a *left cancellable morphism*; and an *epimorphism* as *a right cancellable morphism.*

2.32.Theorem[14]. Every morphism in **L-FCyc** is a monomorphism■

2.33.Remark[14]. All the morphisms in **L-FCyc** are not epimorphisms. We give below an example to prove this.

*2.34.Example.[14].*Let Z_1={0} with the maximum lattice, L_1={(O_1=)I_1}, the one-point lattice; Z_2={0,1} with the maximum lattice, L_2={O_2,I_2} and Z_6={0,1,2,3,4,5} with the maximum lattice L_3={O_3(=1), 2, 3, I_3(=6)}. Define A:$Z_1 \to L_1$, B : $Z_2 \to L_2$ and C : $Z_6 \to L_3$ by
A(0)=I_1
B(0)=I_2,B(1)=O_2 and
C(0)=I_3,C(2)=C(4)=2,C(3)=3 and C(1)=C(5)=O_3.

Define F : $Pr(L_1) \to Pr(L_2)$ by F(A^{-1})(I_1)={0} ; G : $Pr(L_2) \to Pr(L_3)$ by G(B^{-1})(I_2)={0} and G(B^{-1})(O_2)={2,4} and H : $Pr(L_2) \to Pr(L_3)$ by H(B^{-1})(I_2)={0} and H(B^{-1})(O_2)={3}. Define f: $L_1 \to L_2$ by f(I_1)=I_2; g :$L_2 \to L_3$ by g(I_2)=I_3,g(O_2)=2 and h :$L_2 \to L_3$ by h(I_2)=I_3, h(O_2)=3. Then F_f:A$\to$ B, G_g,H_h :B$\to$ C. Here $G_g \neq H_h$. But $G_gF_f = H_hF_f$ ■

2.35.Definition.[15] Let G_1 and G_2 be two finite cyclic groups with maximum lattices L_1 and L_2 respectively. Let f:$L_1 \to L_2$ be an isomorphism. Let A$\in$(G_1,L_1) and B$\in$(G_2,L_2) , each of them saturating its respective lattice. A function F: $Pr(L_1) \to Pr(L_2)$ is said to define an *L-fuzzy isomorphism* from A onto B through f if F (A^{-1} (a)) =B^{-1}(f (a)) for every a $\in L_1$. We write F_f: A $\to$ B is an isomorphism.

2.36. Definition [3]. Let $\mathcal{C}$ be a category and A,B $\in Ob\mathcal{C}$. A morphism f $\in$ Mor (A,B) is called an isomorphism if there is a mapping g $\in$ Mor (B,A) such that fg=1_B and gf = 1_A.

2.37.Theorem[15]. A morphism F_f :A$\to$ B in **L-FCyc** is an isomorphism if and only if the maximum lattices for A and B are isomorphic■

III. Initial objects,final objects,zero objects and zero morphisms in L-FCyc

In this section,we discuss the existence of initial objects,final objects zero objects and zero morphisms in this category.All these notions are introduced first and then discussed.

3.1 Definition[4]. An object A in a category $\mathcal{C}$ is called an *initial object* if Mor(A,B) consists of exactly one element for all B $\in \mathcal{C}$.

3.2 Example. Let Z_1={0} is an initial object in the category **Grp.**

3.3 Theorem. **L-FCyc** has initial objects.

Proof: Let G be a group consisting of one element 0. Its maximum lattice is the singleton lattice L_1= {I_1}. Define A:G$\to L_1$ by A(0)=I_1. Let H be any finite cyclic group and L_2 be the corresponding maximum lattice. Define B:H $\to L_2$ which saturates L_2. Define f:$L_1 \to L_2$ be such that f(I_1)=I_2. Then we can find exactly one morphism $F_f : A \rightarrow B$ given by $F(A^{-1}(I_1))=B^{-1}(f(I_1))$. Hence A is an initial object in **L-FCyc**■

3.4 Definition[4]. An object A in a category $\mathcal{C}$ is called a *final object* if Mor(B,A) consists of exactly one element for all $B \in Ob\mathcal{C}$.

3.5 Proposition. **L-FCyc** has no final objects.

Proof: Suppose, if possible, that A is a final object in **L-FCyc** .Then for any B in **L-FCyc** we must have a morphism, say, $F_f: B \rightarrow A$. Assume that the maximum lattices for B and A are L_1 and L_2 respectively and that L_1 is not embeddable on L_2. Then there cannot exist such a morphism. In other words, Mor(B,A) is empty, and not singleton■

3.6. Definition[4]. An object is called *zero object* if it is an initial and a final object.

3.7. Proposition. **L-FCyc** has no zero objects.

Proof: Follows from proposition 3.5.■

3.8. Definition[4]. A morphism $f: A \rightarrow B$ in $\mathcal{C}$ is called a *left zero morphism* if fg=fh for all g,h $\in$Mor(C,A) and for all $C \in \mathcal{C}$.

3.9. Theorem. The Category **L-FCyc** has left zero morphisms.

Proof: Let G={0} be a group consisting of one element. Let $L_1=\{I_1\}$ be a lattice consisting one point. Define $A: G \rightarrow L_1$ by $A(0)=I_1$. Let H be any finite cyclic group and L_2 be its maximum lattice. Then L_1 is embeddable on L_2. Let $B: H \rightarrow L_2$ be an L-fuzzy group. Then $B(0)=I_2$, the maximum element of L_2. Define $f: L_1 \rightarrow L_2$ by $f(I_1)=I_2$. Define $F : PrL_1 \rightarrow PrL_2$ by $F(A^{-1}(I_1)) = B^{-1}(I_2)$. Then $F(A^{-1}(I_1))=B^{-1}(f(I_1)$. $\therefore F_f : A \rightarrow B$ is a morphism in **L-FCyc.** We will show that F_f is a left zero morphism. Let C be any object of **L-FCyc** and let L_3 be the maximum lattice corresponding to C. There arises two distinct cases. *Case(i): L_3 is isomorphic to L_1*.Then L_3 has only one point say, I_3. Define $g : L_3 \rightarrow L_1$ by $g(I_3)=I_1$. This mapping is unique in the sense that if $h:L_3 \rightarrow L_1$ is another mapping, then g=h. Hence if G_g , H_h $\in$Mor(C,A), then by proposition 2.24, $G_g=H_h$. $\therefore F_fG_g=F_fH_h$. *Case(ii): L_3 is not embeddable on L_1*. In this case, Mor(C, A) is empty and the result holds vacuously■

3.10. Remark. We can also form a category whose objects are the L-fuzzy subgroups of finite cyclic groups which have isomorphic maximum lattices and whose morphisms are the L-fuzzy isomorphisms. This category is a full subcategory of **L-FCyc.**

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