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Numerical Solution of a non-linear Volterra Integro-differential Equation via Runge-Kutta-Verner Method

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Abstract- In this paper a higher-order numerical solution of a non-linear Volterra integro-differential equation is discussed. Example of this question has been solved numerically using the Runge-Kutta-Verner method for Ordinary Differential Equation (ODE) part and Newton-Cotes formulas for integral parts.

Index Terms- A higher-order accuracy, Lagrange interpolating, quadrature formulas, Runge-Kutta methods, non-linear Volterra integro-differential equation.

I. INTRODUCTION

A functional equation in which the unknown function appears in the form of it is a derivative as well as under the integral sign is called an integro-differential equation (see [10, 11, 14, 15, 16]). In this paper we will consider the non-linear Volterra integro-differential equation of the form (see [2, 4, 7, 8])

$$u'(t) = u(t)\ (F(t, u(t), \int_{t_0}^{t} k(t-s)u(s)\,ds)\ ),\ u(t_0) = u_0,\ t \geq t_0. \quad (1)$$

Equation (1) can be solved numerically using various methods (see [6, 9, 10, 11, 12]). In this paper $u(t_n)$ will denote the exact value of u at $t_n = t_0 + nh$. We shall use $\tilde{u}(t_n)$ or $\tilde{u}_n$ to denote a numerical solution u of at t_n. However, in this paper we will construct higher-order numerical method for equation (1). Since the integral cannot be determined explicitly, it may be approximated using familiar numerical integration methods. The Newton-Cotes integration formulae, which include the 2-point closed Newton-Cotes formula is called the trapezoidal rule, the 3-point rule is known as Simpson's 1/3 rule, the 4-point closed rule is Simpson's 3/8 rule, the 5-point closed rule is Boole's rule (Bode's rule), Weddle's rule, higher rules include the 6-point, 7-point and 8-point are well suited here since they use nodes which were given in [1, 10, 13,17] and [4, 12].

II. THE NUMERICAL INTEGRATION OF A NON-LINEAR VOLTERRA INTEGRO-DIFFERENTIAL EQUATION

In general formulae for the numerical solution of integro-differential equations rely upon formulae for the underlying Ordinary Differential Equation (ODE), combined with auxiliary quadrature rules approximation of

$$\tilde{z}(t_n) := h\sum_{j=0}^{n} \omega_{n,j} k(t_n - t_j)\ \tilde{u}(t_j) \approx \int_{t_0}^{t_n} k(t-s)\ u(s)\ ds. \quad (2)$$

Of course, whereas we have defined approximations $\tilde{z}(t_n)$ in terms of quadrature rules that reflect the underlying ODE method, it is in principle possible to "mix and match". The combinations of formulae can be chosen on the basis of order of convergence. The first involves adapting Runge-Kutta methods. We will require to approximate integral term

$$\tilde{z}(t_n) := h\sum_{j=0}^{n} \omega_{n,j}\ k(t_n - t_j)\ \tilde{u}(t_j) \approx \int_{t_0}^{t_n} k(t-s)u(s)ds$$

at selected values at t. Equation (1) can be solved several ways. In this paper we shall focus on higher-order numerical method for equation (1). The integral may be approximated using familiar numerical integration methods. The Newton-Cotes integration formulas, which include left and right rectangle rules, the trapezoidal rule, Simpson's 1/3 rule and Simpson's 3/8 rule are well suited here since they used nodes which were previously calculated [10, 11]:

$$\int_{t_0}^{t_n} k(t-s)\, u(s)\, ds \approx h\sum_{j=0}^{n} \omega_{n,j}\, k(t_n - t_j)\, \tilde{u}(t_j)$$

where $\omega_{n,j}$ are the appropriate coefficients for the composite integration schemes chosen. A combination of integration method may be used.

III. NUMERICAL ROUTINE FOR NON-LINEAR VOLTERRA INTEGRO-DIFFERENTIAL EQUATION

Now consider the non-dimensional problem (1). In order to solve (1) numerically, we purpose the use of two methods familiar to most mathematicians. We consider methods which approximate the solution the initial value problem (IVP)

$$u'(t) = u(t)\, F(t, u(t), \int_0^t k(t-s) u(s)\, ds),\ u(t_0) = u_0,$$

at time $t_n = t_0 + nh$, n=0, 1, 2, 3, …., where $h = t_n - t_{n-1}$ is the constant nodal step-size and, in the Example 3.1,

$$F(t, u(t), \int_{t_0}^{t} k(t-s)\, u(s)\, ds) = r - c\int_0^t k(t-s) u(s)\, ds.$$

For example, the explicit Euler method approximates the solution to Example 3.1 at t_{n+1}

$$\tilde{u}_{n+1} = \tilde{u}_n + h\, \tilde{u}_n \left(r - c\int_0^{t_n} e^{-\delta(t-s)}\, u(s)\, ds \right).$$

The explicit finite difference method given in [11] as applied to equation (1) easily extended to more accurate predictor-corrector method. The predictor step uses ($\tilde{u}_{n+1} = \tilde{u}_n + h\, \tilde{u}_n\, (F(t_n, \tilde{u}_n, \tilde{z}(t_n)))$) to obtain $\tilde{u}^{\kappa}_{n+1}$, which is followed by the corrector step, which uses higher order trapezoidal method

$$\tilde{u}_{n+1} = \tilde{u}_n + h\tilde{u}_n \left(\frac{1}{2} F\left(t_n, \tilde{u}_n, \tilde{z}(t_n)\right) + \frac{1}{2} F(t_{n+1}, \tilde{u}^{\kappa}_{n+1}, \tilde{z}(t_{n+1})) \right). \tag{3}$$

This procedure is known as modified Euler method (second order Runge-Kutta-RK2) and is one order magnitude more accurate than the explicit Euler method.

The fourth order classical Runge-Kutta method (RK4) can also be adapted to the numerical solution of equation (1). Stepping from $\tilde{u}_n$ with step-size h to obtain $\tilde{u}_{n+1}$, the RK4 method as applied to this problem in [10, 11].

The **sixth order** Runge-Kutta-Verner methods [3] may be used but not readily, since the intranodal evaluation points are uniformly spaced. Consequently, the integrals needed during the intermediate calculations to step from t_n to t_{n+1} may require the trapezoidal rule or Lagrange polynomial interpolating integration on a non-uniform partition $[t_n, t_{n+1}]$.

Runge-Kutta-Verner method (RKV) can also be adapted to the numerical solution of (1). Stepping from $\tilde{u}_n$ with step-size h to obtain $\tilde{u}_{n+1}$, the RKV method as applied to this problem may be written as:

$$k_1 = h\tilde{u}_n\, F(t_n, \tilde{u}_n, \tilde{z}(t_n)\ ),$$

$$\tilde{u}^a_{n+1/6} = \tilde{u}_n + \frac{k_1}{6},$$

$$k_2 = h\tilde{u}^a_{n+1/6}\, F(t_{n+1/6}, \tilde{u}^a_{n+1/6}, \tilde{z}_{n+1/6}\ ),$$

$$k_2 = h\tilde{u}^a_{n+1/6}\, F\left(t_{n+1/6}, \tilde{u}^a_{n+1/6}, \tilde{z}_n + \frac{h}{12}\left[\tilde{u}_n + \tilde{u}^a_{n+1/6}\right]\right),$$

$$\tilde{u}^b_{n+4/15} = \tilde{u}_n + \frac{4k_1}{75} + \frac{16k_2}{75},$$

$$k_3 = h\tilde{u}^b_{n+4/15} F(t_{n+4/15}, \tilde{u}^b_{n+4/15}, \tilde{z}_{n+4/15}),$$

$$k_3 = h\tilde{u}^b_{n+4/15} F\left(t_{n+4/15}, \tilde{u}^b_{n+4/15}, \tilde{z}_n + \frac{4h}{30}\left[\tilde{u}_n + \tilde{u}^b_{n+4/15}\right]\right),$$

$$\tilde{u}^c_{n+2/3} = \tilde{u}_n + \frac{5k_1}{6} - \frac{8k_2}{3} + \frac{5k_3}{2},$$

$$k_4 = h\tilde{u}^c_{n+2/3} F(t_{n+2/3}, \tilde{u}^c_{n+2/3}, \tilde{z}_{n+2/3}),$$

$$k_4 = h\tilde{u}^c_{n+2/3} F\left(t_{n+2/3}, \tilde{u}^c_{n+2/3}, \tilde{z}_n + \frac{2h}{6}\left[\tilde{u}_n + \tilde{u}^c_{n+2/3}\right]\right),$$

$$\tilde{u}^d_{n+5/6} = \tilde{u}_n + \frac{165k_1}{64} + \frac{55k_2}{6} - \frac{425k_3}{64} + \frac{85k_4}{96},$$

$$k_5 = h\tilde{u}^d_{n+5/6} F(t_{n+5/6}, \tilde{u}^d_{n+5/6}, \tilde{z}_{n+5/6}),$$

$$k_5 = h\tilde{u}^d_{n+5/6} F\left(t_{n+5/6}, \tilde{u}^d_{n+5/6}, \tilde{z}_n + \frac{5h}{12}\left[\tilde{u}_n + \tilde{u}^d_{n+5/6}\right]\right),$$

$$\tilde{u}^e_{n+1} = \tilde{u}_n + \frac{12k_1}{15} - 8k_2 + \frac{4015k_3}{612} - \frac{11k_4}{36} + \frac{88k_5}{255},$$

$$k_6 = h\tilde{u}^e_{n+1} F(t_{n+1}, \tilde{u}^e_{n+1}, \tilde{z}_{n+1}),$$

$$k_6 = h\tilde{u}^e_{n+1} F\left(t_{n+1}, \tilde{u}^e_{n+1}, \tilde{z}_n + \frac{h}{2}\left[\tilde{u}_n + \tilde{u}^e_{n+1}\right]\right),$$

$$\tilde{u}^f_{n+1/15} = \tilde{u}_n + \frac{8263k_1}{15000} + \frac{124k_2}{75} - \frac{643k_3}{680} - \frac{81k_4}{250} + \frac{2484k_5}{10625},$$

$$k_7 = h\tilde{u}^f_{n+1/15} F(t_{n+1/15}, \tilde{u}^f_{n+1/15}, \tilde{z}_{n+1/15}),$$

$$k_7 = h\tilde{u}^f_{n+1/15} F\left(t_{n+1/15}, \tilde{u}^f_{n+1/15}, \tilde{z}_n + \frac{h}{30}\left[\tilde{u}_n + \tilde{u}^f_{n+1/15}\right]\right),$$

$$\tilde{u}^g_{n+1} = \tilde{u}_n + \frac{3501k_1}{1720} - \frac{300}{43}k_2 + \frac{297275k_3}{52632} - \frac{319k_4}{2322} + \frac{24068k_5}{84065} + \frac{3850k_7}{26703},$$

$$k_8 = h\tilde{u}^g_{n+1} F(t_{n+1}, \tilde{u}^g_{n+1}, \tilde{z}_{n+1}),$$

$$k_8 = h\tilde{u}^g_{n+1} F\left(t_{n+1}, \tilde{u}^g_{n+1}, \tilde{z}_n + \frac{h}{2}\left[\tilde{u}_n + \tilde{u}^g_{n+1}\right]\right),$$

$$\tilde{u}_{n+1} = \tilde{u}_n + \frac{13k_1}{160} + \frac{2375}{5984}k_3 + \frac{5k_4}{16} + \frac{12k_5}{85} + \frac{3k_6}{44}, \tag{4}$$

and

$$\tilde{u}_{n+1} = \tilde{u}_n + \frac{3k_1}{40} + \frac{875}{2244}k_3 + \frac{23k_4}{72} + \frac{264k_5}{1955} + \frac{125k_7}{11592} + \frac{43k_8}{616}. \tag{5}$$

In this example, the trapezoidal rule is used to approximate $\tilde{z}(t_n) \approx \int_{t_0}^{t_n} k(t-s)u(s)\,ds$ on $[t_n, t_{n+1/6}]$, $[t_n, t_{n+4/15}]$, $[t_n, t_{n+2/3}]$, $[t_n, t_{n+5/6}]$, $[t_n, t_{n+1}]$, $[t_n, t_{n+1/15}]$, $[t_n, t_{n+1}]$ in calculating, k_2, k_3, k_4, k_5, k_6, k_7 and k_8 respectively. If desired, the trapezoidal rule may be used on $[t_0, t_n]$ (gives second order accuracy, See Table 1); the trapezoidal rule and Simpson's 1/3 rule (giving third order accuracy, see [10, 11]) may be used on $[t_0, t_n]$.

In order to get **higher-**order accuracy the integral term must be evaluated more accurately on $[t_n, t_{n+1/6}]$, $[t_n, t_{n+4/15}]$, $[t_n, t_{n+2/3}]$, $[t_n, t_{n+5/6}]$, $[t_n, t_{n+1}]$, $[t_n, t_{n+1/15}]$, $[t_n, t_{n+1}]$ in calculating, k_2 , k_3 , k_4 , k_5, k_6 , k_7 and k_8, as shown in (6) , (7), (8), (9), (10), (11), (12) below. The 5-point extended closed rule is Boole's method may be devised on $[t_0, t_n]$ as following:

```
z(1)=0
u(1)=u0
If n=1
z(n+1)= z(n) + h( u(n) + u(n+1) ) /2
elseif n==2
z(n+1)= z(n-1) + h( u(n-1) +4 u(n) + u(n+1) ) /3
elseif n==3
z(n+1)= z(n-2) +3h ( u(n-2) +3 u(n-1) +3  u(n) + u(n+1) ) / 8
elseif n==4
z(n+1)= z(n-3) +2h (7 u(n-3) +32 u(n-2) +12u(n-1) + 32 u(n) +7 u(n+1) ) / 45
elseif n==5
z(n+1)= z(n-4) +5h (19 u(n-4) +75 u(n-3) +50 u(n-2)+50 u(n-1) + 75 u(n) +19 u(n+1) ) / 288
elseif n==6
z(n+1)= z(n-5) + h (41u(n-5)+ 216 u(n-4) +27 u(n-3) +272 u(n-2)+27 u(n-1) + 216 u(n) +41 u(n+1) ) / 140
elseif n==7
z(n+1)= z(n-6) + 7h (751u(n-6)+3577u(n-5)+ 1323 u(n-4) +2989 u(n-3) +2989 u(n-2)+1323 u(n-1) +…
   3577 u(n) +7511 u(n+1) ) / 17280
elseif n==8
z(n+1)= z(n-3) + 2h (7 u(n-3) +32 u(n-2) +12u(n-1) + 32 u(n) +7 u(n+1) ) / 45
elseif mod(n,4)==0
z(n+1)= z(n-3) + 2h (7 u(n-3) +32 u(n-2) +12u(n-1) + 32 u(n) +7 u(n+1) ) / 45
elseif mod(n,4)==1
z(n+1)= z(n-3) + 2h (7 u(n-3) +32 u(n-2) +12u(n-1) + 32 u(n) +7 u(n+1) ) / 45
elseif mod(n,4)==2
z(n+1)= z(n-3) + 2h (7 u(n-3) +32 u(n-2) +12u(n-1) + 32 u(n) +7 u(n+1) ) / 45
elseif mod(n,4)==3
z(n+1)= z(n-3) + 2h (7 u(n-3) +32 u(n-2) +12u(n-1) + 32 u(n) +7 u(n+1) ) / 45
else
z(n+1)= z(n-3) + 2h (7 u(n-3) +32 u(n-2) +12u(n-1) + 32 u(n) +7 u(n+1) ) / 45
```

If we interpolating on $\tilde{u}_{n-2}, \tilde{u}_{n-1}, \tilde{u}_n, \tilde{u}_{n+1/6}$ (special formulae required for the first two steps, for example we can use (4) and (5)) Lagrange's formula for points t=-2, -1, 0, 1/6 gives

$$u(t) = \frac{1}{h^3}\left[-\frac{3}{13}t\ (t-\frac{h}{6})\ (t+h)u_{-2} + \frac{6}{7}t\ (t-\frac{h}{6})(t+2h)u_{-1} - 3(t+h)(t+2h)(t-\frac{h}{6})u_0 + \frac{216}{91}t(t+h)(t+2h)u_{1/6}\right].$$

If we integrate the expression between 0 and h/6, we get

$$\int_0^{h/6} u(s)ds \approx h\left(\frac{13}{168}u_{1/6} + \frac{1}{5184}u_{-2} - \frac{25}{18144}u_{-1} + \frac{469}{5184}u_0\right). \quad (6)$$

Similarly, we can find t= t=-2, -1, 0, 4/15

$$\int_0^{4h/15} u(s)ds \approx h\left(\frac{34}{285}u_{4/15} + \frac{8}{10125}u_{-2} - \frac{1024}{192375}u_{-1} + \frac{1538}{10125}u_0\right), \quad (7)$$

and find t= t=-2, -1, 0, 2/3

$$\int_0^{2h/3} u(s)ds \approx h\left(\frac{4}{15}u_{2/3} + \frac{1}{81}u_{-2} - \frac{28}{405}u_{-1} + \frac{37}{81}u_0\right), \quad (8)$$

and find t= t=-2, -1, 0, 5/6

$$\int_0^{5h/6} u(s)ds \approx h\left(\frac{85}{264}u_{5/6} + \frac{125}{5184}u_{-2} - \frac{3625}{28512}u_{-1} + \frac{3185}{5184}u_0\right), \quad (9)$$

and find t= t=-2, -1, 0, 1

$$\int_0^{h} u(s)ds \approx h\left(\frac{3}{8}u_1 + \frac{1}{24}u_{-2} - \frac{5}{24}u_{-1} + \frac{19}{24}u_0\right), \quad (10)$$

and find t= t=-2, -1, 0, 1/15

$$\int_0^{h/15} u(s)ds \approx h\left(\frac{31}{960}u_1 + \frac{1}{81000}u_{-2} - \frac{61}{648000}u_{-1} + \frac{2791}{81000}u_0\right), \quad (11)$$

and finally find t= t=-2, -1, 0, 1

$$\int_0^{h} u(s)ds \approx h\left(\frac{3}{8}u_1 + \frac{1}{24}u_{-2} - \frac{5}{24}u_{-1} + \frac{19}{24}u_0\right). \quad (12)$$

Therefore the Runge-Kutta-Verner formulae become $n \geq 3$ (for starting values we can use equation (4) and (5))

$$k_1 = h\tilde{u}_n F(t_n, \tilde{u}_n, \tilde{z}(t_n)) ,$$

$$\tilde{u}^a_{n+1/6} = \tilde{u}_n + \frac{k_1}{6},$$

$$k_2 = h\tilde{u}^a_{n+1/6} F(t_{n+1/6}, \tilde{u}^a_{n+1/6}, \tilde{z}_{n+1/6}),$$

$$k_2 = h\tilde{u}^a_{n+1/6} F\left(t_{n+1/6}, \tilde{u}^a_{n+1/6}, \tilde{z}_n + h\left[\frac{13}{168}\tilde{u}^a_{n+1/6} + \frac{1}{5184}\tilde{u}_{n-2} - \frac{25}{18144}\tilde{u}_{n-1} + \frac{469}{5184}\tilde{u}_n\right]\right),$$

$$\tilde{u}^b_{n+4/15} = \tilde{u}_n + \frac{4k_1}{75} + \frac{16k_2}{75},$$

$$k_3 = h\,\tilde{u}^b_{n+4/15} F(t_{n+4/15}, \tilde{u}^b_{n+4/15}, \tilde{z}_{n+4/15}),$$

$$k_3 = h\tilde{u}^b_{n+4/15} F\left(t_{n+4/15}, \tilde{u}^b_{n+4/15}, \tilde{z}_n + h\left[\frac{34}{285}\tilde{u}^b_{n+4/15} + \frac{8}{10125}\tilde{u}_{n-2} - \frac{1024}{192375}\tilde{u}_{n-1} + \frac{1538}{10125}\tilde{u}_n\right]\right),$$

$$\tilde{u}^{c}_{n+2/3} = \tilde{u}_n + \frac{5k_1}{6} - \frac{8k_2}{3} + \frac{5k_3}{2},$$

$$k_4 = h\,\tilde{u}^{c}_{n+2/3}\,F(t_{n+2/3}, \tilde{u}^{c}_{n+2/3}, \tilde{z}_{n+2/3}\ ),$$

$$k_4 = h\,\tilde{u}^{c}_{n+2/3}\,F\left(t_{n+2/3}, \tilde{u}^{c}_{n+2/3}, \tilde{z}_n + h\left[\frac{4}{15}\tilde{u}^{c}_{n+2/3} + \frac{1}{81}\tilde{u}_{n-2} - \frac{28}{405}\tilde{u}_{n-1} + \frac{37}{81}\tilde{u}_n\right]\right),$$

$$\tilde{u}^{d}_{n+5/6} = \tilde{u}_n + \frac{165k_1}{64} + \frac{55k_2}{6} - \frac{425k_3}{64} + \frac{85k_4}{96},$$

$$k_5 = h\tilde{u}^{d}_{n+5/6}\,F(t_{n+5/6}, \tilde{u}^{d}_{n+5/6}, \tilde{z}_{n+5/6}),$$

$$k_5 = h\tilde{u}^{d}_{n+5/6}\,F\left(t_{n+5/6}, \tilde{u}^{d}_{n+5/6}, \tilde{z}_n + h\left[\frac{85}{264}\tilde{u}^{d}_{n+5/6} + \frac{125}{5184}\tilde{u}_{n-2} - \frac{3625}{28512}\tilde{u}_{n-1} + \frac{3185}{5184}\tilde{u}_n\right]\right),$$

$$\tilde{u}^{e}_{n+1} = \tilde{u}_n + \frac{12k_1}{15} - 8k_2 + \frac{4015k_3}{612} - \frac{11k_4}{36} + \frac{88k_5}{255},$$

$$k_6 = h\tilde{u}^{e}_{n+1}\,F(t_{n+1}, \tilde{u}^{e}_{n+1}, \tilde{z}_{n+1}\ ),$$

$$k_6 = h\tilde{u}^{e}_{n+1}\,F\left(t_{n+1}, \tilde{u}^{e}_{n+1}, \tilde{z}_n + h\left[\frac{3}{8}\tilde{u}^{e}_{n+1} + \frac{1}{24}\tilde{u}_{n-2} - \frac{5}{24}\tilde{u}_{n-1} + \frac{19}{24}\tilde{u}_n\right]\right),$$

$$\tilde{u}^{f}_{n+1/15} = \tilde{u}_n + \frac{8263k_1}{15000} + \frac{124k_2}{75} - \frac{643k_3}{680} - \frac{81k_4}{250} + \frac{2484k_5}{10625},$$

$$k_7 = h\tilde{u}^{f}_{n+1/15}\ F(t_{n+1/15}, \tilde{u}^{f}_{n+1/15}, \tilde{z}_{n+1/15}\ ),$$

$$k_7 = h\tilde{u}^{f}_{n+1/15}\,F\left(t_{n+1/15}, \tilde{u}^{f}_{n+1/15}, \tilde{z}_n + h\left[\frac{31}{960}\tilde{u}^{f}_{n+1/15} + \frac{1}{81000}\tilde{u}_{n-2} - \frac{61}{648000}\tilde{u}_{n-1} + \frac{2791}{81000}\tilde{u}_n\right]\right),$$

$$\tilde{u}^{g}_{n+1} = \tilde{u}_n + \frac{3501k_1}{1720} - \frac{300}{43}k_2 + \frac{297275k_3}{52632} - \frac{319k_4}{2322} + \frac{24068k_5}{84065} + \frac{3850k_7}{26703},$$

$$k_8 = h\,\tilde{u}^{g}_{n+1}\,F(t_{n+1}, \tilde{u}^{g}_{n+1}, \tilde{z}_{n+1}\ ),$$

$$k_8 = h\tilde{u}^{g}_{n+1}\,F\left(t_{n+1}, \tilde{u}^{g}_{n+1}, \tilde{z}_n + h\left[\frac{3}{8}\tilde{u}^{e}_{n+1} + \frac{1}{24}\tilde{u}_{n-2} - \frac{5}{24}\tilde{u}_{n-1} + \frac{19}{24}\tilde{u}_n\right]\right), \tag{13}$$

and the sixth-order method

$\tilde{u}_{n+1} = \tilde{u}_n + \frac{3k_1}{40} + \frac{875}{2244}k_3 + \frac{23k_4}{72} + \frac{264k_5}{1955} + \frac{125k_7}{11592} + \frac{43k_8}{616}$ is used to estimate the error in the fifth-order method $\tilde{u}_{n+1} = \tilde{u}_n + \frac{13k_1}{160} + \frac{2375}{5984}k_3 + \frac{5k_4}{16} + \frac{12k_5}{85} + \frac{3k_6}{44}$.

In Example 3.1, we have used Runge-Kutta-Verner methods and numerical quadrature, trapezoidal rule, the 3-point rule is known as Simpson's 1/3 rule, the 4-point closed rule is Simpson's 3/8 rule, the 5-point closed rule is Boole's rule (Bode's rule), Weddle's rule, higher rules include the 6-point, 7-point and 8-point and their combinations.

Example 3.1: Consider a first order non-Linear Volterra integro-differential equation of the form

$$u'(t) = u(t)\left(r - c\int_0^t e^{-\delta(t-s)}u(s)\,ds\right),\ t \ge 0; \qquad u(0) = u_0. \tag{14}$$

For analytical solution of equation (14), take $\delta = 0$ in (14). In equation (14), if we choose $r = 2,\ c = 2,\ u_0 = 0.1$ we will get exact solution as

$u(t) = \dfrac{22(21+2\sqrt{110})e^{\sqrt{\frac{22}{5}}t}}{5\left(21+2\sqrt{110})e^{\sqrt{\frac{22}{5}}t}\right)^2}$. Now, writing $m(t) = \int_0^t u(s)ds$, this is the same as the differential equation

$$u'(t) = u(t)(r - cm(t)),$$

$$u'(t) = ru(t) - cm(t)u(t),$$

If $m(t) = \int_0^t u(s)ds$ then $m'(t) = u(t)$ and $m''(t) = u'(t)$. Additionally, $m(0) = 0$ and $m'(0) = u(0) = u_0$.

$$m''(t) = r\,m'(t)(r - cm(t)),$$

$$m''(t) = r\,m'(t) - c\,m'(t)\,m(t),$$

Equivalent to the modified logistic equation for $m(t)$,

$$m'(t) = r\,m(t) - \frac{c\,m^2(t)}{2} + m'(0),$$

$$m'(t) = r\,m(t) - \frac{c\,m^2(t)}{2} + u(0),$$

$$m'(t) = -\frac{c\,m^2(t)}{2} + r\,m(t) + u_0,$$

$$m'(t) = -\frac{c}{2}(m^2(t) - \frac{2r\,m(t)}{c} - \frac{2u_0}{c}),$$

$$m'(t) = -\frac{c}{2}(m(t) - a)(m(t) + b),$$

(15)

where a and b are roots of $m^2(t) - \dfrac{2r\,m(t)}{c} - \dfrac{2u_0}{c}$.

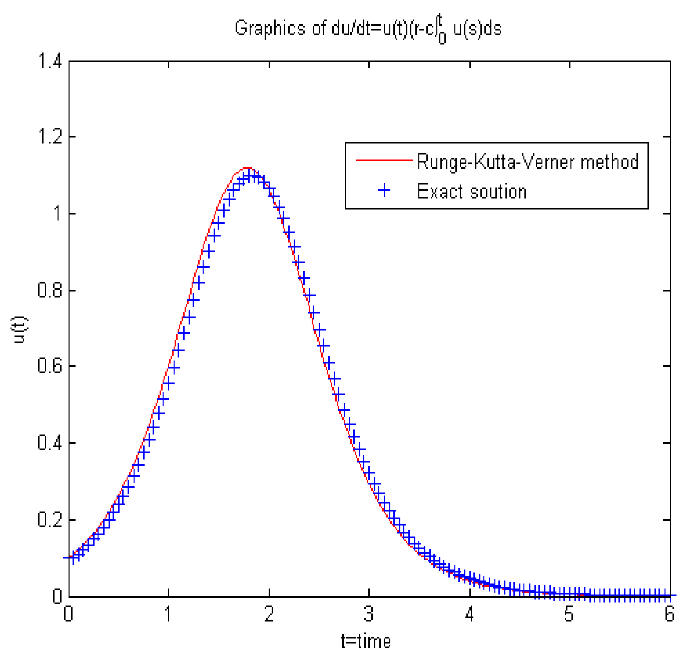
If we solve equation (15) with initial condition $m(0) = 0$ we get

$$m(t) = \frac{a\,e^{\frac{cat+cbt}{2}} - a}{e^{\frac{cat+cbt}{2}} + \frac{a}{b}},$$

After rearranging above solution we obtain

$$m(t) = \frac{a\,b(e^{t\frac{c}{2}(a+b)} - 1)}{b\,e^{t\frac{c}{2}(a+b)} + a},$$

$m(t) = \dfrac{a\,b(e^{\gamma t} - 1)}{b\,e^{\gamma t} + a}$, where $\gamma = \dfrac{c}{2}(a+b)$ and



$ab = \frac{2u_0}{c}$. We know that our exact solution was $u(t)$. When $m'(t) = u(t)$, $u(t) = \frac{u_0 (a+b)^2 e^{\gamma t}}{(a + b e^{\gamma t})^2}$ analytical solution of equation (14). Here a and b are roots of the equation $m^2(t) - \frac{2r\, m(t)}{c} - \frac{2u_0}{c}$, (where $u_0 > 0$) so that a is approached by m at large values of t. The exponent γ is defined by $\gamma = \frac{c}{2}(a+b)$.

Table 1: Numerical *solutions (14) for RKV method (* $r = 2,\ c = 2,\ u_0 = 0.1,\ \delta = 0,\ t_{\max} = 6$*).*

t	*Numerical solution h=0.00625*	*Numerical solution h=0.003125*	*Actual Solution Results*	*Error with h=0.00625*	*Error with h=0.003125*
1.0	0.5994306	0.5991739	0.5989171	5.1351e-04	2.5687e-04
2.0	1.0465870	1.0454132	1.0442420	2.3451e-03	1.1712e-03
3.0	0.2939327	0.2940388	0.2941451	2.1246e-04	1.0632e-04
4.0	0.0409158	0.0410287	0.0411417	2.2590e-04	1.1302e-04
5.0	0.0050827	0.0051087	0.0051348	5.2069e-05	2.6078e-05
6.0	0.0006223	0.0006269	0.0006316	9.2785e-06	4.6520e-06

Runge-Kutta-Verner method (RKV) and numerical quadrature rules results.

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