The Gumbel Marshall-Olkin-Weibull distribution: Properties and applications

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Abstract- In this paper, a new Gumbel generated Weibull distribution is introduced. The statistical properties including the quantile function, moments about the origin, incomplete moment and entropy of the new distribution are studied. The unknown parameters of the new distribution are estimated using the maximum likelihood approach. Furthermore, the asymptotic consistency behavior of the maximum likelihood estimates of the new distribution is evaluated through Monte Carlo simulation. Finally, the applicability and the potentials of the new distribution are illustrated in two real-life data sets.

Index Terms- Gumbel distribution, Marshall-Olkin distribution, Moments, Reliability, Maximum likelihood

I. INTRODUCTION

To provide adequate fit for real life data sets using classical distributions has proved difficult due to the heterogeneity of such data sets from assumed homogeneous population. Many of the classical distributions have been made more flexible to provide adequate fit to data sets by extending or introduction of additional parameter. There is increasing number of new distributions in the literature; however, of great importance is the ability to provide adequate fit to ever dynamic sample data from many different areas of study.

A class of distributions called a new Gumbel generated family of distribution having density function (pdf) given in (1) was proposed by [21].

$$G(x) = exp\left[-Bp^{\frac{1}{\sigma}}\left(\frac{F(x)}{1-F(x)}\right)^{-\frac{1}{\sigma}}\right]; \qquad (1)$$

where $B = exp\left(\frac{\mu}{\sigma}\right)$.

Our intention is to extend the Weibull distribution using equation (1). However, there are good number of distributions in the literature that were generated by either extending or introducing additional parameter to Weibull distribution such as but not limited to exponentiated Weibull proposed by ([5],[14]), Marshall-Olkin extended Weibull proposed by [10], beta- Weibull introduced by [7], Kumaraswamy Weibull introduced by [6], Gamma-Weibull introduced by [8], Transmuted Weibull by [12], Gumbel-Weibull [9], The Topp-Leone generated Weibull introduced by [11], the Lindley Weibull by [13].

This paper aims at generating a flexible distribution with unimodal, bimodal, skewed, monotone increasing and decreasing pdf shapes, and among many other failure rate shapes, bathtub shape as shown in Figure 1 and 3.

The importance and areas of applicability of bimodal distribution were highlighted by [2] and references therein. Furthermore, [3] and [2] noted that class of lifetime distribution having a bathtub shape failure rate function is very important in the study of lifetime of electronic, electromechanical, mechanical products, and survival analysis; see also [4].

The remainder of paper is structured as follows. The new distribution is derived in section 2. In section 3, we considered moments of the new distribution. Reliability functions are derived in section 4 while the entropy is considered in section 5. The parameter estimation is considered in section 6. In section 7, real-life applications are considered. Finally, section 8 is the concluding remark.

II. THE NEW DISTRIBUTION

Consider the pdf $F(x)$ of a random variable defined by

$$F(x) = 1 - exp\left[-\left(\frac{x}{\theta}\right)^{\lambda}\right]; \; x, \theta, \lambda \in \mathbb{R}^{+}. \qquad (2)$$

The cdf of the new distribution called the Gumbel Marshall-Olkin Weibull is generated by substituting equation (2) in equation (1), and it is given by

$$G(x) = exp\left[-Bp^{\frac{1}{\sigma}}\left(\exp\left[-\left(\frac{x}{\theta}\right)^{\lambda}\right]-1\right)^{-\frac{1}{\sigma}}\right] \qquad (3)$$

The density function (pdf) of GMO-W is obtained by taking the derivative of equation (3) with respect to (w.r.t) x and it is given by

$$g(x) = \frac{Bp^{\frac{1}{\sigma}}\lambda x^{\lambda-1}\exp\left[\left(\frac{x}{\theta}\right)^{\lambda}\right]}{\sigma\theta^{\lambda}\left\{\exp\left[\left(\frac{x}{\theta}\right)^{\lambda}\right]-1\right\}^{\frac{1}{\sigma}+1}} \times$$
$$exp\left[-Bp^{\frac{1}{\sigma}}\left(\exp\left[\left(\frac{x}{\theta}\right)^{\lambda}\right]-1\right)^{-\frac{1}{\sigma}}\right] \qquad (4)$$

Proposition 1: If random variable Y follows Gumbel distribution, then random variable $X = \theta[\ln(p\ e^{Y}+1)]^{\frac{1}{\lambda}}$ follows Gumbel Marshall-Olkin Weibull distribution.

Proof

The pdf of random variable Y is given by

$$f(y) = \frac{exp\left(\frac{y-\mu}{\sigma}\right) exp\left[-exp\left(\frac{y-\mu}{\sigma}\right)\right]}{\sigma}$$

We have that if $x = \theta[\ln(p\ e^{y}+1)]^{\frac{1}{\lambda}}$ then $y = log\left[p^{-1}\left(\exp\left[\left(\frac{x}{\theta}\right)^{\lambda}\right]-1\right)\right]$ and $\frac{dy}{dx} = \frac{\lambda x^{\lambda-1}}{\theta^{\lambda}\left(1-\exp\left[\left(\frac{x}{\theta}\right)^{\lambda}\right]\right)}$.

By transformation, the pdf of x is defined as

$$g(x) = f(y)\left|\frac{dy}{dx}\right| \qquad (5)$$

Substituting the values of y and $\left|\frac{dy}{dx}\right|$ in equation (5) yields

$$g(x) = \frac{Bp^{\frac{1}{\sigma}}\left\{\exp\left[\left(\frac{x}{\theta}\right)^{\lambda}\right]-1\right\}^{-\frac{1}{\sigma}}}{\sigma} \times$$
$$exp\left(-Bp^{\frac{1}{\sigma}}\left\{\exp\left[\left(\frac{x}{\theta}\right)^{\lambda}\right]-1\right\}^{-\frac{1}{\sigma}}\right)\frac{\lambda x^{\lambda-1}}{\theta^{\lambda}\left(1-\exp\left[\left(\frac{x}{\theta}\right)^{\lambda}\right]\right)}$$

Simplifying further, we obtain the pdf of X as

$$g(x) = \frac{Bp^{\frac{1}{\sigma}}\lambda x^{\lambda-1}\exp\left[\left(\frac{x}{\theta}\right)^{\lambda}\right]}{\sigma\theta^{\lambda}\left\{\exp\left[\left(\frac{x}{\theta}\right)^{\lambda}\right]-1\right\}^{\frac{1}{\sigma}+1}} \times$$
$$exp\left[-Bp^{\frac{1}{\sigma}}\left(\exp\left[\left(\frac{x}{\theta}\right)^{\lambda}\right]-1\right)^{-\frac{1}{\sigma}}\right].$$

End of proof.

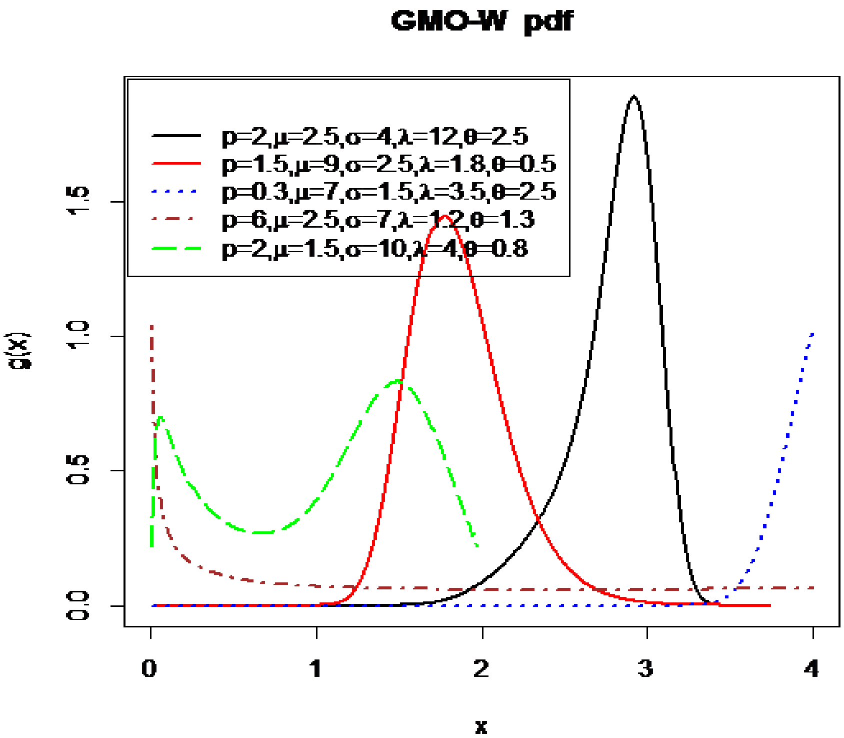

Figure 1. Plots of GMO-W pdf for different parameter values.

III. STATISTICAL PROPERTIES

In this section, we derived some of the statistical properties of the GMO-W distribution.

A. Quantile Function

The quantile function of GMO-W denoted by $Q(u)$ is obtained by applying the probability integral transform on equation (3) and it is given by

$$Q(u) = \theta(\ln\{1 - pB^{\sigma}[\ln(u)]^{-\sigma}\})^{\frac{1}{\lambda}}, \quad u \in (0,1). \qquad (6)$$

We can obtain the α^{th} percentile by substituting appropriately the corresponding values in equation (6). For instance, the 50^{th}percentile which is the median (M) can be obtained by substituting the corresponding value $u = 0.5$ in equation (5), this yields $M = \theta(\ln\{1 - pB^{\sigma}[\ln(0.5)]^{-\sigma}\})^{\frac{1}{\lambda}}$.

By equation (6), we can obtain an alternative measure of kurtosis proposed by [15] and defined as

$$k = \frac{Q\left(\frac{7}{8}\right) - Q\left(\frac{5}{8}\right) + Q\left(\frac{3}{8}\right) - Q\left(\frac{1}{8}\right)}{Q\left(\frac{6}{8}\right) - Q\left(\frac{2}{8}\right)}$$

Figure 1 shows the plots of GMO-W and Weibull kurtosis at respective parameter values $p = 0.5, \mu = 0.5, \sigma = 3, \lambda = 0.5\ to\ 3, \theta = 0.1\ to\ 10.1$ to 1and $\lambda = 0.5\ to\ 3, \theta = 0.1\ to\ 1$. Figure 1 plot reveals that GMO-W has heavier tail than the Weibull distribution.

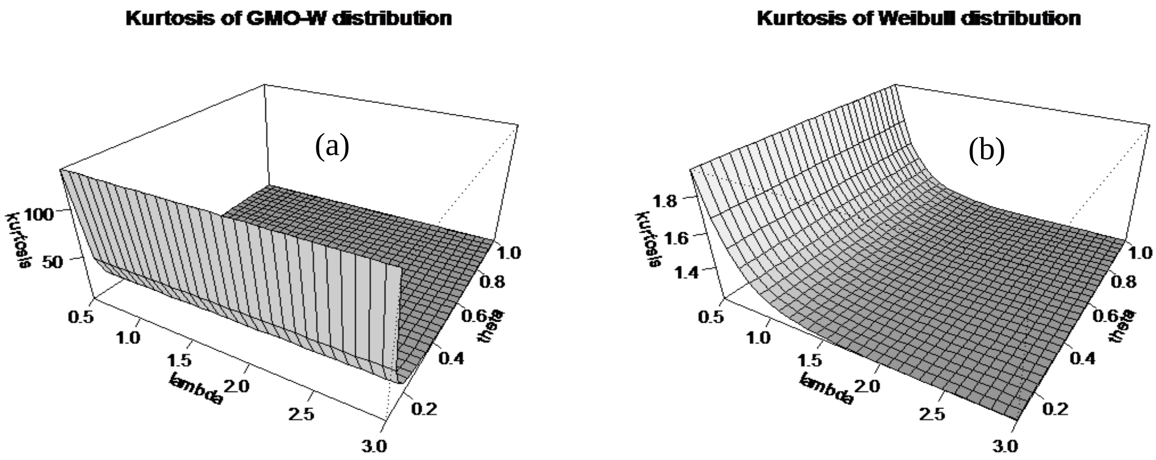

Figure 2. Plots ofKurtosis a) GMO-W b) Weibull distribution.

B. Moments

The n^{th} moment about the origin say μ_n' of a random variable X having GMO quantile function of GMO-W is defined as

$$\mu_n' = E(X^n) = \int_0^\infty x^n g(x)dx \qquad (7)$$

where $g(x)$ is as defined as in equation (4). Substituting equation (4) in (7) and applying the general binomial series expansion that

$$\mu_n' = \frac{Bp^{\frac{1}{\sigma}}}{\sigma\theta^\lambda} \sum_{i,j=0}^{\infty} \frac{(-1)^i}{i!} \left(Bp^{\frac{1}{\sigma}}\right)^i \binom{\frac{i}{\sigma}+\frac{1}{\sigma}+j}{j} \times$$
$$\int_0^\infty x^{n+\lambda-1} exp\left[-\left(\frac{i}{\sigma}+\frac{1}{\sigma}+j\right)\left(\frac{x}{\theta}\right)^\lambda\right]dx \qquad (8)$$

Letting $y = \left(\frac{i}{\sigma}+\frac{1}{\sigma}+j\right)\left(\frac{x}{\theta}\right)^\lambda$, implies that $x = \frac{\theta y^{\frac{1}{\lambda}}}{\left(\frac{i}{\sigma}+\frac{1}{\sigma}+j\right)^{\frac{1}{\lambda}}}$ and $dx = \frac{\theta y^{\frac{1}{\lambda}-1}}{\lambda\left(\frac{i}{\sigma}+\frac{1}{\sigma}+j\right)^{\frac{1}{\lambda}}}dy$. By change of variable, equation (8) becomes

$$\mu_n' = \frac{Bp^{\frac{1}{\sigma}}\theta^n}{\sigma\left(\frac{i}{\sigma}+\frac{1}{\sigma}+j\right)^{\frac{n}{\lambda}+1}} \sum_{i,j=0}^{\infty} \frac{(-1)^i}{i!} \left(Bp^{\frac{1}{\sigma}}\right)^i \binom{\frac{i}{\sigma}+\frac{1}{\sigma}+j}{j} \times$$
$$\int_0^\infty y^{\frac{n}{\lambda}} exp(-y)dy$$
$$= \frac{Bp^{\frac{1}{\sigma}}\theta^n}{\sigma\left(\frac{i}{\sigma}+\frac{1}{\sigma}+j\right)^{\frac{n}{\lambda}+1}} \sum_{i,j=0}^{\infty} \frac{(-1)^i}{i!} \left(Bp^{\frac{1}{\sigma}}\right)^i \binom{\frac{i}{\sigma}+\frac{1}{\sigma}+j}{j} \times$$
$$\Gamma\left(\frac{n}{\lambda}+1\right)$$

The n^{th}incomplete moment was proposed by [16] to determine the shape distribution. The n^{th} incomplete moment for a random variable X with pdf $f(x)$ is defined as
$I(w;n) = \int_{-\infty}^{w} x^n f(x)dx$
If X follows GMO-W, then the n^{th} incomplete moment is defined by

$$I(w;n) = \int_0^w x^n g(x)\,dx\,; \qquad (9)$$

where $g(x)$ is defined as in equation (4). Applying the general binomial series expansion to equation (9) yields

$$I(w;n) = \frac{B\lambda p^{\frac{1}{\sigma}}}{\sigma\theta^{\lambda}} \sum_{i,j=0}^{\infty} \frac{(-1)^i}{i!} \left(Bp^{\frac{1}{\sigma}}\right)^i \binom{\frac{i}{\sigma}+\frac{1}{\sigma}+j}{j} \times$$
$$\int_0^w x^{n+\lambda-1} exp\left[-\left(\frac{i}{\sigma}+\frac{1}{\sigma}+j\right)\left(\frac{x}{\theta}\right)^{\lambda}\right] dx \qquad (10)$$

Applying change of variable in the integral above by letting $q = \left(\frac{i}{\sigma}+\frac{1}{\sigma}+j\right)\left(\frac{x}{\theta}\right)^{\lambda}$, equation (10) becomes

$$I(w;n) = \frac{B\lambda p^{\frac{1}{\sigma}}\theta^n}{\sigma\left(\frac{i}{\sigma}+\frac{1}{\sigma}+j\right)^{\frac{n}{\lambda}+1}} \sum_{i,j=0}^{\infty} \frac{(-1)^i}{i!}\left(Bp^{\frac{1}{\sigma}}\right)^i \times$$
$$\binom{\frac{i}{\sigma}+\frac{1}{\sigma}+j}{j} \int_0^w q^{\frac{n}{\lambda}} exp(-q) dq$$
$$= \frac{B\lambda p^{\frac{1}{\sigma}}\theta^n}{\sigma\left(\frac{i}{\sigma}+\frac{1}{\sigma}+j\right)^{\frac{n}{\lambda}+1}} \sum_{i,j=0}^{\infty} \frac{(-1)^i}{i!}\left(Bp^{\frac{1}{\sigma}}\right)^i \times$$
$$\binom{\frac{i}{\sigma}+\frac{1}{\sigma}+j}{j}\left(\Gamma\left(\frac{n}{\lambda}+1\right) - \Gamma\left(\frac{n}{\lambda}+1,w\right)\right).$$

For $n = 0$, equation (9) becomes equation (3).

C.TRIMMED-L MOMENT

The Trimmed-L (TL) moments was proposed by [17] as a robust method of estimation for population central moments. The n^{th} TL-moment of a random variable X with respective cdf $F(x)$ and pdf $f(x)$ for $t_1 = t_2 = t$ is given by

$$\lambda_n^t = \frac{1}{n}\sum_{k=0}^{n-1}(-1)^k \binom{n-1}{k} E(Y_{n+t-k:n+2t}); \; n = 1,2,\ldots \quad (11)$$

where

$$E(Y_{i:m}) = \frac{m!}{(i-1)!\,(m-i)!}\sum_{j=0}^{m-1}(-1)^j\binom{m-1}{i} \times$$
$$\int_{-\infty}^{\infty} x f(x) F(x)^{i+j-1}\, dx$$

Substituting $f(x)$ and $F(x)$ in equation (11) with equations (4) and (3) respectively, applying the general binomial series expansion, the n^{th} TL moment of GMO-W can be expressed as

$$\lambda_n^t = \frac{B\lambda p^{\frac{1}{\sigma}}}{n\sigma\theta^{\lambda}}\sum_{k=0}^{n-1}(-1)^k\binom{n-1}{k}\frac{(n+2t)!}{(n+t-k-1)!\,(t+k)!} \times$$
$$\sum_{i=0}^{t+k}(-1)^i\binom{t+k}{i}\sum_{j,m=0}^{\infty}\frac{(-1)^j}{j!}\left[(n-t-k+i)Bp^{\frac{1}{\sigma}}\right]^j \times$$
$$\binom{\frac{j}{\sigma}+\frac{1}{\sigma}+m}{m}\int_0^{\infty} x^{\lambda} exp\left[\left(\frac{j}{\sigma}+\frac{1}{\sigma}+m\right)\left(\frac{x}{\theta}\right)^{\lambda}\right] dx \qquad (12)$$

By change of variable in the integral of equation (12), the n^{th} TL moment of GMO-W is given by

$$\lambda_n^t = \frac{B\theta p^{\frac{1}{\sigma}}}{n\sigma}\sum_{k=0}^{n-1}(-1)^k\binom{n-1}{k}\frac{(n+2t)!}{(n+t-k-1)!\,(t+k)!} \times$$
$$\sum_{i=0}^{t+k}(-1)^i\binom{t+k}{i}\sum_{j,m=0}^{\infty}\frac{(-1)^j}{j!\left(\frac{j}{\sigma}+\frac{1}{\sigma}+m\right)^{\frac{1}{\lambda}+1}} \times$$

$$\left[(n-t-k+i)Bp^{\frac{1}{\sigma}}\right]^{j}\binom{\frac{j}{\sigma}+\frac{1}{\sigma}+m}{m}\Gamma(\lambda+1)$$

At $t = 1$ and $n = 1,2,3,4$, the TL-moment corresponds to measures of location, scale, skewness and kurtosis respectively.

IV. RELIABILITY

The hazard rate function (hrf) of a random variable X is defined by $hrf = f(x)/1 - F(x)$. The corresponding hrf of GMO-W random variable is given by

$$hrf = \frac{Bp^{\frac{1}{\sigma}}\lambda x^{\lambda-1}\exp\left[\left(\frac{x}{\theta}\right)^{\lambda}\right]}{\sigma\theta^{\lambda}\left\{\exp\left[\left(\frac{x}{\theta}\right)^{\lambda}\right]-1\right\}^{\frac{1}{\sigma}+1}}\times \frac{exp\left[-Bp^{\frac{1}{\sigma}}\left(\exp\left[\left(\frac{x}{\theta}\right)^{\lambda}\right]-1\right)^{-\frac{1}{\sigma}}\right]}{1-exp\left[-Bp^{\frac{1}{\sigma}}\left(\exp\left[\left(\frac{x}{\theta}\right)^{\lambda}\right]-1\right)^{-\frac{1}{\sigma}}\right]}.$$

Figure 3 shows the plots of GMO-W hazard rate function with different parameter values.

Suppose the life of a component which has a random strength X_1 and it is subjected to random stress X_2. The component survives the stress whenever $X_1 > X_2$ hence, $R = P(X_2 < X_1)$ measures the reliability of the component.
Let X_1 and X_2 follow GMO-Weibull, then

$$R = \int_0^{\infty} \frac{Bp^{\frac{1}{\sigma}}\lambda x^{\lambda-1}\exp\left[\left(\frac{x}{\theta}\right)^{\lambda}\right]}{\sigma\theta^{\lambda}\left\{\exp\left[\left(\frac{x}{\theta}\right)^{\lambda}\right]-1\right\}^{\frac{1}{\sigma}+1}}\times exp\left[-2Bp^{\frac{1}{\sigma}}\left(\exp\left[\left(\frac{x}{\theta}\right)^{\lambda}\right]-1\right)^{-\frac{1}{\sigma}}\right] \qquad (13)$$

Simplifying equation (13) yields

$$R = \frac{Bp^{\frac{1}{\sigma}}\lambda}{\sigma\theta^{\lambda}}\sum_{i,j=0}^{\infty}\frac{(-1)^i}{i!}\left(2Bp^{\frac{1}{\sigma}}\right)^{i}\binom{\frac{2}{\sigma}+j}{j}\times \int_0^{\infty} exp\left[-\left(\frac{2}{\sigma}+j\right)\left(\frac{x}{\theta}\right)^{\lambda}\right]dx$$

$$= \frac{\theta^{-\lambda+1}Bp^{\frac{1}{\sigma}}\lambda}{\sigma}\sum_{i,j=0}^{\infty}\frac{(-1)^i}{i!}\left(2Bp^{\frac{1}{\sigma}}\right)^{i}\binom{\frac{2}{\sigma}+j}{j}\frac{\Gamma\left(\frac{1}{\lambda}+1\right)}{\left(\frac{2}{\sigma}+j\right)^{\frac{1}{\lambda}}}$$

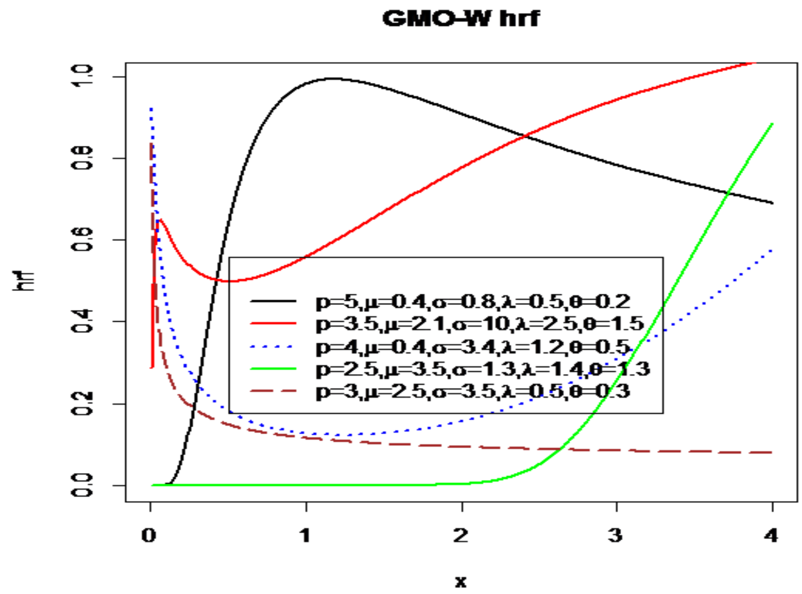

Figure 3. Plots of GMO-W hrf for different parameter values.

V. ENTROPY

The measure of uncertainty of random variable X having GMO-W pdf is determined using Renyi entropy. For a random variable X having pdf $f(x)$, the Renyi entropy introduced by [18] is defined by

$$I_R(\gamma)=\frac{1}{1-\gamma}log\left(\int_{-\infty}^{\infty}f^{\gamma}(x)\,dx\right) \qquad (14)$$

Substituting equation (4) in equation (14) yields equation (15) on simplification

$$I_{RGMO-W}(\gamma)=\frac{1}{1-\gamma}log\left\{\left(\frac{Bp^{\frac{1}{\sigma}}\lambda}{\sigma\theta^{\lambda}}\right)^{\gamma}\sum_{i,j=0}^{\infty}\frac{(-1)^i}{i!}\times\left(\gamma Bp^{\frac{1}{\sigma}}\right)^i\binom{\gamma\left(\frac{1}{\sigma}+1\right)+\frac{i}{\sigma}+j-1}{j}\times\int_{-\infty}^{\infty}x^{\gamma(\lambda-1)}exp\left[\left(\frac{\gamma}{\sigma}+\frac{i}{\sigma}+j\right)\left(\frac{x}{\sigma}\right)^{\lambda}\right]dx\right\} \qquad (15)$$

Letting $w=\left(\frac{\gamma}{\sigma}+\frac{i}{\sigma}+j\right)\left(\frac{x}{\sigma}\right)^{\lambda}$ and by change of variable,
the integral in equation (15) becomes

$$\frac{\theta^{\gamma(\lambda-1)+1}}{\lambda\left(\frac{\gamma}{\sigma}+\frac{i}{\sigma}+j\right)^{\frac{1}{\lambda}[1+\gamma(\lambda-1)]}}\Gamma\left(\frac{1}{\lambda}[1+\gamma(\lambda-1)]\right)$$

and the Renyi entropy for GMO-W becomes

$$I_{RGMO-W}(\gamma)=\frac{1}{1-\gamma}log\left\{\left(\frac{Bp^{\frac{1}{\sigma}}}{\sigma}\right)^{\gamma}\sum_{i,j=0}^{\infty}\frac{(-1)^i}{i!}\times\left(\gamma Bp^{\frac{1}{\sigma}}\right)^i\binom{\gamma\left(\frac{1}{\sigma}+1\right)+\frac{i}{\sigma}+j-1}{j}\times\frac{\gamma^{\gamma-1}\theta^{1-\gamma}}{\left(\frac{\gamma}{\sigma}+\frac{i}{\sigma}+j\right)^{\frac{1}{\lambda}[1+\gamma(\lambda-1)]}}\Gamma\left(\frac{1}{\lambda}[1+\gamma(\lambda-1)]\right)\right\}.$$

VI. ESTIMATION

Let $X_1, X_2, \ldots, X_n$ be a random sample of size n drawn from GMO-W population. The log-likelihood function of the sample is given by

$$\ell(\Theta) = \frac{n\mu}{\sigma} + n\,log\left(p^{\frac{1}{\sigma}}\right) + n\,log(\lambda) + (\lambda - 1)\sum_{i=1}^{n} \log(x_i) + \sum_{i=1}^{n}\left(\frac{x_i}{\theta}\right)^{\lambda} - exp\left(\frac{\mu}{\sigma}\right)p^{\frac{1}{\sigma}}\sum_{i=1}^{n}\left(exp\left[\left(\frac{x_i}{\sigma}\right)^{\lambda}\right] - 1\right)^{-\frac{1}{\sigma}} - n\,log(\sigma) - n\,\lambda\,log(\theta) - \left(\frac{1}{\sigma} + 1\right)\sum_{i=1}^{n} log\left(exp\left[\left(\frac{x_i}{\sigma}\right)^{\lambda}\right] - 1\right). \quad (16)$$

The score functions $\left(\frac{\partial\ell(\Theta)}{\partial p}, \frac{\partial\ell(\Theta)}{\partial\mu}, \frac{\partial\ell(\Theta)}{\partial\sigma}, \frac{\partial\ell(\Theta)}{\partial\theta}, \frac{\partial\ell(\Theta)}{\partial\lambda}\right)'$ associated to equation (16) are given as follows

$$\frac{\partial\ell(\Theta)}{\partial p} = \frac{n}{\sigma p} - \frac{exp\left(\frac{\mu}{\sigma}\right)}{\sigma}p^{\frac{1}{\sigma}-1}\sum_{i=1}^{n}\left(exp\left[\left(\frac{x_i}{\sigma}\right)^{\lambda}\right] - 1\right)^{-\frac{1}{\sigma}}$$

$$\frac{\partial\ell(\Theta)}{\partial\mu} = \frac{n}{\sigma} - \frac{exp\left(\frac{\mu}{\sigma}\right)}{\sigma}p^{\frac{1}{\sigma}}\sum_{i=1}^{n}\left(exp\left[\left(\frac{x_i}{\sigma}\right)^{\lambda}\right] - 1\right)^{-\frac{1}{\sigma}}$$

$$\frac{\partial\ell(\Theta)}{\partial\sigma} = \frac{\mu exp\left(\frac{\mu}{\sigma}\right)}{\sigma^2}p^{\frac{1}{\sigma}}\sum_{i=1}^{n}\left(exp\left[\left(\frac{x_i}{\sigma}\right)^{\lambda}\right] - 1\right)^{-\frac{1}{\sigma}} - \frac{n}{\sigma} + \frac{log(p)p^{\frac{1}{\sigma}}}{\sigma^2}exp\left(\frac{\mu}{\sigma}\right)\sum_{i=1}^{n}\left(exp\left[\left(\frac{x_i}{\sigma}\right)^{\lambda}\right] - 1\right)^{-\frac{1}{\sigma}} - \frac{p^{\frac{1}{\sigma}}}{\sigma^2}exp\left(\frac{\mu}{\sigma}\right)\sum_{i=1}^{n} log\left(exp\left[\left(\frac{x_i}{\sigma}\right)^{\lambda}\right] - 1\right) \times \left(exp\left[\left(\frac{x_i}{\sigma}\right)^{\lambda}\right] - 1\right)^{-\frac{1}{\sigma}} + \frac{1}{\sigma}\sum_{i=1}^{n} log\left(exp\left[\left(\frac{x_i}{\sigma}\right)^{\lambda}\right] - 1\right)$$

$$\frac{\partial\ell(\Theta)}{\partial\theta} = -\lambda\theta^{-(\lambda+1)}\sum_{i=1}^{n} x_i^{\lambda} + \frac{exp\left(\frac{\mu}{\sigma}\right)p^{\frac{1}{\sigma}}\lambda\theta^{-(\lambda+1)}}{\sigma} \times \sum_{i=1}^{n} x_i^{\lambda}\,exp\left[\left(\frac{x}{\theta}\right)^{\lambda}\right]\left(exp\left[\left(\frac{x_i}{\sigma}\right)^{\lambda}\right] - 1\right)^{-\left(\frac{1}{\sigma}+1\right)} - \frac{n\lambda}{\theta} + \left(\frac{1}{\sigma} + 1\right)\lambda\theta^{-(\lambda+1)}\sum_{i=1}^{n}\frac{x_i^{\lambda}exp\left[\left(\frac{x}{\theta}\right)^{\lambda}\right]}{exp\left[\left(\frac{x_i}{\sigma}\right)^{\lambda}\right] - 1}$$

$$\frac{\partial\ell(\Theta)}{\partial\lambda} = \frac{n}{\lambda} + \sum_{i=1}^{n} log(x_i) + \sum_{i=1}^{n}\left(\frac{x}{\theta}\right)^{\lambda} log\left(\frac{x}{\theta}\right) + \frac{exp\left(\frac{\mu}{\sigma}\right)}{\sigma} \times p^{\frac{1}{\sigma}}\sum_{i=1}^{n}\left(exp\left[\left(\frac{x_i}{\sigma}\right)^{\lambda}\right] - 1\right)^{-\left(\frac{1}{\sigma}+1\right)} exp\left[\left(\frac{x}{\theta}\right)^{\lambda}\right]\left(\frac{x}{\theta}\right)^{\lambda} log\left(\frac{x}{\theta}\right) - n\,log(\theta) + \left(\frac{1}{\sigma} + 1\right)\sum_{i=1}^{n}\frac{exp\left[\left(\frac{x}{\theta}\right)^{\lambda}\right]\left(\frac{x}{\theta}\right)^{\lambda} log\left(\frac{x}{\theta}\right)}{exp\left[\left(\frac{x_i}{\sigma}\right)^{\lambda}\right] - 1}$$

The maximum likelihood estimators for the unknown parameters of GMO-W distribution can be obtained by equating the score functions to zero and solving simultaneously for the respective parameters. However, the resultant equations are non-linear hence; the

maximum likelihood estimates can be evaluated by using non-linear iterative optimization method such as BFGS which is implementable in R statistical software.
The Fisher Information Matrix ($I_{\_GMOW}(\Theta)$) can be obtain by taking the negative expectation of the second partial derivative of the log-likelihood with respect to parameter given by $-E\left(\frac{\partial^2 \ell(\Theta)}{\partial\Theta_i\partial\Theta_j}\right)$. Under certain regularities $\sqrt{n}(\widehat{\Theta}-\Theta)$ follows multivariate normal $N_5(0,V)$; where variance-covariance V is the inverse of the Fisher Information Matrix denoted by $I^{-1}_{GMOW}(\Theta)$.
The asymptotic consistency of the maximum likelihood is evaluated through simulation study. Random samples of sizes $n = 50, 75, 100, 150, 00, 150, 250$ are generated using equation (6) for $N = 500$ times with parameter values $\Theta = (p = 2, \mu = 1.8, \sigma = 0.3, \lambda = 1.3, \theta = 0.2)$. For each N_i $i = 1,2, \dots, 500$, parameter estimates $\widehat{\Theta}$ are determined and mean estimate, biases and mean square errors (mse) computed using :

$$mean = \frac{1}{N}\sum_{i=1}^{N}\widehat{\Theta}$$

$$bias = \frac{1}{N}\sum_{i=1}^{N}(\widehat{\Theta}-\Theta)$$

and

$$mse = \frac{1}{N}\sum_{i=1}^{N}(\widehat{\Theta}-\Theta)^2$$

Table 1 shows the results of the simulation study. The values in Table 1 indicate that as the sample size increases, the mean estimates tends to the true parameter values and the mse decreases as expected.

Table1. Result of simulation study

Sample size ($\boldsymbol{n}$)	Parameter	Mean value	Bias	Mse
50	$p = 2$	2.0564	0.0564	0.7337
	$\mu = 1.8$	1.4452	-0.3548	4.4357
	$\sigma = 0.3$	0.6245	0.3245	0.8087
	$\lambda = 1.3$	4.5204	3.2204	96.0671
	$\theta = 0.2$	0.2027	0.0027	0.0199
75	$p = 2$	2.0661	0.0661	0.6418
	$\mu = 1.8$	1.5329	-0.2033	3.9769
	$\sigma = 0.3$	0.5329	0.2329	0.5287
	$\lambda = 1.3$	3.4518	2.1518	63.0859
	$\theta = 0.2$	0.1922	-0.0078	0.0163
100	$p = 2$	2.0318	0.0319	0.6341
	$\mu = 1.8$	1.5863	-0.2137	3.2444
	$\sigma = 0.3$	0.5011	0.2011	0.4342
	$\lambda = 1.3$	3.2395	1.9395	54.6662
	$\theta = 0.2$	0.1981	-0.0019	0.0141
150	$p = 2$	2.090	0.0981	0.3614
	$\mu = 1.8$	1.8360	0.0360	1.4709
	$\sigma = 0.3$	0.3966	0.0966	0.1409
	$\lambda = 1.3$	2.0468	0.7469	15.2519
	$\theta = 0.2$	0.1876	-0.0124	0.0091
250	$p = 2$	2.1436	0.1436	0.1809
	$\mu = 1.8$	2.0611	0.2611	0.6481
	$\sigma = 0.3$	0.3530	0.0530	0.0512
	$\lambda = 1.3$	1.5180	0.2182	3.5316
	$\theta = 0.2$	0.1718	-0.0282	0.0065

VII. APPLICATIONS

Two real-life data sets are used to illustrate the applicability and potentials of GMOW distribution to provide better fit among other existing distributions in the literature; Beta-Weibull (BW), Gamma-Weibull (GW), Exponentiated-Weibull (ExpW), Marshall-Olkin extended Weibull (MOW), Kumaraswamy-Weibull (KW), and Weibull distribution. The Cramer von Mises, Anderson darling, Akaike Information Criterion (AIC), and Kolmogorov-Siminov (K-S) goodness-of-fit statistics are used to determine the model of best fit. Generally, model with the least value of goodness-of-fit statistics is adjudged the best.

The first data set is on the lifetime of 50 industrial device put on life test. The data set is reported by [19]. The data set is given as Data set 1: 0.1, 0.2, 1.0, 1.0, 1.0, 1.0, 1.0, 2.0, 3.0, 6.0, 7.0, 11.0, 12.0, 18.0, 18.0, 18.0, 18.0, 18.0, 21.0, 32.0, 36.0, 40.0, 45.0, 45.0, 47.0, 50.0, 55.0, 60.0, 63.0, 63.0, 67.0, 67.0, 67.0, 67.0, 72.0, 75.0, 79.0, 82.0, 82.0, 83.0, 84.0, 84.0, 84.0, 85.0, 85.0, 85.0, 85.0, 85.0, 86.0, 86.0.
The second data set represent observed survival times (weeks) for AG positive. The data set is concave-convex and it is reported by [20]. The data set is given as data set 2: 65, 156, 100, 134, 16, 108, 121, 4, 39, 143, 56, 26, 22, 1, 1, 5, 65.
The parameter estimates and associated standard errors of the competing distributions are determined for the two data sets (Data set 1 and Data set 2); they are given respectively in Tables 2 and 4. Furthermore, the estimates goodness-of-fit statistics are given respectively in Tables 3 and 5 for Data set 1 and Data set 2.

Table 2. Parameter estimate (standard errors) for Data set 1

Distribution	Parameter Estimate (Standard errors)				
$GMOW(p,\mu,\sigma,\lambda,\theta)$	0.1713 (1.0837)	-0.2879 (6.8924	18.2782 (2.0306)	5.0512 (0.0389	40.8369 (0.0389)
$BW(a,\beta,\lambda,\theta)$	0.2216 (0.0543)	0.0555 (0.0084	1.6764 (0.0032)	10.3829 (0.003)	
$GW(a,\lambda,\theta)$	1.4616 (0.1437)	0.6872 (0.0828)	69.5254 (22.1679)		
$ExpW(a,\lambda,\theta)$	0.1378 (0.0195)	5.1300 (0.0059)	87.1016 (0.9588)		
$MOW(p,\lambda,\theta)$	6.6762 (5.2876)	0.6994 (0.1499)	13.9565 (8.8993)		
$KW(a,b,\lambda,\theta)$	0.1557 (0.0212)	0.0634 (0.0091)	1.5445 (0.0123)	9.3740 (0.0132)	
$Weibull(\lambda,\theta)$	0.9489 (0.1195)	44.8762 (6.9396)			

Table 3. Goodness-of-fit statistics for Data set 1

Distribution	CvM	An	AIC	K-S
$GMOW(p,\mu,\sigma,\lambda,\theta)$	0.0974	0.8255	445.315	0.1289
$BW(a,\beta,\lambda,\theta)$	0.3000	1.9339	470.6071	0.1798
$GW(a,\lambda,\theta)$	0.5729	3.4035	462.8996	0.4953
$ExpW(a,\lambda,\theta)$	0.2459	1.6586	463.6542	0.2252
$MOW(p,\lambda,\theta)$	1.5570	7.9348	481.4148	0.9262
$KW(a,b,\lambda,\theta)$	0.3422	2.1868	476.6580	0.1732
$Weibull(\lambda,\theta)$	0.4947	3.0004	485.9592	0.1931

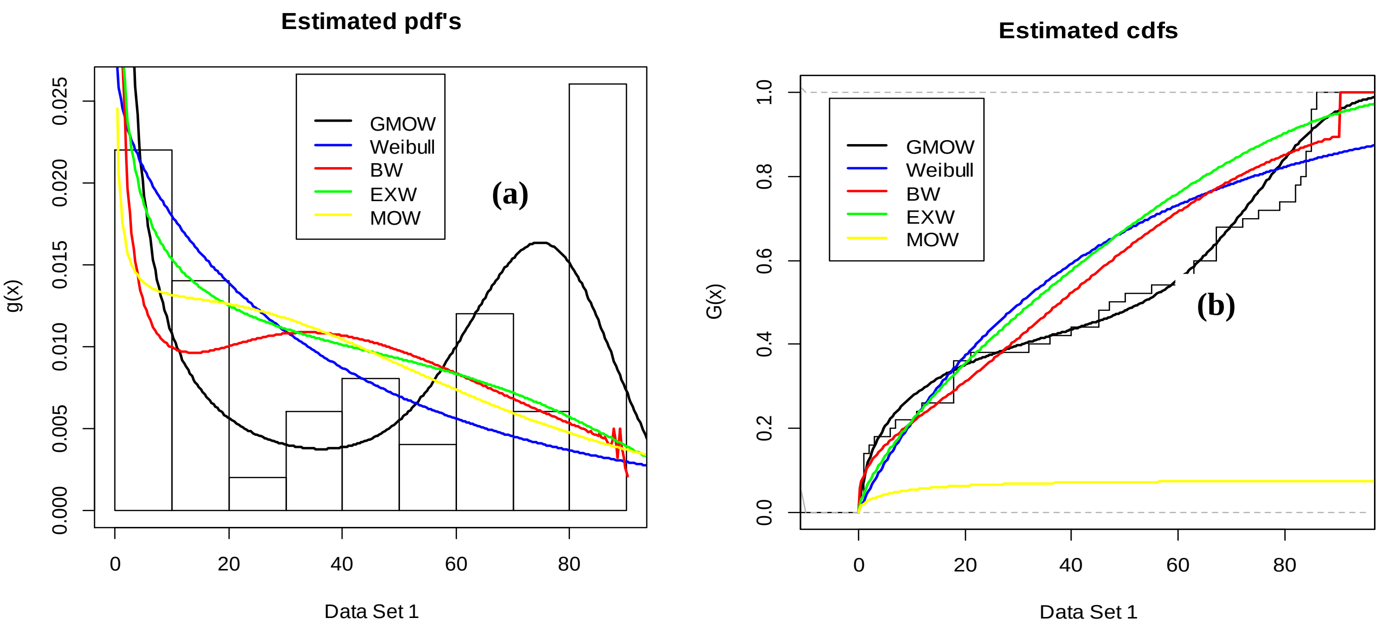

Figure 4. Estimated plots of Data set 1 a) competing pdfs b) competing cdfs with the empirical cdf.

The values in Tables 3 and 5 clearly indicate that GMOW distribution has the least goodness-of-fit statistics for the two data sets hence, provided the best fit among the competing distributions. The plots of the estimated pdfs and estimated cdfs of some the competing distributions are shown in Figures 4 and 5 for Data set 1 and Data set 2 respectively. The plots in the two Figures also show that the GMOW distribution provided the best fit for the data sets.

Table 4. Parameter estimate (standard errors) for Data set 2

Distribution	Parameter Estimate (Standard errors)				
$GMOW(p,\mu,\sigma,\lambda,\theta)$	0.0041	1.4494	9.4383	3.6756	63.5884
	(0.0221)	(8.3423)	(6.2459)	(2.7448)	(31.750)
$BW(a,\beta,\lambda,\theta)$	0.7967	0.0638	0.8714	2.5268	
	(0.4953)	(0.0161)	(0.0024)	(0.0024)	
$GW(a,\lambda,\theta)$	0.9004	0.9343	63.4599		
	(1.9671)	(1.5478)	(173.246)		
$ExpW(a,\lambda,\theta)$	1.6149	0.6012	31.1669		
	(1.7986)	(0.4083)	(49.3803)		
$MOW(p,\lambda,\theta)$	3.7857	0.6904	24.1952		
	(7.8594)	(0.3627)	(41.5819)		
$KW(a,b,\lambda,\theta)$	0.7599	0.0679	0.9365	3.6398	
	(0.0326)	(0.0165)	(0.0078)	(0.0079)	
$Weibull(\lambda,\theta)$	0.8841	59.1653			
	(0.1831)	(16.936)			

Table 5. Goodness-of-fit statistics for Data set 2

Distribution	CvM	An	AIC	K-S
$GMOW(p,\mu,\sigma,\lambda,\theta)$	0.0235	0.1806	176.9677	0.1021
$BW(a,\beta,\lambda,\theta)$	0.0659	0.4791	181.2613	0.1515
$GW(a,\lambda,\theta)$	0.0678	0.4916	180.0657	0.1668
$ExpW(a,\lambda,\theta)$	0.0890	0.6165	181.5105	0.1698
$MOW(p,\lambda,\theta)$	0.3714	2.1015	179.8503	0.8697
$KW(a,b,\lambda,\theta)$	0.0647	0.4732	181.8574	0.1364
$Weibull(\lambda,\theta)$	0.0704	0.5076	178.2193	0.1491

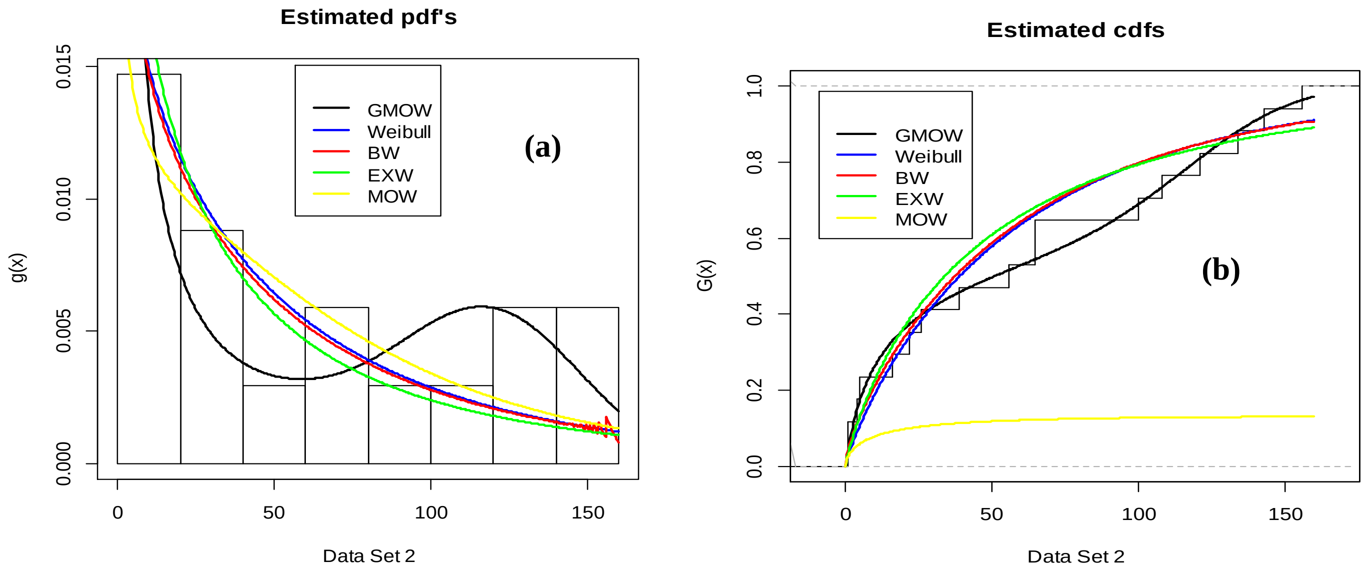

Figure 5. Estimated plots for Data set 2 a) competing pdfs b) competing cdf with the empirical cdf.

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VIII. CONCLUSION

In this paper, we introduced a new flexible extended Weibull distribution called Gumbel Marshall-Olkin Weibull. The new distribution has bimodal pdf shape and bathtub hazard rate function shape among other shapes. The statistical properties of the new distribution were derived and the unknown parameters of the new distribution were estimeated using the maximum likelihood method. Finally, the new distribution and other existing distributions in the literature were used to model two real-life data sets. The computed goodness-of-fit statistics show that the new distribution provided the best fit for the two data sets.

REFERENCES

[1] F. Famoye, C. Lee, and N. Eugene (2004). Beta-Normal distribution: Bimodality properties and application. *Journal of Modern Applied Statistical Methods*, 3: 85- 103.

[2] S. Nadarajah (2009). Bathtub-Shape failure rate function. *Qual Quant,* 43: 855-863

[3] C.D Lai, M. Xie and D.N.P. Murthy (2001). Bathtub-Shape Failure Rate Life Distributions. Handbook of Statistics, N. Balakrishnan and C.R. Rao eds, 20.

[4] M. Meniconi and D. M. Barry (1996). The power function distribution: A useful and simple distribution to assess electrical component reliability. *Microlectron Reliab,* 36(9): 1207-1212.

[5] G.S. Mudholkar, D.K. Srivastava and M. Freimer (1995). The Exponentiated Weibull family: A reanalysis of the Bus-Motor failure data. *Technometrics,* 37(4): 436-445.

[6] G.M. Cordeiro, E.M.M Ortega and S. Nadarajah (2010). The Kumaraswamy Weibull distribution with application to failure data. *Journal of the Franklin institute*, 347: 1399-1429.

[7] C. Lee, F. Famoye and O. Olumolade (2007). Beta-Weibull distribution: Some properties and applications to censored data. *Journal of Modern Applied Statistical methods*, 6:173-186.

[8] S. B. Provost, Abdus Saboor and Munir Ahmad (2011). The Gamma-Weibull distribution. *Pak. J. Statist*., 27(2): 111-131.

[9] R. Al-Aqtash, C. Lee and F. Famoye (2014). Gumbel-Weibull distribution: properties and apllications. *Journal of Modern Statistical Methods*, 13(2): 201-225.

[10] M.E. Ghitany, E.K. Al-Hussaini and R.A. Al-Jarallah (2006). Marshall-Olkin extended Weibull distribution and its application to censored data. *Journal of Applied Statistics*, 32(10): 1025-1034.

[11] G. R. Aryal, E.M.M. Ortega, G.G. Hamedani and H.M. Yousef (2017). The Topp-Leone generated Weibull distribution: Regression Model, Characterizations and applications. *International journal of statistics and probability*, 6:126-141.

[12] G.R. Aryal and C. P. Tsokos (2011). Transmuted Weibull distribution: A generalization of the Weibull probability distribution. *European Journal of pure and applied mathematics,* 4(2): 89-102.

[13] G. M. Cordeiro, A. Afify, H. H. Yousof, S. Cakmakyapan and G. Ozel (2018). The Lindley Weibull distribution; properties and applications. *Annals of the Brazilian Academy of Science,* 90(3): 2579-2598.

[14] G. S. Mudholkar, D. K. Srivastava and G. D. Kollia (1996). A generalization of the Weibull distribution with application to the analysis of survival data. *Journal of the Ameri. Stat. Associa.*, 19: 1575-1583.

[15] J. J. A. Moors (1988). A Quantile alternative for Kurtosis. *Statistician,* 37:25-32

[16] R. J. Butler and J. B. McDonald (1989). Using Incomplete moments to measure inequality. *Journal of Econometrics*, 42: 109-119.

[17] E. A. H. Elamir and A. H. Seheult (2003). Trimmed L-moments. *Computational Statistics & data analysis*, 43:299-314.

[18] A. Renyi (1961). On measure of entropy and information. Fourth Berkeley symposium on Math. Statist. And Prob, 548-561.

[19] G.M. Cordeiro, M. Alizadeh, G. Ozel, B. Hosseini, E. M. M. Ortega and E. Altum (2016). The generalized odd log-logistics family if distribution: properties, regression models and applications. *Journal of Statistical computation and simulation*, DOI: 10.1080/00949655.2016.1238088.

[20] M. Alizadeh, F. Lak, M. Rasekhi, T. G. Ramires, H. M. Yousof and E. Altum(2017). The odd log-logistic Topp-Leone G family of distribution: heteroscedastic regression models and applications. *Compt. Stat.,* DOI: 10.1001/s00180.017.0780-9.

[21] E. E. Nwezza, C. V. Ogbuehi, U. U. Uwadi, and C.O. Omekara (2020).A New Gumbel generated family of distributions: Properties, bivariate distribution and application. *American journal of applied mathematics and statistics*, 8:9-20.

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