ISSN 2250-3153

Application of Adomian Decomposition method to seepage flow derivatives in porous media using fractional calculus

F.A.Faisal [*], M.A.Bashir [**]

[*] Department of Mathematics, University of Hafr ALBatin-Kindom of Saudi Arabia, P.o.box:1803, Email: fafaysal@uod.edu.sa
[**] college of mathematical sciences and statistics, ALNelein university ,Sudan, Email: info@aes.edu.sd.

Abstract- In this paper, we have solved seepage flow derivatives in porous media by using the Adomian Decomposition method (ADM). Our solution proved rapid convergence to the exact solution.

Index Terms- Adomian Decomposition method; Fractional Differential Equation system; fractional calculus.

I. Introduction

In fact it's very difficult to solve or to approximate nonlinear problems. Common analytic procedures linearize the problem or assume the nonlinearities insignificant. Such procedures change the actual problem or lead to lose some important information. The Adomian Decomposition method(ADM) was successfully applied to autonomous ordinary and partial differential equation. The method provides a solution without linearization, Perturbation, or unjustified assumption for linear and nonlinear differential equation.

II. Preliminaries

11.1 Some formulae of fractional derivatives

Let us first start with Liouville's first formula with the known result $D^n e^{ax} = a^n e^{ax}$ where $D = \frac{d}{dx}, n \in N$, and extended it at first in the particular case $\alpha = \frac{1}{2}, a = 2$ and then to arbitrary order α (rational, irrational or complex) by

$$D^\alpha e^{ax} = a^\alpha e^{ax} \tag{1}$$

He assumed the series representation for $f(x)$ as $f(x) = \sum_{k=0}^{\infty} c_k e^{a_k x}$ and defined the derivative of arbitrary order α by

$$D^\alpha f(x) = \sum_{k=0}^{\infty} c_k a_k^\alpha e^{a_k x} \tag{2}$$

Secondly the above formula was applied to the explicit function $x^{-\alpha}$. He considered the integral

$$I = \int_0^\infty u^{\beta-1} e^{-xu} du \tag{3}$$

Substituting $xu = t$ gives t he result

$$I = x^{-\beta} \int_0^\infty t^{\beta-1} e^{-t} dt = x^{-\beta}\Gamma(\beta),,,,, \mathrm{Re}\,\alpha \succ 0 \tag{4}$$

ISSN 2250-3153

Operating on both sides of $x^{-\beta} = \frac{1}{\Gamma(\beta)}$ whit D^{α} with respect to x He obtained

$$\Gamma(\beta)D^{\alpha}x^{-\beta} = \int_0^{\infty} u^{\beta-1}D^{\alpha}e^{-xu}du$$

$$D^{\alpha}(e^{-xu}) = (-1)^{\alpha}u^{\alpha}e^{-xu}$$

$$D^{\alpha}x^{-\beta} = (-1)^{\alpha}\frac{\Gamma\alpha+\beta}{\Gamma\beta}x^{-\alpha-\beta} \tag{5}$$

Liouville used the latter in his investigation of potential theory

II.1. The methodology

To give a clear overview of Adomian decomposition method, we first consider the linear differential equation written in an operator form by

$$Lu + Ru = g, \tag{6}$$

where L is, mostly, the lower order derivative which is assumed to be invertible, R Is other linear differential operator , and g is a source term [1-3]. We apply the inverse operator L^{-1} to both sides of equation (6) and using given condition to obtain

$$u = f - L^{-1}(Ru) \tag{7}$$

Where the function f represents the terms arising from integrating the source term g and from using the given conditions that are assumed to be prescribed. As indicated before. Adomian decomposition method defines the solution u by an infinite series of components given by

$$(8)$$

Where the components $u_0, u_1, u_2 \ldots$ are usually recurrently determined. Substituting (8) into both side of (7) leads to

$$\sum_{n=0}^{\infty}u_n = f - L^{-1}(R(\sum_{n=0}^{\infty}u_n)) \tag{9}$$

For simplicity. Equation (9) can be rewritten as

$$u_0 + u_1 + u_2 + \ldots = f - L^{-1}(R(u_0 + u_1 + u_2 + \ldots)). \tag{10}$$

To construct the recursive relation needed for the components $u_0, u_1, u_2 \ldots$

It is important to note that Adomian decomposition method suggests that zeroth component u_0 is usually defined by the function f described above, i.e.by all terms, that are not included under the inverse operator L^{-1}, which arise from the initial data and from integrating the inhomogeneous term. Accordingly, the formal recursive relation is defined by

$$u_0 = f,$$
$$u_{k+1} = -L^{-1}(R(u_k)), k \geq 0, \tag{11}$$

Or equivalently

$$u_0 = f,$$
$$u_1 = -L^{-1}(R(u_0)),$$

$$u_2 = -L^{-1}(R(u_1)), \tag{12}$$
$$u_3 = -L^{-1}(R(u_2)),$$
$$.$$
$$.$$
$$.$$

It is clearly seen that the relation (12) reduced the differential equation under consideration into an elegant determination of computable components. Having determined these components, we then substitute it into (8) to obtain the solution a series form.

III. Seepage flow derivatives in porous media

We shall treat the above mention problem using Adomian decomposition method (ADM). The problem is modelled by the fractional partial differential equation (FPDE):

$$\frac{\partial^\alpha p(x,y,t)}{\partial x^\alpha} + \frac{\partial^\alpha p(x,y,t)}{\partial y^\alpha} - \frac{1}{v}\frac{\partial p(x,y,t)}{\partial t} = 0$$
$$p(0,y,,t) = 1 + e^y + e^t$$
$$p(x,0,t) = 1 + e^x + e^t$$
$$p(x,y,0) = 1 + e^x + e^y$$
$$p_0(x,y,t) = 1 + e^y + e^z \tag{13}$$

Solution:

$$L_x^\alpha p(x,y,t) = \frac{1}{v} L_t(p(x,y,t) - L_y^\alpha(p(x,y,t)), \tag{14}$$

Where

$$L_t = \frac{\partial}{\partial t}, L_x^\alpha = \frac{\partial^\alpha}{\partial x^\alpha}, L_y^\alpha = \frac{\partial^\alpha}{\partial y^\alpha} \tag{15}$$

We assume that the inverse of the operator exists as the $L_x^{-\alpha} = J_x^\alpha$

Applying the inverse operator J_x^α to both sides of $(14),$ and using the given condition $p_0(x,y,t) = 1 + e^y + e^t$ yield

$$p(x,y,t) = p_0(x,y,t) + J_x^\alpha\left(\frac{1}{v} L_t(p(x,y,t)) - L_y^\alpha(p(x,y,t))\right).$$

$$p(x,y,t) = 1 + e^y + e^t + J_x^\alpha\left(\frac{1}{v} L_t(p(x,y,t)) - L_y^\alpha(p(x,y,t))\right) \tag{16}$$

As mentioned above, the decomposition method sets the solution $p(x,y,t)$ in an series form by

$$p(x,y,t) = \sum_{n=0}^{\infty} p_n(x,y,t) \tag{17}$$

Inserting (17) into both sides of the $(16),$ we obtain

$$\sum_{n=0}^{\infty} p_n(x,y,t) = 1 + e^y + e^t +$$
$$J_x^{\alpha}\left(\frac{1}{v}L_t\left(\sum_{n=0}^{\infty} p_n(x,y,t)\right) - L_y^{\alpha}\left(\sum_{n=0}^{\infty} p_n(x,y,t)\right)\right). \qquad (18)$$

Using few terms only for simplicity reasons, we obtain

$$p_0 + p_1 + p_2 + ... = 1 + e^y + e^t$$
$$+ J_x^{\alpha}(\frac{1}{v}L_t(p_0 + p_1 + p_2 + ...) - L_y^{\alpha}(p_0 + p_1 + p_2 + ...)), \qquad (19)$$

Proceeding as before, we identify the zeroth component $p_0(x,y,t),$ by

$$p_0(x,y,t) = 1 + e^y + e^t, \qquad (20)$$

Having identifies the zeroth component $p_0(x,y,t),$ we obtain the recursive scheme

$$p_0(x,y,t) = 1 + e^y + e^t, \qquad (21)$$
$$p_{k+1}(x,y,t) =$$
$$J_x^{\alpha}\left(\frac{1}{v}L_t(p_k(x,y,t)) - L_y^{\alpha}(p(x,y,t))\right), k \geq 0,$$

The components $p_0, p_1, p_2, ...$ are thus determined as follows:

$$p_0(x,y,t) = 1 + e^y + e^t,$$

$$p_1(x,y,t) =$$
$$= J_x^{\alpha}\left(\frac{1}{v}L_t(1$$
$$= J_x^{\alpha}\left(\frac{1}{v}(e^t)\right.$$
$$p_1(x,t) = x^{\alpha}$$

$$p_2(x,y,t) = J_x^{\alpha}\left(\frac{1}{v}L_t(p_1(x,y,t)) - L_y^{\alpha}(p_1(x,y,t))\right)$$
$$= J_x^{\alpha}\left(\begin{array}{l}\frac{1}{v}L_t\left(x^{\alpha}\left(\frac{1}{v\Gamma(\alpha+1)}e^t - \frac{1}{\Gamma(\alpha+1)}e^y\right)\right) - \\ L_y^{\alpha}\left(x^{\alpha}\left(\frac{1}{v\Gamma(\alpha+1)}e^t - \frac{1}{\Gamma(\alpha+1)}e^y\right)\right)\end{array}\right)$$
$$= J_x^{\alpha}\left(\frac{1}{v}\left(\frac{x^{\alpha}}{v\Gamma(\alpha+1)}e^t\right) - \left(-\frac{x^{\alpha}}{\Gamma(\alpha+1)}e^y\right)\right)$$
$$= \left(\frac{1}{v}\left(\frac{x^{2\alpha}\Gamma(\alpha+1)}{\Gamma(\alpha+)\Gamma(2\alpha+1)}e^t\right) + \left(\frac{x^{2\alpha}\Gamma(\alpha+1)}{\Gamma(\alpha+1)\Gamma(2\alpha+1)}e^y\right)\right)$$
$$p_2(x,y,t) = x^{2\alpha}\left(\frac{1}{v^2\Gamma(2\alpha+1)}e^t + \frac{1}{\Gamma(2\alpha+1)}e^y\right),$$

$$p_3(x,y,t) = J_x^{\alpha}\left(\frac{1}{v}L_t(p_2(x,y,t)) - L_y^{\alpha}(p_2(x,y,t))\right)$$

$$= J_x^{\alpha}\left(\begin{array}{l}\frac{1}{v}L_t\left(x^{2\alpha}\left(\frac{1}{v^2\Gamma(2\alpha+1)}e^t + \frac{1}{\Gamma(2\alpha+1)}e^y\right)\right) - \\ L_y^{\alpha}\left(x^{2\alpha}\left(\frac{1}{v^2\Gamma(2\alpha+1)}e^t + \frac{1}{\Gamma(2\alpha+1)}e^y\right)\right)\end{array}\right)$$

$$= J_x^{\alpha}\left(\frac{1}{v}\left(\frac{x^{2\alpha}}{v^2\Gamma(2\alpha+1)}e^t\right) - \left(\frac{x^{2\alpha}}{\Gamma(2\alpha+1)}e^y\right)\right)$$

$$= \left(\frac{1}{v}\left(\frac{x^{3\alpha}\Gamma(\alpha+1)}{v^2\Gamma(\alpha+)\Gamma(2\alpha+1)}e^t\right) - \left(\frac{x^{3\alpha}\Gamma(\alpha+1)}{\Gamma(\alpha+1)\Gamma(2\alpha+1)}e^y\right)\right)$$

$$= \left(\frac{x^{3\alpha}}{v^3\Gamma(2\alpha+2)}e^t - \left(\frac{x^{3\alpha}}{\Gamma(2\alpha+2)}e^y\right)\right)$$

$$p_3(x,y,t) = x^{3\alpha}\left(\frac{1}{v^3\Gamma(2\alpha+2)}e^t - \frac{1}{\Gamma(2\alpha+2)}e^y\right),$$

It is obvious that all components, $p_k = 0, k \geq 1$ .consequently, the solution is given by

$$p(x,y,t) = p_0 + p_1 + p_2 \ldots$$

$$p(x,y,t) = p_0 + p_1 + p_2 + \ldots$$

$$p(x,y,t) = 1 + e^y + e^t + \left(\frac{x^{\alpha}}{v\Gamma(\alpha+1)}e^t - \frac{x^{\alpha}}{\Gamma(\alpha+1)}e^y\right) +$$

$$\left(\frac{x^{2\alpha}}{v^2\Gamma(2\alpha+1)}e^t + \frac{x^{2\alpha}}{\Gamma(2\alpha+1)}e^y\right) + \left(\frac{x^{3\alpha}}{v^3\Gamma(2\alpha+2)}e^t - \frac{x^{3\alpha}}{\Gamma(2\alpha+2)}e^y\right),$$

$$p(x,y,t) = 1 + \left[1 - \frac{x^{\alpha}}{\Gamma(\alpha+1)} + \frac{x^{2\alpha}}{\Gamma(2\alpha+1)} - \frac{x^{3\alpha}}{\Gamma(2\alpha+2)} + \ldots\right]e^y$$

$$+\left[1 + \frac{x^{\alpha}}{v\Gamma(\alpha+1)} + \frac{x^{2\alpha}}{v^2\Gamma(2\alpha+1)} + \frac{x^{3\alpha}}{v^3\Gamma(2\alpha+2)} + \ldots\right]e^t$$

The exact solution obtained by $\alpha = 1$.

$$p(x,y,t)=1+\left[1-\frac{x}{1!}+\frac{x^2}{2!}-\frac{x^3}{3!}+...\right]e^y+\left[1+\frac{x}{v1!}+\frac{x^2}{v^2 2!}+\frac{x^3}{v^3 3!}+...\right]e^t$$

$$p(x,y,t)=1+\left[e^{-x}e^y\right]+\left[e^{\frac{x}{v}}e^t\right]=1+e^{y-x}+e^{t+\frac{x}{v}} \qquad (22)$$

IV. CONCLUSION

The fundamental goal of this work has been to construct an approximate solution of seepage flow derivatives in porous media. The goal has been achieved by using the (ADM). The method was used in a direct way without using linearization, perturbation or restrictive assumptions. Comparing this method with others, such as variational iteration, we consider this method to be more effective.

REFERENCES

[1] Abdul-majid-wazwaz-partial differential equation and solitary-waves theory nonlinear physical science-2009.
[2] Debnath L. Bhatta D. –integral transforms and their applications (2ed,crc,2007) ISBN.
[3] J.H,He, Approximate analytical solution for seepage flow with fractional derivatives in porous media, computer meth apple mesh, Eng. 167(1998),57-68.
[4] J.H.He,variational iteration method for autonomous ordinary differential systems ,Apple .mesh comput,114,(2000),115-123.
[5] J.H.He,variatianl iteration method-a kind of non-linear analytical technique some examples, int j. nonlin-mech.34(1999),699-708.
[6] Kimeu.Joseph Fractional calculus and application,2009.
[7] Shaher Momani.Salah Abuasad, Zaid obibat ,variational iteration method for solving boundary value problems 2006.

AUTHORS

First Author – F.A.Faisal , Department of Mathematics, university of Hafr ALBatin-Kindom of Saudi Arabia, P.o.box:1803 ,Email: fafaysal@uod.edu.sa.

Second Author – M.A.Bashir,collge of mathematical sciences and statistics, ALNelein university ,Sudan, Email:info@aes.edu.sd.
.
Correspondence Author – .A.Faisal , Department of Mathematics, university of Hafr ALBatin-Kindom of Saudi Arabia, P.o.box:1803 ,Email: fafaysal@uod.edu.sa, tell:00249918276132.