Some Partial Differential Equations In $\mathbb{R}^n$

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Abstract- In this paper, the initial-value problems are studied and solved for homogeneous and non-homogeneous transport equations. Moreover, the fundamental solution and mean value theorem are used to derive Laplace's equation. Finally, general solution of one-dimensional wave equation is established to apply d'Alembert's formula.
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***Index Terms*-** homogeneous transport, non-homogenous transport, Laplace equation, Wave equation, d'Alembert's formula

I. Introduction

A partial differential equation (PDE) is an equation involving an unknown function of two or more variables and certain of its partial derivatives.

Using the notation explained in a typical PDE, fix an integer k ≥ 1 and let *U* denote an open subset of R^n.

An expression of the form

$$F(D^k u(x), D^{k-1}u(x), \ldots\ldots Du(x), u(x), x) = 0 \ (x \epsilon U)$$

u called a k order partial differential equation, where

$$F: R^{n^k} \times R^{n^{k-1}} \times \ldots . . \times R^n \times R \times U \to R$$

In this paper, three fundamental linear partial differential equations for which various explicit formulas for solutions are available. These are

the transport equation $u_t + b. Du = 0$

Laplace's equation $\Delta u = 0$

the wave equation $u_{tt} - \Delta u = 0$

The geometric notations are that

(i) A point in $\mathbb{R}^{n+1}$ will often be denoted as (x, t) = $(x_1,...,x_n, t)$, and we usually interpret $t = x_{n+1}$ as time.

A point $x \in \mathbb{R}i^n$ will sometimes be written $x = (x', x_n)$ for $x' = (x_1, \ldots, x_{n-1}) \in \mathbb{R}^{n-1}$

(ii) U, V and W usually denote open subsets of i^n We write $V \subset\subset U$ if $V \subset \bar{V} \subset U$ $\bar{V}$ is compact, and say V is compactly contained in U.

(iii) ∂U = boundary of U,

$\bar{U} = U \cup \partial U$ closure of U.

(iv) $U_T = U \times (0, T)$

(v) $B^0 = \{y \in \mathbb{R}^n \big| |x - y| < r\}$ = open ball with center x and radius r > 0.

(vi) B(x, r) = closed ball with center x and radius r > 0.

(vii) α(n) = volume of unit ball B(0, 1) in $\mathbb{R}^n$

$$= \frac{\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}+1\right)} \text{ where } \Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\,dt$$

nα(n) = surface area of unit sphere ∂B(0, 1) in $\mathbb{R}^n$

(viii) If a = $(a_1,...,a_n)$ and b = $(b_1,...,b_n)$ belong to i^n

$$a.b = \sum_{i=1}^n a_i b_i, |a| = \left(\sum_{i=1}^n a_i^2\right)^{\frac{1}{2}}, |b| = \left(\sum_{i=1}^n b_i^2\right)^{\frac{1}{2}}$$

The notations of functions are that

(i) If $u: U \to \mathbb{R}$ we write

$(x) = \left(u^1(x), \ldots, u^m(x)\right) \quad (x \in U)$

we say u is smooth provided u is infinitely differentiable.

(ii) If u, v are two functions, we write $u \equiv v$ to mean that u is identically equal to v; that is, the functions u, v agree for all values of their arguments. We use the notation $u := v$ to define u as equaling v.

(iii) If $u: U \to \mathbb{R}$ we write

$u(x) = \left(u^1(x), \ldots, u^m(x)\right) \quad (x \in U)$

The function u^k is the k^{th} component to u, k = 1,...,m.

(iv) Averages:

$⨍_{B(x,r)} f\,dy = \frac{1}{\alpha(n)r^n} ⨍_{B(x,r)} f\,dy$ = average of f over the ball B(x, r)

and

$⨍_{B(x,r)} f\,dS := \frac{1}{n\alpha(n)r^{n-1}} \int_{\partial B(x,r)} f\,dS$= average of f over the sphere ∂B(x, r).

Then, the partial differential equations are used to solve the transport equation, Laplace's equation and the wave equation for initial value problems.

II. TRANSPORT EQUATION

One of the simplest partial differential equations is the transport equation with constant coefficients. This is the PDE

$$u_t + b. D u = 0 \text{ in } \mathbb{R}^n \times (0, \infty) \qquad (1)$$

where b is a fixed vector in $\mathbb{R}^n, b = (b_1, \ldots., b_2) and\ u: \mathbb{R}^n \times [0, \infty) \to \mathbb{R}$ is the unknown, u = u(x, t). Here $x = (x_1, \ldots., x_2) \in \mathbb{R}^n$ denotes a typical point in space, and $t \geq 0$ denotes a typical time. We write $Du = D_x u = (u_{x_1}, \ldots., u_{x_n})$ for the gradient of u with respect to spatial variables x.

For any point $(x, t) \in \mathbb{R}^n \times (0, \infty)$we define

$$z(s) := u(x + sb, t + s) \quad (s \in \mathbb{R})$$

We then calculate

$$z(s) = Du(x+sb, t+s).b + u_t(x+sb, t+s) = 0$$

the second equality holding owing to (1). Thus z(.) is a constant function of s, and consequently for each point (x, t), u is constant on the line through (x, t) with the direction $(b, 1) \in \mathbb{R}^{n+1}$ Hence if we know the value of u at any point on each such line,know its value everywhere in $\mathbb{R}^n \times (0, \infty)$

III. Initial Value problem and nonhomogeneous problem of SOME PARTIAL DIFFERENT EQUATIONS

Now it is the time to articulate the research work with ideas gathered in above steps by adopting any of below suitable approaches:

A. Initial Value Problem

We consider the initial-value problem

$$\left.\begin{aligned} u_t + b.Du = 0 \; in \; R^n \times (0,\infty) \\ u = g \; on \; R^n \times (0,\infty) \end{aligned}\right\}. \quad (2)$$

Here $b \in i^n$ and $g: i^n \to i$ are known, and the problem is to compute u. Given (x, t) as above, the line through (x, t) with direction (b, 1) is represented parametrically by $(x+sb, t+s)$ $(s \in R)$.This line hits the plane $\Gamma := \mathbb{R}^n \times \{t=0\}$ when s = –t, at the point$(x-tb, 0)$ .Since u is constant on the line and $u(x-tb, 0) = g(x-tb)$, we deduce

$$u(x,t) = g(x-tb) \quad (x \in \mathbb{R}^n, t \geq 0) \; (3)$$

So, if (2) has a sufficiently regular solution u, it must certainly be given by (3). And conversely, it is easy to check directly that if g is C^1 then u defined by (3) is indeed a solution of (2).

B. Nonhomogeneous Problem

Next we consider the associated nonhomogeneous problem

$$\left.\begin{aligned} u_t + b.Du = 0 \; in \; R^n \times (0,\infty) \\ u = g \; on \; R^n \times (0,\infty) \end{aligned}\right\}. \quad (4)$$

As before fix $(x,t) \in \mathbb{R}^{n+1}$ and, inspired by the calculation above, set $z(s) := u(x+sb, t+s)$ for $s \in \mathbb{R}$ Then

$$z(s) = Du(x+sb, t+s).b + u_t(x+sb, t+s) = f(x+sb, t+s)$$

Consequently

$$u(x,t) - g(x-t_b) = z(0) - z(-t)$$

$$= \int_{-t}^{0} z(s)ds$$

$$= \int_{-t}^{0} f(x+sb, t+s)ds$$

By the change of variables t + s = s_1, we get

$$u(x,t) - g(x-t_b) = \int_0^t f(x+(s-t)b, s_1)ds_1$$

$$u(x,t) - g(x-tb) = \int_0^t f(x+,(s-t)b, s)ds$$

and so

$$u(x,t) = g(x-t_b) + \int_0^t f(x+,(s-t)_b, s)ds$$

$$,(x \in \mathbb{R}^n, t \geq 0) \quad (5)$$

Among the most important of all partial differential equations are undoubtedly Laplace's equations

$$\Delta u = 0 \quad (6)$$

and Poisson's equation

$$-\Delta u = f. \quad (7)$$

In both (6) and (7), $x \in U$and the unknown is$u: \overline{U} \to \mathbb{R}$ u = u(x), where $U \subset R^n$ is a given open set. In (7), the function $f: U \to \mathbb{R}$ is also given.

IV. Derivation of solution for LAPLACE'S EQUATION AND WAVE EQUATION

Let u be a solution of Laplace's equation (6) in U = $U = \mathbb{R}^n$ of the form

$$u(x) = v(r),$$

where $r = |x| = (x_1^2 + L + x_n^2)^{\frac{1}{2}}$ and v is to be selected (if possible) so that Δu = 0 holds. First note for i = 1,...,n that

$$\frac{\partial r}{\partial x_i} = \frac{1}{2}(x_1^2 + L \; + x_n^2)^{-\frac{1}{2}} \cdot 2x_i = \frac{x_i}{|x|} = \frac{x_i}{r}.$$

We thus have

$$u_{x_i} = \frac{\partial u}{\partial x_i} = \frac{dv}{dr} \cdot \frac{\partial r}{\partial x_i} = v'(r)\frac{x_i}{r},$$

$$u_{x_i x_i} = v''(r)(\frac{x_i}{r})^2 + \frac{v'(r)}{r} + v'(r)x_i(-\frac{1}{r^2}) \cdot \frac{x_i}{r}$$

$$= v''(r)\frac{x_i^2}{r^2} + v'(r)(\frac{1}{r} - \frac{x_i^2}{r^3}),$$

$$\sum_{i=1}^{n} u_{x_i x_i} = \sum_{i=1}^{n} v''(r)\frac{x_i^2}{r^2} + \sum_{i=1}^{n} v'(r)\left(\frac{1}{r} - \frac{x_i^2}{r^3}\right)$$

$$= \frac{v''(r)}{r^2}(x_1^2 + \dots + x_n^2) + \frac{v'(r)}{r}\cdot n - \frac{v'(r)}{r^3}(x_1^2 + \dots + x_n^2).$$

Hence $\Delta \mathbf{u} = \frac{v''(r)}{r^2}\cdot r^2 + \frac{v'(r)}{r}\cdot n - \frac{v'(r)}{r^3}\cdot r^2$

$$= v''(r) + \frac{v'(r)}{r}(n-1).$$

Since $\Delta u = 0$,

$$v''(r) + \frac{n-1}{r}v'(r) = 0, \quad (8)$$

$$v'' = \frac{1-n}{r}v',$$

$$\frac{v''}{v'} = \frac{1-n}{r}.$$

Integrating both sides**,**

$$\log|v'| = (1-n)\log r + \log a,$$

$$v' = ar^{1-n}.$$

If $n \geq 3$

$$v = a\frac{r^{2-n}}{2-n} + c$$

$$= \frac{a}{2-n}r^{2-n} + c$$

where $b = \frac{a}{2-n}$

If n =2, $v'(r) = \frac{a}{r}$

v= a log r + c

$$v(r) = \begin{cases} a\log r + c\ (n = 2) \\ \dfrac{b}{r^{n-2}} + c\ \ (n \geq 3) \end{cases}$$

where b and c are constants.

The function is

$$\emptyset(x) = \begin{cases} -\frac{1}{2\pi}\log|x| & (n = 2) \\ \frac{1}{n(n-2)\alpha(n)|x|^{n-2}} & (n \geq 3) \end{cases} \quad (9)$$

that defined for $x \in \mathbb{R}^n$, $x \neq 0$, is the fundamental solution of Laplace's equations.

The wave equation is a simplified model for a vibrating string (n = 1), membrane (n = 2), or elastic solid (n = 3). In these physical interpretation u(x, t) represents the displacement in some direction of the point x at time $t \geq 0$.

the wave equation is $u_{tt} - \Delta u = 0$, (10)

and the nonhomogeneous wave equation is

$$u_{tt} - \Delta u = f \quad (11)$$

subject to appropriate initial and boundary conditions. Hence t > 0 and $x \in U$ where $U \subset R^n$ is open. The unknown is $u: \overline{U} \times [0,\infty) \to \mathbb{R}$, u = u(x, t), and the Laplacian Δ is taken with respect to the spatial variables $x = (x_1, \dots, x_n)$. In (11) the function $f: U \times [0,\infty) \to \mathbb{R}$ is given.

Let V represent any smooth subregion of U. The acceleration within V is then

$$\frac{d^2}{dt^2}\int_V u dx = \int_V u_{tt} dx$$

and the net contact force is

$$-\int_{\partial V} \mathbf{F}\cdot\boldsymbol{\upsilon} dS,$$

where **F** denotes the force acting on V through ∂V and the mass density is taken to be unity.

Newton's law asserts that the mass times the acceleration equals the net force:

$$\int_v u_{tt} dx = -\int_{\partial v} \mathbf{F}\cdot\boldsymbol{\upsilon}\ dS.$$

This identity obtains for each subregion V and so

$$u_{tt} = -\text{div}\ \mathbf{F}.$$

For elastic bodies, **F** is a function of the displacement gradient Du, whence

$$u_{tt} + \text{div}\ \mathbf{F}(Du) = 0$$

for small Du, the linearization $\mathbf{F}(Du) \approx a\,Du$ is often appropriate; and so

$$u_{tt} - a\,\Delta u = 0.$$

This is the wave equation if a = 1.

This physical interpretation strongly suggests it will be mathematically appropriate to specify two initial conditions, on the displacement u and the velocity u_t, at time t = 0.

V. Derivation of D'Alembert's formula

We consider the initial-value problem for the one-dimensional wave equation in all of P:

$$u_{tt} - u_{xx} = 0 \text{ in } R^n \times (0,\infty)$$
$$u = g, u_t = h \text{ on } R^n \times (0,\infty) \quad (12)$$

where g, h are given. We desire to derive a formula for u in terms of g and h.

Note that the PDE in (12) can be "factored" to read

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right)u = 0. \quad (13)$$

Let

$$v(x,t) := \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right)u(x,t). \quad (14)$$

Then (13) says

$$v_t(x,t) + v_x(x,t) = 0 \qquad (x \in \mathbb{R}, t > 0)$$

This is a transport equation with constant coefficients. Applying formula (3) (with n = 1, b = 1), we find

$$v(x, t) = a(x - t) \quad (15)$$

for $a(x) := v(x, 0)$.Combining now (13) – (15), we obtain

$$u_t(x,t) - u_x(x,t) = a(x-t) \qquad in\ \mathbb{R} \times (0,\alpha)$$

This is a nonhomogeneous transport equation; and so formula (5) (with n = 1, b = – 1, f(x, t) = a (x – t)) implies for $b(x); = u(x, 0)$that

$$u(x,t)=\int_0^t a(x+(t-s)-s)ds+b(x+t)$$

$$=\frac{1}{2}\int_{x-t}^{x+t} a(y)dy+b(x+t). \qquad (16)$$

We lastly invoke the initial conditions in (12) to compute a and b. The first initial condition in (12) gives

$$b(x) = g(x) \; x \in \mathbb{R}$$

whereas the second initial condition and (26) imply

$$a(x) = v(x, 0) = u_t(x, 0) - u_x(x, 0)$$
$$= h(x) - g'(x) \quad (x \in \mathbb{R})$$

Our substituting into (16) now yields

$$u(x,t)=\frac{1}{2}\int_{x-t}^{x+t} h(y)-g'(y)dy+g(x+t).$$

Hence

$$u(x,t) = \frac{1}{2}[g(x+t)+g(x-t)]+\frac{1}{2}\int_{x-t}^{x+t} h(y)dy \; ,$$
$(x \in \mathbb{R}, t \geq 0)$ (17)

This is d'Alembert's formula.
To solve the solution of wave equation for n = 1,
Assume $g \in C^2(\mathbb{R})$, $h \in C^1(\mathbb{R})$ and define u by d'Alembert's formula. Then

(i) $u \in C^2(\mathbb{R} \times [0,\infty))$

(ii) $u_{tt} - u_{xx} = 0 \; in \; \mathbb{R} \times (0,\infty)$ and

(iii) $\lim\limits_{\substack{(x,t)\to(x^0,0)\\ t>0}} u(x,t)=g(x^0), \quad \lim\limits_{\substack{(x,t)\to(x^0,0)\\ t>0}} u_t(x,t)=h(x^0),$

for each point $x^0 \in \mathbb{R}$

The proof is

We have

$$u(x,t) = \tfrac{1}{2}[g(x+t)+g(x-t)]+\tfrac{1}{2}\int_{x-t}^{x+t} h(y)dy$$
$(x \in \mathbb{R}, t \geq 0)$ (18)

since $g \in C^2(\mathbb{R})$, $h \in C^1(\mathbb{R})$,$u \in C^2(\mathbb{R} \times [0,\infty))$).

Next, we differentiate (18) with respect to x, we get

$$u_x=\frac{1}{2}\big[g'(x+t)+g'(x-t)\big]+\frac{1}{2}\big[h(x+t)-h(x-t)\big].$$

Again,

$$u_{xx}=\frac{1}{2}\big[g''(x+t)+g''(x-t)\big]+\frac{1}{2}\big[h'(x+t)-h'(x-t)\big]. \qquad (19)$$

In addition, we differentiate (18) with respect to t, we get

$$u_t=\frac{1}{2}\big[g'(x+t)-g'(x-t)\big]+\frac{1}{2}\big[h(x+t)+h(x-t)\big]. \qquad (20)$$

Again,

$$u_{tt}=\frac{1}{2}\big[g''(x+t)+g''(x-t)\big]+\frac{1}{2}\big[h'(x+t)-h'(x-t)\big]. \qquad (21)$$

So, we get

$$u_{tt}-u_{xx}=0.$$

Finally, we get

$$\lim_{(x,t)\to(x^0,0)} u(x,t)=u(x^0,0)=\frac{1}{2}\Big[2g(x^0)\Big]=g(x^0).$$

Then, from (20),

$$\lim_{\substack{(x,t)\to(x^0,0)\\ t>0}} u_t(x,t)=u_t(x^0,0)$$

$$=\frac{1}{2}\Big[g'(x^0)-g'(x^0)\Big]+\frac{1}{2}\Big[h(x^0)+h(x^0)\Big]=h(x^0).$$

VI. CONCLUSION

This paper contains a brief summary of the work done and some recommendations for future research directions. These some partial differential equations are based on the computation of inequalities and integration. Some of the application areas include these equations such as Laplace's equation, Wave equation.

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