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# Important Summaries and Graphs of the Mixture of Two Logistic Distributions

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***Abstract-*** The mixture of distributions is a good model to some realities where the population consists of heterogeneous components. This paper presents some of the important summaries of the mixture of two logistic distributions such as the Mean, Variance and Moment Generating Function. Moreover, various graphs of the mixture with different values of the mixing parameters are presented.

***Index Terms-*** Mixture of Distributions, Logistic Distribution and Moment Generating Functions, Mean and Variance.

## I. Introduction

The first one to introduce the logistic function as a model of population growth was the Belgian Mathematician Pierre Francois Verhulst (1804-1849) in 1838.

In probability theory and statistics, the logistic distribution is a continuous probability distribution. The logistic distribution has been used for various growth models, and is used in a certain type of regression, known appropriately as logistic regression. Its cumulative distribution function is the logistic function, which appears in logistic regression and feed forward neural networks [10]. The logistic distribution is very similar in shape to the normal distribution because its density function is symmetric and has a bell shape but has heavier tails (higher kurtosis). Besides the maximum difference between two distribution functions can be less than 0.01. So the logistic distribution has a close approximation to the normal distribution [8].

The logistic probability density function is defined as

$$f(x) = \frac{e^{\frac{(x-a)}{b}}}{b\left[1+e^{\frac{(x-a)}{b}}\right]^2},$$

with location parameter $a$ and scale parameter $b$ [4].

There are some who argue that the logistic distribution is inappropriate for modeling lifetime data because the left-hand limit of the distribution extends to negative infinity. This could conceivably result in modeling negative times-to-failure. However, provided that the distribution in question has a relatively high mean and a relatively small location parameter, the issue of negative failure times should not present itself as a problem [5].

Mixture models have provided a mathematical-based approach to the statistical modeling of data generated from some underlying source. Mixture models based on probability density function (pdf) have been used successfully on a number of applications ranging from speaker recognition to bioinformatics. Comprehensive surveys of estimation techniques, discussions and applications are available in the work by [11].

The mixture of logistic distributions provides a more robust model to the fitting of normal mixture models, as observations that a typical of a component are given reduced weight in the calculation of its parameters, for logistic distribution has similar shape with the normal distribution but has heavier tails and high kurtosis as compared to the exponentially decaying tails of a Gaussian [11].

The formula for the mixture of two logistic distributions is defined as

$$f(x) = \varphi_1 f_1(x) + \varphi_2 f_2(x) \text{ , where } \varphi_1 + \varphi_2 = 1,$$

$$f_1(x) = \frac{e^{\frac{(x-a_1)}{b_1}}}{b_1\left[1+e^{\frac{(x-a_1)}{b_1}}\right]^2} \quad \text{and} \quad f_2(x) = \frac{e^{\frac{(x-a_2)}{b_2}}}{b_2\left[1+e^{\frac{(x-a_2)}{b_2}}\right]^2}.$$

In this paper, some important properties of the mixture of two logistic distributions such as the mean and variance and the moment generating function are derived and presented. Moreover, various graphs of the mixture are presented.

## II. IMPORTANT SUMMARIES

In this section we present some of the important summaries such as the mean, variance and the Moment Generating Function of the Mixture.

**Theorem 1** If $X$ is a random variable of the mixture of two logistic distribution, the mean of $X$ denoted by $E[X]$ is

$$E[x] = \varphi_1 a_1 + \varphi_2 a_2.$$

*Proof.*

Let $f(x) = \varphi_1 f_1(x) + \varphi_2 f_2(x)$, where $\varphi_1 + \varphi_2 = 1$. Then,

$$E[x] = \int_{-\infty}^{\infty} x \left[ \varphi_1 \frac{e^{\frac{(x-a_1)}{b_1}}}{b_1 \left[1 + e^{\frac{(x-a_1)}{b_1}}\right]^2} + \varphi_2 \frac{e^{\frac{(x-a_2)}{b_2}}}{b_2 \left[1 + e^{\frac{(x-a_2)}{b_2}}\right]^2} \right] dx$$

$$= \int_{-\infty}^{\infty} \left[ \varphi_1 \frac{x}{b_1 \left[1 + e^{\frac{(x-a_1)}{b_1}}\right]^2} e^{\frac{(x-a_1)}{b_1}} + \varphi_2 \frac{x}{b_2 \left[1 + e^{\frac{(x-a_2)}{b_2}}\right]^2} e^{\frac{(x-a_2)}{b_2}} \right] dx$$

$$= \varphi_1 \int_{-\infty}^{\infty} \frac{x}{b_1 \left[1 + e^{\frac{(x-a_1)}{b_1}}\right]^2} e^{\frac{(x-a_1)}{b_1}} dx + \varphi_2 \int_{-\infty}^{\infty} \frac{x}{b_2 \left[1 + e^{\frac{(x-a_2)}{b_2}}\right]^2} e^{\frac{(x-a_2)}{b_2}} dx$$

$$= \varphi_1 a_1 + \varphi_2 a_2.$$

**Lemma 1** If $X$ is a random variable of the mixture of two logistic distributions then

$$E[x^2] = \varphi_1 \left( {a_1}^2 + \frac{\pi^2 {b_1}^2}{3} \right) + \varphi_2 \left( {a_2}^2 + \frac{\pi^2 {b_2}^2}{3} \right).$$

*Proof.*

Let $f(x) = \varphi_1 f_1(x) + \varphi_2 f_2(x)$, where $\varphi_1 + \varphi_2 = 1$. Then,

$$E[x^2] = \int_{-\infty}^{\infty} x^2 \left[ \varphi_1 \frac{e^{\frac{(x-a_1)}{b_1}}}{b_1 \left[1 + e^{\frac{(x-a_1)}{b_1}}\right]^2} + \varphi_2 \frac{e^{\frac{(x-a_2)}{b_2}}}{b_2 \left[1 + e^{\frac{(x-a_2)}{b_2}}\right]^2} \right] dx$$

$$= \int_{-\infty}^{\infty} \left[ \varphi_1 \frac{x^2}{b_1 \left[1 + e^{\frac{(x-a_1)}{b_1}}\right]^2} e^{\frac{(x-a_1)}{b_1}} + \varphi_2 \frac{x^2}{b_2 \left[1 + e^{\frac{(x-a_2)}{b_2}}\right]^2} e^{\frac{(x-a_2)}{b_2}} \right] dx$$

$$= \varphi_1 \int_{-\infty}^{\infty} \frac{x^2}{b_1 \left[1 + e^{\frac{(x-a_1)}{b_1}}\right]^2} e^{\frac{(x-a_1)}{b_1}} dx + \varphi_2 \int_{-\infty}^{\infty} \frac{x^2}{b_2 \left[1 + e^{\frac{(x-a_2)}{b_2}}\right]^2} e^{\frac{(x-a_2)}{b_2}} dx$$

$$= \varphi_1 \left( {a_1}^2 + \frac{\pi^2 {b_1}^2}{3} \right) + \varphi_2 \left( {a_2}^2 + \frac{\pi^2 {b_2}^2}{3} \right).$$

**Theorem 2**The variance of the mixture of two logistic distributions is

$$Var[x] = \varphi_1 {a_1}^2 + \varphi_1 \frac{\pi^2 {b_1}^2}{3} + \varphi_2 {a_2}^2 + \varphi_2 \frac{\pi^2 {b_2}^2}{3} - {\varphi_1}^2 {a_1}^2 - 2\varphi_1 a_1 \varphi_2 a_2 - {\varphi_2}^2 {a_2}^2.$$

*Proof.* The variance is defined by,

$$var[x] = \mathrm{E}[x^2] - (\mathrm{E}[x])^2 .$$

By Theorem 1 and Lemma 1 we have,

$$var[x] = \varphi_1\left({a_1}^2 + \frac{\pi^2 {b_1}^2}{3}\right) + \varphi_2\left({a_2}^2 + \frac{\pi^2 {b_2}^2}{3}\right) - (\varphi_1 a_1 + \varphi_2 a_2)^2$$

$$= \varphi_1 {a_1}^2 + \varphi_1 \frac{\pi^2 {b_1}^2}{3} + \varphi_2 {a_2}^2 + \varphi_2 \frac{\pi^2 {b_2}^2}{3} - {\varphi_1}^2 {a_1}^2 - 2\varphi_1 a_1 \varphi_2 a_2 - {\varphi_2}^2 {a_2}^2.$$

**Theorem 3** The moment generating function (mgf) of the mixture of two logistic distributions is

$$M_X(t) = \varphi_1(e^{a_1 t})\left(\frac{\pi b_1 t}{\sin(\pi b_1 t)}\right) + \varphi_2(e^{a_2 t})\left(\frac{\pi b_2 t}{\sin(\pi b_2 t)}\right).$$

*Proof.* The moment generating function (mgf) is

$$M_X(t) = \int_{-\infty}^{+\infty} e^{tx} f(x) dx ,$$

where$f(x)$ is the probability density function of the continuous function.

Let $f(x)$=$\varphi_1 f_1(x) + \varphi_2 f_2(x)$, where $\varphi_1 + \varphi_2 = 1$.Then,

$$M_X(t) = \int_{-\infty}^{+\infty} e^{tx}[\varphi_1 f_1(x) + \varphi_2 f_2(x)]dx$$

$$= \int_{-\infty}^{\infty} e^{tx}\left[\varphi_1 \frac{1}{b_1\left[1 + e^{\frac{(x-a_1)}{b_1}}\right]^2} e^{\frac{(x-a_1)}{b_1}} + \varphi_2 \frac{1}{b_2\left[1 + e^{\frac{(x-a_2)}{b_2}}\right]^2} e^{\frac{(x-a_2)}{b_2}}\right]dx$$

$$= \int_{-\infty}^{\infty}\left[\varphi_1 \frac{1}{b_1\left[1 + e^{\frac{(x-a_1)}{b_1}}\right]^2} e^{tx\frac{(x-a_1)}{b_1}} + \varphi_2 \frac{1}{b_2\left[1 + e^{\frac{(x-a_2)}{b_2}}\right]^2} e^{tx\frac{(x-a_2)}{b_2}}\right]dx$$

$$= \varphi_1 \int_{-\infty}^{\infty} \frac{1}{b_1\left[1 + e^{\frac{(x-a_1)}{b_1}}\right]^2} e^{tx\frac{(x-a_1)}{b_1}} dx + \varphi_2 \int_{-\infty}^{\infty} \frac{1}{b_2\left[1 + e^{\frac{(x-a_2)}{b_2}}\right]^2} e^{tx\frac{(x-a_2)}{b_2}} dx$$

$$= \varphi_1(e^{a_1 t})\left(\frac{\pi b_1 t}{\sin(\pi b_1 t)}\right) + \varphi_2(e^{a_2 t})\left(\frac{\pi b_2 t}{\sin(\pi b_2 t)}\right)$$

## III. GRAPHS OF THE MIXTURE

The following are the graphs of the Mixtures of Two Logistic Distributions with different values of their parameters where $a_1 \neq a_2$ as well as $b_1 \neq b_2$.

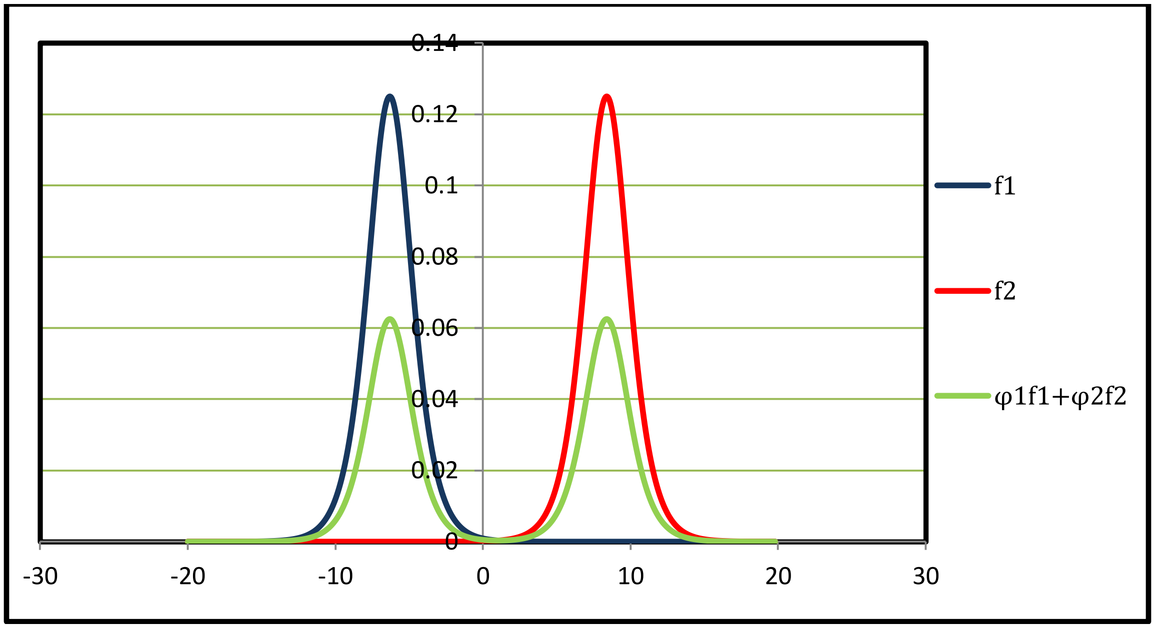


Figure 1. Graph of the mixture of two logistic distributions
$a_1 = -7, a_2 = 7, b_1 = 2, b_2 = 4, \varphi_1 = 0.5, \varphi_2 = 0.5$

Figure 1 shows the graph of two logistic distributions and the mixture of two logistic distributions. $f_1$ is the graph of the logistic distribution with $a_1 = -7$ and $b_1 = 2$. $f_2$ is the graph of logistic distribution with $a_2 = 7$ and $b_2 = 4$. $\varphi_1 f_1 + \varphi_2 f_2$ is the graph of the mixture of two logistic distribution. The graph $\varphi_1 f_1 + \varphi_2 f_2$ of is the average of $f_1$ and $f_2$, since $\varphi_1 = \varphi_2 = 0.5$ .

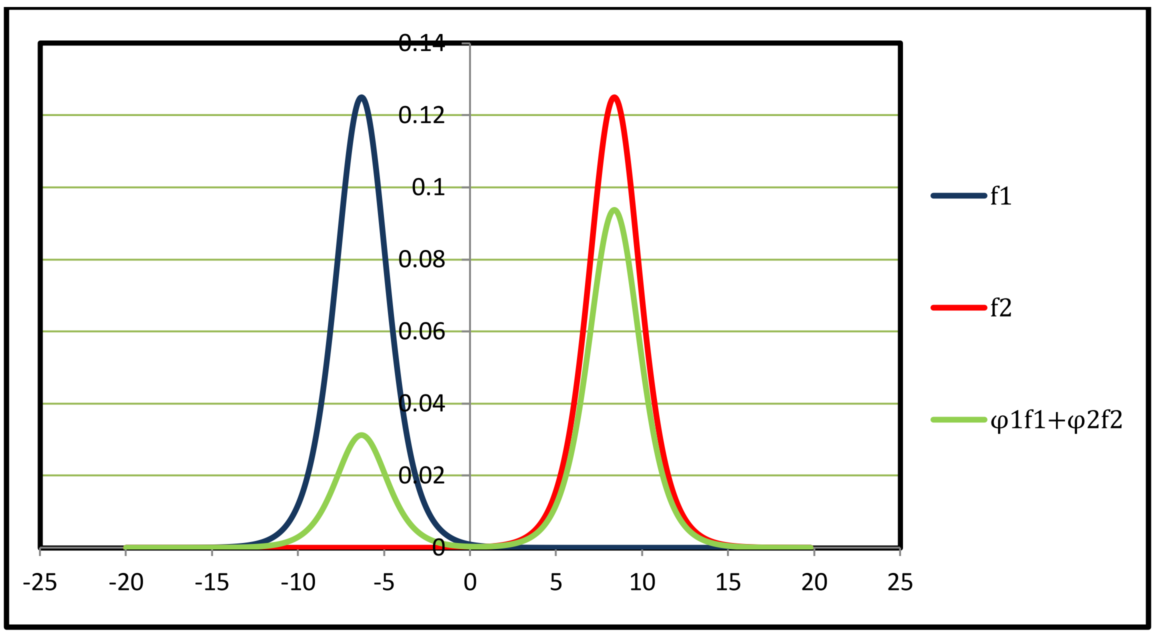


Figure 2. Graph of the mixture of two logistic distribution
$a_1 = -7, a_2 = 7, b_1 = 2, b_2 = 4, \varphi_1 = 0.25, \varphi_2 = 0.75$

Figure 2 shows the graph of two logistic distributions and the mixture of two logistic distributions. $f_1$ is the graph of the logistic distribution with $a_1 = -7$ and $b_1 = 2$. $f_2$ is the graph of logistic distribution with $a_2 = 7$ and $b_2 = 4$. $\varphi_1 f_1 + \varphi_2 f_2$ is the graph of the mixtures of two logistic distribution. The graph of $\varphi_1 f_1 + \varphi_2 f_2$ tends to $f_2$, since $\varphi_2 > \varphi_1$, $\varphi_1 = 0.25$ and $\varphi_2 = 0.75$.

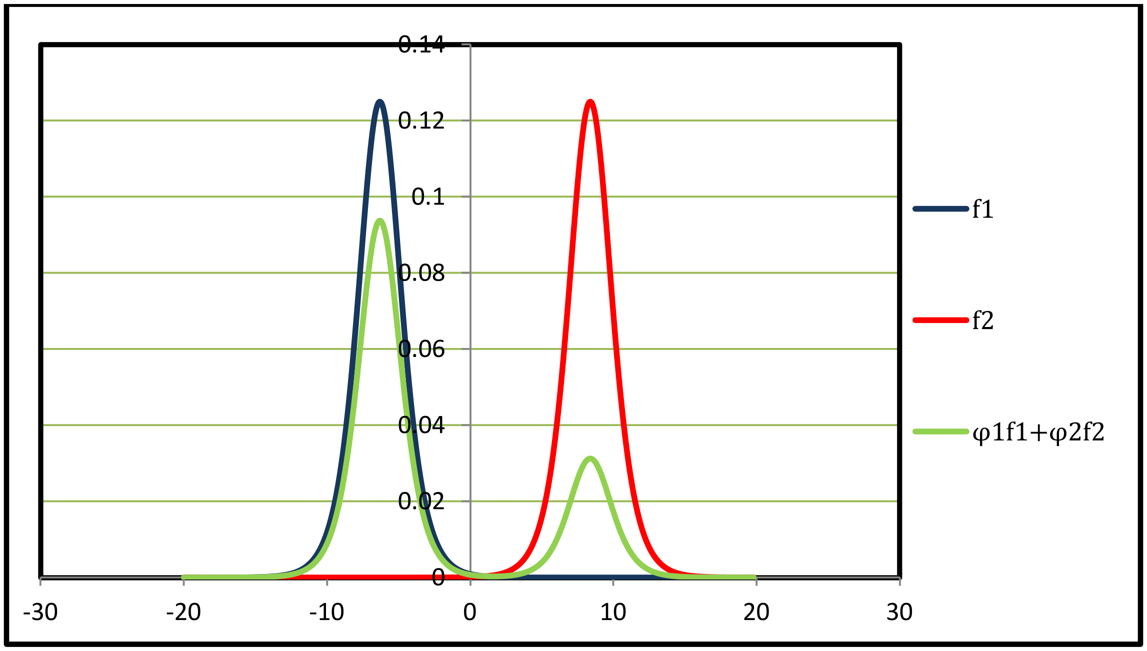


Figure 3. Graph of the mixture of two logistic distributions
$a_1 = -7, a_2 = 7, b_1 = 2, b_2 = 4, \varphi_1 = 0.75, \varphi_2 = 0.25$

Figure 3 shows the graph of two logistic distributions and the mixture of two logistic distributions. $f_1$ is the graph of the logistic distribution with $a_1 = -7$ and $b_1 = 2$. $f_2$ is the graph of logistic distribution with $a_2 = 7$ and $b_2 = 4$. $\varphi_1 f_1 + \varphi_2 f_2$ is the graph of the mixtures of two logistic distributions. The graph of $\varphi_1 f_1 + \varphi_2 f_2$ tends to $f_1$, since $\varphi_1 > \varphi_2$, $\varphi_1 = 0.75$ and $\varphi_2 = 0.25$.

Thus, the three figures show that if $\varphi_1 = \varphi_2$, the graph of mixture of two logistic distributions is the average of $f_1$ and $f_2$. The graph of mixture of two logistic distributions tends to $f_2$, if $\varphi_2 > \varphi_1$ and the graph of mixture of two logistic distributions tends to $f_1$ , if $\varphi_1 > \varphi_2$. The graph of the mixture of two logistic distributions tends to the graph of the logistic distribution that has a greater value of weight $\varphi$.

## IV. CONCLUDING REMARKS

Based on the graphs, if $\varphi_1 = \varphi_2$, the graph of mixture of two logistic distributions is the average of $f_1$ and $f_2$. If $\varphi_2 > \varphi_1$, the graph of mixture of two logistic distributions tends to $f_2$. And if $\varphi_1 > \varphi_2$, the graph of mixture of two logistic distributions tends to $f_1$. The graph of the mixture of two logistic distributions tends to the graph of the logistic distribution that has a greater value of weight $\varphi$.

Studies on the mixture of more than two logistic distributions, on the estimation of parameters of the mixture of two logistic distributions, and on the estimation of parameters of the mixture of more than two logistic distributions are highly recommended.

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