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Ring of Adjacency Matrices of Graphs

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Abstract- This paper presents a new ring consisting of adjacency matrices of graphs. New definitions of binary sum and product of adjacency matrices are being introduced. The paper shows the basic properties for a ring are being satisfied.

Index Terms- Algebraic Structures, Rings, Adjacency Matrices and Graphs

I. INTRODUCTION

The most general algebraic structure with two binary operations is the ring (Fraleigh, 1989).

The theory of rings grew out of the study of two particular classes of rings, polynomial rings in n variables over the real or complex numbers and the "integers" of an algebraic number field (Fraleigh, 1989).

It is known that the set of all $n \times n$ matrices having elements of $R = set\ of\ real\ numbers$ as entries, under the usual addition and multiplication of matrices is a ring.

The adjacency matrix is the matrix associated with graphs and digraphs. Graphs and digraphs have many realistic applications. They are used as models in Sociology, Communication and Transportation.

There are many examples of rings being presented and studied. The rational, real and complex numbers form rings and they even form a field and likewise they are commutative rings. The Gaussian integers as do the Eisenstein integers. The set of integers modulo n forms also a ring. The power set of S becomes a ring if we define addition to be the symmetric differences of sets and multiplication to be the intersection. This is an example of a Boolean ring. A Ring R is said to be a Boolean ring if $a^2 = a$ for every $a \in R$ .The set of all continuous real-valued functions defined on the interval $[a,b]$ forms a ring (even an associative algebra) the operations are addition and multiplication of functions. We have also the ring of Cauchy Sequences.

In this paper, we show that the set of $n \times n$ adjacency matrices with a new defined binary operations being introduced and presented would constitute as a ring.

II. RESULTS AND DISCUSSION

The adjacency matrix of a graph that has n vertices is the $n \times n$ matrix whose i, jth element is 1 if there is at least one edge between P_i and P_j and zero otherwise, where P_i and P_j are vertices of a graph. Note that the adjacency matrices in this paper are symmetric matrices consisting of zeros and ones.

We consider the following definitions of sum and product of two adjacency matrices.

Definition 1. If $A = \lfloor a_{ij} \rfloor$ and $B = \lfloor b_{ij} \rfloor$ are $n \times n$ adjacency matrices of graphs G_1 and G_2, then the sum $A \oplus B$ is the $n \times n$ adjacency matrix $C = \lfloor c_{ij} \rfloor$ of a new graph G_3, defined by

$$c_{ij} = a_{ij} + b_{ij} = \begin{cases} 0, \text{if } a_{ij} + b_{ij} = 0 \text{ or } a_{ij} + b_{ij} > 1 \\ 1, \text{if } a_{ij} + b_{ij} = 1 \end{cases}.$$

The matrix C is obtained by adding corresponding elements of A and B with the corresponding definition of the sum of values. This makes the sum of two adjacency matrices to be an adjacency matrix of another graph.

Definition 2. If $A = \lfloor a_{ij} \rfloor$ and $B = \lfloor b_{ij} \rfloor$ are $n \times n$ adjacency matrices of graphs G_1 and G_2, then the product $A \otimes B$ is the $n \times n$ adjacency matrix $C = \lfloor c_{ij} \rfloor$ of a new graph G_3, defined by

$$c_{ij} = a_{ij} b_{ij} = \begin{cases} 0, \text{if } a_{ij} = 0 \text{ or } b_{ij} = 0, \text{or } a_{ij} = 0 \text{ and } b_{ij} = 0 \\ 1, \text{if } a_{ij} = 1 \text{ and } b_{ij} = 1 \end{cases}.$$

The matrix C is obtained by multiplying corresponding elements of A and B with the corresponding definition of the product. This makes the product of two adjacency matrices to be an adjacency matrix of another graph.

With the above two definitions of sum and product, we can now show that the set of $n \times n$ adjacency matrices together with these two binary operations constitute a ring.

Theorem 1. The set R of $n \times n$ adjacency matrices together with the sum $\oplus$ in **Definition 1** and product $\otimes$ in **Definition 2** is a ring.

Proof: To prove we must satisfy the eight (8) properties of a ring. Note that R is a non empty set.

Property 1. The operation $\oplus$ is a binary operation.

From **Definition 1,** it is clear that the sum of two $n \times n$ adjacency matrices is also an adjacency matrix. The sum of two symmetric matrices is a symmetric matrix and the entries are either zeros or ones.

Property 2. The binary operation $\oplus$ is associative.

Let $A = \lfloor a_{ij} \rfloor$, $B = \lfloor b_{ij} \rfloor$ and $C = \lfloor c_{ij} \rfloor$ be $n \times n$ adjacency matrices of graphs G_1, G_2 and G_3, respectively. Then $A \oplus [B \oplus C] = [A \oplus B] \oplus C$. This follows from the fact that addition of $n \times n$ matrices is associative.

Property 3. The existence of an identity element E in R such that $E \oplus A = A \oplus E = A$ for all $A \in R$. (This element E is an identity element for $\oplus$ on R).

The identity element in this case is the adjacency matrix whose entries are zeros.

That is $E = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$.

Property 4. The existence of additive inverse.

The elements of the additive inverse of A, an $n \times n$ adjacency matrix are defined as follows: Suppose a_{ij} are the elements of A, we let $-a_{ij}$ to be the corresponding elements of $-A$, where $-A$ is the additive inverse of A. Then, we have:

$-a_{ij} = \begin{cases} 0, & \text{if } a_{ij} = 0 \\ 1, & \text{if } a_{ij} = 1 \end{cases}$. This gives $a_{ij} + (-a_{ij}) = 0$. Thus, the inverse of A is A itself.

As an example: Suppose $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$, then $-A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$. Then $A \oplus -A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

Property 5. The binary operation $\oplus$ is commutative.

The commutativity follows from the fact that addition of matrices is commutative.

Property 6. The operation $\otimes$ *is a binary operation.*

From **Definition 2,** it is clear that the product of two $n \times n$ adjacency matrices is also an adjacency matrix. The product of two symmetric matrices under this definition is also a symmetric matrix. Moreover, the entries are either zeros or ones.

Property 7. The binary operation $\otimes$ *is associative.*

Let $A = \lfloor a_{ij} \rfloor$, $B = \lfloor b_{ij} \rfloor$ and $C = \lfloor c_{ij} \rfloor$, then we have

$$\begin{aligned} A \otimes (B \otimes C) &= \lfloor a_{ij} \rfloor \otimes \{\lfloor b_{ij} \rfloor \otimes \lfloor c_{ij} \rfloor\} \\ &= [a_{ij}] \otimes [b_{ij} c_{ij}] \\ &= [a_{ij}(b_{ij} c_{ij})] \\ &= [(a_{ij} b_{ij}) c_{ij}] \\ &= [a_{ij} b_{ij}] \otimes [c_{ij}] \\ &= \{[a_{ij}] \otimes [b_{ij}]\} \otimes [c_{ij}] \\ &= (A \otimes B) \otimes C. \end{aligned}$$

Property 8. The Distributive Laws hold.

For all A, B, C $\in R$ *, we have*

$A \otimes (B \oplus C) = (A \otimes B) \oplus (A \otimes C)$ *and* $(A \oplus B) \otimes C = (A \otimes B) \oplus (A \otimes C)$.

Let $A = \lfloor a_{ij} \rfloor$, $B = \lfloor b_{ij} \rfloor$ and $C = \lfloor c_{ij} \rfloor$, then we have

$$\begin{aligned} A \otimes (B \oplus C) &= \lfloor a_{ij} \rfloor \otimes \{\lfloor b_{ij} \rfloor \oplus \lfloor c_{ij} \rfloor\} \\ &= [a_{ij}] \otimes [b_{ij} + c_{ij}] \\ &= [a_{ij}(b_{ij} + c_{ij})] \\ &= [a_{ij} b_{ij} + a_{ij} c_{ij}] \\ &= [a_{ij} b_{ij}] \oplus [a_{ij} c_{ij}] \\ &= (A \otimes B) \oplus (A \otimes C). \end{aligned}$$

The same way can be used to show that the right distributive law is true.

With the above definitions of a binary sum and product, and we have shown that all the properties are being satisfied, we are able to form a new ring consisting of all $n \times n$ adjacency matrices of a graph.

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