Utility of irreducible group representations in differential equations

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Abstract- Group representations play a central role in the classification of problems with group symmetries. In this paper we deal with the algebraic setup that provides this classification. In particular we relate the irreducible representations to the conjugacy classes of the symmetry group.

Index Terms- conjugacy classes,irreducible group representations

I. Introduction

Conjugacy classes and irreducible representations of Lie groups:

1.1 *Definition*: an equivalence relation on a group$G$, its equivalence classes partition G. The equivalenceclasses under this relation are called the conjugacy classes of G. So, the conjugacy class of $g \in G$ is

$C[g] = \{g = xgx^{-1} \mid x \in g\}$ (2.1)

conjugacy classes C[a],C[b] are equal if and only if a and b are conjugate i.e. $gag^{-1} = b$, $g \in G$, and *disjoint otherwise* .
for some $x \in G$. *The relation is symmetric, since* $g = yhy^{-1}$*with* $y = x^{-1}$. *When* $xgx^{-1} = h$, *we say* x *conjugates* g to h.

Example: .When G is abelian, each element is its own conjugacy class. *1.1*

1.2 *Example:*The conjugacy class of (12) in$S_3$is $\{(12),(13),(23)\}$, as we saw In S_3, the conjugates of (12) , When we make a table of $\sigma(12)\sigma^{-1}$ forall $\sigma \in S_3$.

σ	(1)	(12)	(13)	(23)	(123)	(132)
$\sigma(12)\sigma^{-1}$	(12)	(12)	(23)	(13)	(23)	(13)

Similarly, the reader can check the conjugacy class of (123) is $\{(123),(132)\}$. The conjugacy class of (1) is just $\{(1)\}$. So S_3 has three conjugacy classes:
$\{(1)\},\{(12),(13),(23)\},\{(123),(132)\}$.

3 Example: In $D_4 = \langle r,s\rangle$, there are five conjugacy classes: *1*
$\{1\},\{r^2\},\{s,r^2s\},\{r,r^3\},\{rs,r^3s\}$.

The geometric effect on a square of the members of a conjugacy class of D_4 is the same: a90 degree rotation in some direction, a reflection across a diagonal, or a reflection across an edge bisector.

1.4 Example: There are five conjugacy classes in $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$:
$\{1\},\{-1\},\{i,-i\},\{j,-j\},\{k,-k\}$.
Example*:* There are four conjugacy classes in A_4: 1.5
$\{(1)\},\{(12)(34),(13)(24),(14)(23)\},\{(123),(243),(134),(142)\},\{(132),(234),(143),(124)\}$.

Notice the 3-cycles (123) and (132) are not conjugate in A_4. All 3-cycles are conjugate inthe larger group S_4, e.g., (132) = $(23)(123)(23)^{-1}$ and the conjugating permutation (12) isnot in A_4.[2].

II. Some basic properties of conjugacy classes

2.1 Theorem : Any two elements of a conjugacy class have the same order.

Proof. This is saying g and xgx^{-1} have the same order. That follows from the formula $(xgx^{-1})^n = xg^nx^{-1}$, which shows $(xgx^{-1})^n = 1$ if and only if $g^n = 1$.

A naive converse to Theorem 3.1 is false: elements of the same order in a group need not be conjugate. This is clear in abelian groups, where different elements are never conjugate.

Looking at the nonabelian examples , in D_4there are five elements of order two spread across 3 conjugacy classes. Similarly, there are examples of non-conjugate elements of equal order in $Q_8$and A_4. But in S_3, elements of equal order in S_3 are conjugate. Amazingly,this is the largest example of a finite group where this property holds: up to isomorphism, the only nontrivial finite groups where all elements of equal order are conjugate are Z/(2)and S_3.

A proof is given in [1] and [2], and depends on the classification of finite simple groups. A conjugacy problem about S_3 that remains open, as far as I know, is the conjecture that $S_3$is the only nontrivial finite group (up to isomorphism) in which different conjugacy classes always have different sizes.

Let's verify the observation in Section 2 that different conjugacy classes in a group are disjoint.

2.2 Theorem: Let G be a group and $g,h \in G$. If the conjugacy classes of g and h overlap,then the conjugacy classes are equal.

Proof. We need to show every element conjugate to g is also conjugate to h, and vice versa.Since the conjugacy classes overlap, we have $xgx^{-1} = yhy^{-1}$ for some x and y in the group.Therefore
$g = x^{-1}yhy^{-1}x = (x^{-1}y)h(x^{-1}y)^{-1}$,

so$g$ is conjugate to h. Any element conjugate to g is zgz^{-1} for some $z \in G$, and

$zgz^{-1} = z(x^{-1}y)h(x^{-1}y)^{-1}z^{-1} = (zx^{-1}y)h(zx^{-1}y)^{-1}$,
which shows any element conjugate to g is conjugate to h. To go the other way, write
$h = (y^{-1}x)g(y^{-1}x)^{-1}$ and carry out a similar calculation.□

2.3 Theorem : Eeach element of a group belongs to just one conjugacy class. We call any element of a conjugacy class a representative of that class.

A conjugacy class consists of all xgx^{-1} for fixed g and varying x. Instead we can look at all $xgx^{-1}$for fixed x and varying g. That is, instead of looking at all the elements conjugate to g we look at all the ways x can conjugate the elements of the group. This "conjugate-by-x" function is denoted $\gamma_x : G \to G$, so
$\gamma_x(g) = xgx^{-1}$.

2.4 Theorem : Each conjugation function $\gamma_x : G \to G$, is an automorphism of G.

Proof. For any g and h in G,
$\gamma_x(g)\,x(h) = xgx^{-1}xhx^{-1} = xghx^{-1} = \gamma_x(gh)$,
so$x$ is a homomorphism. Since $h = xgx^{-1}$ if and only if $g = x^{-1}hx$, the function γ_x hasinverse $\gamma_{x^{-1}}$, so x is an automorphism of G.□

2.5 Theorem: explains why conjugate elements in a group are "the same except for the point of view": there is an automorphism of the group taking an element to any of its conjugates, namely one of the maps γ_x.

Automorphisms of G having the form $\gamma_x$are called inner automorphisms. That is, an Inner automorphism of G is a conjugation-by-x operation on G, for some $x \in G$. Inner automorphisms are about the only examples of automorphisms that can be written down without knowing extra information about the group (such as being told the group is abelian or that it is a particular matrix group). For some groups every automorphism is an inner automorphism. This is true for the groups S_n when $n \neq 2,6$ (that's right: S_6 is the only nonabelian symmetric group with an automorphism that isn't conjugation by a permutation). The groups $GL_n(\mathbb{R})$ when $n \geq 2$ have extra automorphisms: since$(AB)^T = B^TA^T$and $(AB)^{-1} = B^{-1}A^{-1}$, the function $f(A) = (A^T)^{-1}$ on $GL_n(\mathbb{R})$ is an automorphism andit is not inner.

Here is a simple result where inner automorphisms tell us something about all automorphisms of a group.

2.6 Theorem : If G is a group with trivial center, then the group $Aut(G)$ also has trivial center.

Proof. Let $\varphi \in Aut(G)$ and assume φ commutes with all other automorphisms. We will seewhat it means for φ to commute with an inner automorphism γ_x. For $g \in G$,
$(\varphi \circ \gamma_x)(g) = \varphi(x(g)) = \varphi(xgx^{-1}) = \varphi(x)\varphi(g)\varphi(x)^{-1}$
and
$(\gamma_x \circ \varphi)(g) = \gamma_x(\varphi(g)) = \gamma_x\varphi(g)x^{-1}$,

so having φ and γ_x commute means, for all $g \in G$, that
$\varphi(x)\varphi(g)\varphi(x)^{-1} = \gamma_x\varphi(g)x^{-1} \Leftrightarrow x^{-1}\varphi(x)\varphi(g) = \varphi(g)x^{-1}\varphi(x)$,

so$x^{-1}\varphi(x)$ commutes with every value of φ. Since φ is onto, $x^{-1}\varphi(x) \in Z(G)$. The centerof G is trivial, so $\varphi(x) = x$.

This holds for all $x \in G$, so φ is the identity automorphism.

We have proved the center of $Aut(G)$ is trivial.[2]

Let Gdenotes a reductive algebraic group, possibly non-connected.We use the notation $\mathfrak{g}$for *Lie* G; likewise for closed subgroups of G.Frequently, we consider the diagonal action of Gon G^n, the *n*-fold cartesian product of $G$with itself, by simultaneous conjugation:
$$g.(x_1,\dots,x_n) := (gx_1g^{-1},\dots,gx_ng^{-1}); \quad (2.2)$$

for all $g \in G$and $(x_1,\dots,x_n) \in G^n$.Note that any subgroup $H$of Gacts on G^nin the sameway. We also consider the action of Gon $\mathfrak{g}^n$by diagonal simultaneous adjoint action.If $H$is a closed subgroup of G and $x = (x_1,\dots,x_n) \in G$for some $n \in \mathbb{N}$, then we say $H$is *topologically generated by* x (or *by* $x_1,\dots,x_n$) if $H = \overline{\langle x_1,\dots,x_n\rangle}$.[10].

III. Irreducible representations of Lie group

Given a real Lie algebra $\mathfrak{g}$, its complexification $\mathfrak{g} \otimes \mathbb{C}$ is a complex Lie algebra. Consider in particular the real Lie algebra $\mathfrak{su}(n)$. Its complexification is the Lie algebra $\mathfrak{sl}(n,\mathbb{C})$. Indeed,
$\mathfrak{sl}(n,\mathbb{C}) = \mathfrak{su}(n) \oplus i\mathfrak{su}(n)$
is the decomposition of a trace-free complex matrix into its skew-adjoint and self-adjoint part.

3.1 Remark: Of course, $\mathfrak{sl}(n,\mathbb{C})$ is also the complexification of $\mathfrak{sl}(n,\mathbb{R})$. We have encountered a similar phenomenon for the symplectic groups: The complexification of $\mathfrak{sp}(n)$is $\mathfrak{sp}(n,\mathbb{C})$,which is also the complexification of $\mathfrak{sp}(n,\mathbb{R})$.

We will be interested in representations of Lie algebra $\mathfrak{g}$ on complex vector spaces , i.e. Lie algebra morphisms $\mathfrak{g} \to End_{\mathbb{C}}(V)$. Equivalently, this amounts to a morphism of complex Lie algebras $\mathfrak{g} \otimes \mathbb{C} \to End_{\mathbb{C}}(V)$. If V is obtained by complexification of a real **g**-representation ,then$V$ carries an **g**-equivariant conjugate linear complex conjugation map $\mathbb{C} : V \to V$. Conversely,we may think of real **g**-representations as complex **g**-representations with the additional structure of a **g**-equivariant conjugate linear automorphism of V .

Given a $\mathfrak{g}$-representation on , one obtains representations on various associated spaces. For instance, one has a representation on the symmetric power $S^k(V)$ by
$$\pi(\xi)(v_1\cdots v_k) = \sum_{j=1}^{k} v_1\cdots\pi(\xi)v_j\dots v_k$$

and on the exterior power $\wedge^k(V)$ by
$$\pi(\xi)(v_1\wedge\cdots\wedge v_k) = \sum_{j=1}^{k} v_1\wedge\cdots\pi(\xi)v_j\dots\wedge v_k.$$

A similar formula gives a representation on the tensor powers$\otimes^k(V)$, and both $S^k(V)$,$\wedge^k(V)$are quotients of the

representation on$\otimes^k(V)$. One also obtains a dual representation on$V^* = Hom(V, \mathbb{C})$, by
$\langle \pi(\xi)\alpha, v\rangle = -\langle \alpha, \pi(\xi)(v)\rangle,\ \ \alpha \in V^*, v \in V$. [11]

3.1 Definition : A non-zero representation $V$of Gor $\mathfrak{g}$ is called *irreducible* or *simple* if it has no sub representations other than 0, V. Otherwise $V$is called *reducible*.

3.1 Example : Space $\mathbb{C}^n$, considered as a representation of $SL(n, \mathbb{C})$, is irreducible.

If a representation V is not irreducible (such representations are called *reducible*), then it has a nontrivial subrepresentation $W$and thus, Vcan be included in a short exact sequence $0 \to W \to V \to V/W \to 0$, thus, in a certain sense it is built out of simpler pieces. The natural question iswhether this exact sequence splits, i.e. whether we can write $V = W \oplus (V/W)$ as a representation.If so then repeating this process, we can write Vas a direct sum of irreducible representations.

3.2 Definition: A representation is called completely reducible (or *semi-simple*) if it is isomorphic to a direct sum of irreducible representations: $V \simeq \bigoplus V_i$, $V_i$irreducible.

In such a case one usually groups together isomorphic summands writing $V \simeq \bigoplus n_i V_i$,
$n_i \in \mathbb{Z}_+$, where V_i are pairwise non-isomorphic irreducible representations. The numbers n_i are called *multiplicities.*

However, as the following example shows, not every representation is completely reducible.

3.2 Example: Let $= \mathbb{R}$, so $\mathfrak{g} = \mathbb{R}$. Then a representation of $\mathfrak{g}$ is the same as a vector space V with a linear map $\mathbb{R} \to End(V)$, obviously, every such map is of the form $t \mapsto tA$ for some$A \in End(V)$ which can be arbitrary. The corresponding representation of the group $\mathbb{R}$ is given by $t \mapsto exp(tA)$. Thus, classifying representations of $\mathbb{R}$ is equivalent to classifying linear maps $V \to V$up to a change of basis. Such a classification is known (Jordan normal form) but non-trivial.

If v is an eigenvector of A then the one-dimensional space $\mathbb{C}v \subset V$is invariant under Aandthus a subrepresentation? . Since every operator in a complex vector space has an eigenvector, this shows that every representation of $\mathbb{R}$ is reducible, unless it is one-dimensional. Thus, the only irreducible representations of $\mathbb{R}$ are one-dimensional.

Now one easily sees that writing a representation given by $t \mapsto exp(tA)$ as a direct sum of irreducible ones is equivalent to diagonalizing A. So a representation is completely reducible iff$A$is diagonalizable. Since not every linear operator is diagonalizable, not every representation is completely reducible. Thus, more modest goals of the representation theory would be answering the following questions:

(1) For a given Lie group G, classify all irreducible representations of G.

(2) For a given representation $V$of a Lie group G, given that it is completely reducible, find explicitly the decomposition of Vinto direct sum of irreducibles.

(3) For which Lie groups Gall representations are completely reducible?

One tool which can be used in decomposing representations into direct sum is the use of central elements.

3.3 Lemma:Let V be a representation of G(respectively, $\mathfrak{g}$) and $A: V \to V$ a diagonalizable intertwining operator. Let $V_\lambda \subset V$ be the eigenspace for A with eigenvalue λ. Then each V_λ is asubrepresentation, so $V = \bigoplus V_\lambda$ as a representation of G(respectively $\mathfrak{g}$).

3.4 Lemma: Let V be a representation of G and let $Z \in Z(G)$be a central element of G suchthat $\rho(Z)$is diagonalizable. Then as a representation of G, $V = \bigoplus V_\lambda$, whereV_λ is the eigenspace for $\rho(Z)$with eigenvalue λ. Similar result also holds for central elements in $\mathfrak{g}$.

Of course, there is no guarantee that $V_\lambda$will be an irreducible representation; moreover, in many cases the Lie groups we consider have no central elements at all.

3.5 Example: Consider action of $GL(n, \mathbb{C})$on $\mathbb{C}^n \otimes \mathbb{C}^n$. Then the permutation operator $P: v \otimes w \mapsto w \otimes v$commutes with the action of $GL(n, \mathbb{C})$, so the subspaces$S^2\mathbb{C}^n$, $\Lambda^2\mathbb{C}^n$of symmetric and skew symmetric tensors (which are exactly the eigenspaces of P) are $GL(n, \mathbb{C})$-invariant $\mathbb{C}^n \otimes \mathbb{C}^n = S^2\mathbb{C}^n \oplus \Lambda^2\mathbb{C}^n$as a representation. In fact, both $S^2\mathbb{C}^n$, $\Lambda^2\mathbb{C}^n$are irreducible (this is not obvious but can be proved by a lengthy explicit calculation; later we will discuss better ways of doing this).Thus, $\mathbb{C}^n \otimes \mathbb{C}^n$is completely reducible.[12] .

3.1 Theorem: *Any finite-dimensional representation of a compact Lie group is unitary and thus completely reducible.*

IV. Orthogonality Relations for Finite Groups

Let v_i be a basis in a representation. Writing the operator $\rho(g): V \to V$in the basis v_i, we get a matrix-valued function on G. Equivalently, we can consider each entry $\rho_{ij}(g)$ as a scalar-valued function on G. Such functions are called *matrix coefficients* (of the representation).

4.1 Theorem:.

(1) *Let , W be non-isomorphic irreducible representations of G. Choose bases $v_i \in V$, $i = 1, \ldots, n$ and $w_\alpha \in W, \alpha = 1 \ldots m$. Then for any i, j, α, β, the matrix coefficients $\rho^V_{ij}(g)$, $\rho^W_{\alpha\beta}$ are orthogonal: $\langle \rho^V_{ij}(g), \rho^W_{\alpha\beta}\rangle = 0$, where $\langle\,,\rangle$ is the inner product on $\mathbb{C}^\infty(G, \mathbb{C})$ given by*

$$\langle f_1, f_2\rangle = \int_G f_1(g)\overline{f_2(g)}\,dg.$$

(2) *Let V be an irreducible representation of G and let $v_i \in V$ be an orthonormal basis with respect to a G-invariant inner product* (*which exists by Theorem* 2.19). *Then the matrix coefficients $\rho^V_{ij}(g)$ are pairwise orthogonal, and each has norm squared $\frac{1}{\dim V}$.*

$$\langle \rho^V_{ij}(g), \rho^V_{kl}\rangle = \frac{\delta_{ik}\delta_{jl}}{\dim V}$$

V. Conjugacy Classes for Lie Algebra

5.1 Definition:Let $\mathfrak{g}$be a Lie algebra and Gits adjoint group. Let X be any element of $\mathfrak{g}$.

Then, the conjugacy class of $X$is denoted by $\mathcal{O}_X$and is given by
$Ox = \{Y \in g | \exists h \in G\ (ad(h)Y = X)\}$

Note. When we say *orbit* we mean *adjoint orbit*. In the context of Lie algebras, we use the words *orbit* and *conjugacy class* interchangeably.

Finally, an adjoint orbit$\mathcal{O}_X$is called *nilpotent* (resp. *semisimple*), if $X$is nilpotent (resp. semisimple). That the orbits should fall naturally into these two categories can be seen by an example.

5.1 Example : Suppose $X$is in$\mathfrak{sl}(2)$ and is of the form$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then, over $\mathbb{F}$, its characteristicpolynomial is $(t-\lambda_1)(t-\lambda_2)$. Corresponding to the cases $\lambda_1 = \lambda_2$or $\lambda_1 \neq \lambda_2$, we conclude that any $X$is similar to either

$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ or $\begin{pmatrix} a & 0 \\ o & -a \end{pmatrix}$

for some a. In the former case, $X$is nilpotent and in the latter, it is semi-simple. Clearly, there are infinitely many semisimple classes.

Thus the semisimple orbits in $\mathfrak{sl}(2)$ can be parameterized by the set $K/(a \sim -a)$. Recall that the Weyl group $\mathcal{W} = \{1; s_\alpha\}$ acts on a Cartan subalgebra h by reflecting the origin.

Thus we may identify $K/(a \sim -a)$with $\mathfrak{h}/\mathcal{W}$. This result extends to any semisimple algebra.

5.1 Theorem : (Classification). *: Let*$\mathfrak{g}$*be a semisimple Lie algebra ,* $\mathfrak{h}$ *a Cartan subalgebra, and* $\mathcal{W}$ *the associated Weyl group. Then the set of semisimple orbits is in bijective correspondence with* $\mathfrak{h}/\mathcal{W}$. *In particular, there are infinitely many semisimple classes.*

Although the set of all semisimple orbits is infinite, it has a finite subset of a certain semisimple orbit called *distinguished*. The so-called weighted Dynkin diagrams that parameterize them.

We will define them later for nilpotent elements. The work of Jacobson-Morozov and Dynkin-Kostant proves that there is a bijective correspondence between the set of nilpotent orbits and the set of distinguished semisimple orbits in g. In particular, the nilpotent orbits in $\mathfrak{g}$ are finite. Richardson proves the result generally.

5.2 Theorem: (Richardson's Finiteness Theorem). *: Let* H*be a closed subgroup of* $G = GL(n, K)$, *and* $\mathfrak{h}$ *its Lie algebra. Let* $\mathfrak{h}$ *satisfy the following condition:*

(*) There exists a subspace m*of , stable under* $Ad(H)$, *for which* $\mathfrak{g} = \mathfrak{h} \oplus m$

Then, any G*-orbit in*$\mathfrak{g}$*intersects* $\mathfrak{h}$ *in only finitely many* H*-orbits. In particular,* $\mathfrak{h}$*has only finitely many nilpotent orbits.*[6] .

Note that condition (*) is satisfied by all reductive (hence semisimple) Lie algebras.

The following theorem in positive characteristic is due to Lusztig

5.3 Theorem: *If* $\mathfrak{g}$ *is simple and char* K *is positive, then* g *has only finitely many nilpotent orbits.*

In classical cases, the weighted Dynkin diagrams can be replaced by partitions. Now we quote some parameterization results with examples.

5.4 Theorem: In $\mathfrak{sl}(n)$, *there is a one-to-one correspondence between the set of nilpotent orbits and the set of partitions of* n. *The correspondence sends a nilpotent element X to the partition determined by the block sizes in its Jordan normal form.*

Here is the explicit correspondence: Let $[d_1, d_2, \dots, d_k]$ be a partition of n satisfying the conditions $d_1 + d_2 + \dots + d_k = n$ and $d_1 \leq d_2 \leq \dots \leq d_k > 0$.

Definition : Let J_i be the i by i *5.2*

matrix given by

J_i is called a Jordan

$$\begin{pmatrix} 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & 0 & \dots & 0 & 0 \end{pmatrix}.$$

Then
$X_{[d_1, d_2, \dots, d_k]} = J_{d_1} \oplus J_{d_2} \oplus \dots \oplus J_{d_k}$

is a nilpotent element of $\mathfrak{sl}(n, k)$. Henceforth, we will refer to $X_{[d_1, d_2, \dots, d_k]}$ as the nilpotent element associated with the partition $d_1, d_2, \dots, d_k$. Two distinct partitions give disjoint nilpotent classes.

5.2 Example: In $\mathfrak{sl}(3)$, *n* = 3. The partitions of 3 are: [3], [1 2], [1 1 1].

The corresponding nilpotent elements (representatives) are given by,
$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$
respectively.

5.5 Theorem: Nilpotent orbits in $\mathfrak{so}(n)$, n *odd are in one-to-one correspondence with the set of partitions of* n *in which even parts occur with even multiplicity .*

*5.3 Example:*In $\mathfrak{so}(5)$, *n* = 5. The partitions that give nilpotent orbits are
[1 1 1 1 1], [1 1 3], [1 2 2], [6].

5.4 Lemma: *Let* $\mathcal{O}_X$ *be the orbit containing the element* X. *Then* $\mathfrak{g}^X$*is a subalgebra of* $\mathfrak{g}$*and*
$dim(\mathcal{O}_X) = dim(\mathfrak{g}) - dim(\mathfrak{g}^X)$. [6]
n-tuples in $L(G)$ and conjugacy classes of subalgebras .

Let $\mathfrak{g}$ denotes a finite-dimensional Liealgebra over $\mathbb{F}$ and G is a closed subgroup of the algebraic group $Aut(\mathfrak{g})$ of all Lie algebra automorphisms of $\mathfrak{g}$. Recall that if $= (x_1, \dots, x_n) \in \mathfrak{g}^n$, then $c(x)$ denotes the Lie subalgebra of generated by $\{x_1, \dots, x_n\}$.

Two subalgebras $\mathfrak{a}$ and $\mathfrak{b}$ of are G-conjugate if there exists $g \in G$ such that $g.\mathfrak{a} = \mathfrak{b}$.

We let $L_n = L(X_1, \dots, X_n)$ denote the free Lie algebra over $\mathbb{F}$ on the indeterminates $X_1, \dots, X_n$. Each n-tuple $x = (x_1, \dots, x_n) \in \mathfrak{g}^n$ determines a Lie algebra homomorphism $\eta_x: L_n \to \mathfrak{g}$ given by $\eta_x(X_i) = x_i\ (i = 1, \dots, n)$. The image of $\eta_x$is$c(x)$. For each $a \in L_n$ the map $x \to \eta_x(a)$ is a morphism from $\mathfrak{g}^n$ to $\mathfrak{g}$.

Let δ denote the integer-valued function of $\mathfrak{g}^n$ given by $\delta(x) = \dim \mathrm{c(x)}$.

5.1 PROPOSITION : δ is a lower semicontinuous function.

Proof Let $x \in g^n$ and let $d = \delta(x)$. Then there exist dements $a_1, \dots, a_d$ ofL_n such that $(\eta_x(a_1), \dots, \eta_x(a_d))$ is an F-basis of $c(x)$. The map $: g^n \to g^d$, given by $\tau(y) = (\eta_y(a_1), \dots, \eta_y(a_d))$ is a morphism of algebraic varieties. Thus the set U of all $y \in g^n$ such that the vectors $\eta_y(a_1), \dots, \eta_y(a_d)$ are linearlyindependent is an open neighborhood of $x$in g^n. Clearly, if $y \in U$, then$\delta(y) \geq \delta(x)$. This proves that δ is lower semicontinuous.

5.1 COROLLARY: For each $d > 0$, let $V(d) = \{x \in g^n | dim\, c(x) \leq d\}$ and let $V(d)' = \{x \in g^n | dim\, c(x) = d\}$. Then $V(d)$ is a closed subset of g^n and $V(d)'$ is relatively open in $V(d)$.

Let $Gr_d(g)$ be the Grassmann variety of all d-dimensional subspaces of g andlet $A_d = A_d(g)$ be the closed subvariety of $Gr_d(g)$ consisting of all d-dimensional sub algebras of g. Let $X = \{(a, x) \in A_d \times g^n | x \in \mathfrak{a}^n\}$; X is a closedsubvariety of $A_d \times g^n$. Let $p_1: X \to A_d$ and$p_2: X \to g^n$ denote the restrictionsto X of the projections $pr_1: A_d \times g^n \to A_d$ and $pr_2: A_d \times g^n \to g^n$. Then $p_1$defines X as a vector bundle over A_d; the fibre over a $\mathfrak{a} \in A_d$ is isomorphic to $\mathfrak{a}^n$.

It is clear that $V(d) \supset p_2(X)$. Let $X' = p_2^{-1}(V(d)')$. Then X' is an open subset of X. Let $\pi_2: X' \to V(d)'$ denote the restiction of p_2.Then π_2 is a bijection.

The inverse bijection $\psi\, V(d)' \to X'$ is given by $\psi(x)\ (c(x), x)$,[1]

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