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Integral Solutions of the Sextic Equation with Five Unknowns

$$x^6 - 6w^2(xy+z) + y^6 = 2(y^2+w)T^4$$

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Abstract- We obtain infinitely many non-zero integer quintuples (x, y, z, w, T) satisfying the non-homogeneous sextic equation with five unknowns. Various interesting properties among the values of x, y, z, w and T are presented. Some relations between the solutions and special numbers are exhibited.

Index Terms- Integral solutions, sextic equation with five unknowns, special numbers.

MSC 2000 Mathematics subject classification: 11D41.

NOTATIONS:

$T_{m,n}$ -Polygonal number of rank n with size m

P_n^m - Pyramidal number of rank n with size m

SO_n -Stella octangular number of rank n

S_n -Star number of rank n

PR_n - Pronic number of rank n

OH_n - Octahedral number of rank n

J_n -Jacobsthal number of rank of n

j_n - Jacobsthal-Lucas number of rank n

KY_n -keynea number of rank n

$CP_{n,3}$ - Centered Triangular pyramidal number of rank n

$CP_{n,6}$ - Centered hexagonal pyramidal number of rank n

$F_{4,n,5}$ -Four Dimensional Figurative number of rank n whose generating polygon is a pentagon

$F_{4,n,3}$ -Four Dimensional Figurative number of rank n whose generating polygon is a triangle

I. Introduction

The theory of Diophantine equations offers a rich variety of fascinating problems [1-4]. Particularly, in [5,6], sextic equations with 3 unknowns are studied for their integral solutions. [7-12] analyse sextic equations with 4 unknowns for their non-zero integer solutions. [13, 14] deals with sextic equation with 5 unknowns.

This communication analyses a sextic equation with 5 unknowns given by $x^6 - 6w^2(xy+z) + y^6 = 2(y^2+w)T^4$. Infinitely many non-zero integer quintuples (x, y, z, w, T) satisfying the above equation are obtained. Various interesting properties among the values of x, y, z, w and T are presented.

II. METHOD OF ANALYSIS

The sextic equation with five unknowns to be solved is

$$x^6 - 6w^2(xy+z) + y^6 = 2(y^2+w)T^4 \qquad (1)$$

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The introduction of the linear transformations,

$$x = u+v, y = u-v, w = 2uv, z = 2v^2 \qquad (2)$$

in (1) leads to

$$u^2 + v^2 = T^2 \qquad (3)$$

The above equation (3) is solved through different approaches and thus, one obtains

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different sets of solutions to (1)

Approach12:

The solution to the Pythagorean equation (3) is,

$$u = 2pq, \quad v = p^2 - q^2, \quad T = p^2 + q^2 \tag{4}$$

In view of (2) and (4), the corresponding values of x, y, z, w and T are represented by

$$\left.\begin{aligned} x(p,q) &= 2pq - p^2 - q^2 \\ y(p,q) &= 2pq - p^2 + q^2 \\ z(p,q) &= 2(p^2 - q^2)^2 \\ w(p,q) &= 4pq(p^2 - q^2) \\ T(p,q) &= p^2 + q^2 \end{aligned}\right\} \tag{5}$$

Properties:

1. $x(a(a+1),a) + y(a(a+1),a) = 4(6p_a^3 - 6T_{3,a} - 2SO_a + 4CP_{a,6})$

2. The following expressions are nasty numbers:
   (a) $30[5S_a + 90(OH_a) - 60CP_{a,6} - x(2a,a) + y(2a,a)]$
   (b) $3[x(a,b) + y(a,b) + 2T(a,b)]$

3. .The following expressions are cubic integers:
   (a) $2T(a,b)[(x(a,b) + y(a,b))^2 + 2z(a,b)]$
   (b). $9[T(2^{2n}, 2^{2n}) - 2KY_{2n} + j_{2n+2}]$

4. $4[w(a,1) + z(a,1) + 2T(a,1) - (8CP_{a,3} + 48F_{4,a,3} - 72P_a^3 + 24T_{3,a} + 2T_{12,a} - 10T_{4,a})]$
   is a biquadratic integer

5. $$432F_{4,a,3} - 504P_a^3 + 134T_{3,a} - [x(2a,a) + y(2a,a) + z(2a,a) + w(2a,a) + T(2a,a)] \equiv 0 \pmod 7$$

6. $x(a,a).y(a,a) + z(a,a).w(a,a) + T(a,a) - 2CP_{a,6}(3T_{4,a} - T_{8,a}) - T_{6,a} - 2T_{3,a} + T_{4,a} = 0$

7. $S_a + 9T_{4,a} - 2T_{10,a} - 2T_{4,a}.T_{3,a} + T_{4,a} + CP_{a,6} - x(a,1).y(a,1) - T(a,a) = 1$.

8. $w(2a,a).T(2a,a) - 120[6F_{4,a,5} - 18P_a^3 + 14T_{3,a} - 3(OH_a) + 2CP_{a,6}] = 0$

Remark:

The equation (4) can also be written as

$$u = p^2 - q^2, \quad v = 2pq, \quad T = p^2 + q^2 \tag{6}$$

and the corresponding solution can be obtained.

Approach2:
(3) can be written as

$$T^2 - v^2 = u^2 \tag{7}$$

Writing (18) as a set of double equations in two different ways as shown below:

Set1: $T + v = u^2, T - v = 1,$

Set2: $T + v = 1, T - v = u^2,$

Solving **set1,** the corresponding values of u, v *and* T are given by

$$v = 2k^2 + 2k, u = 2k+1, T = 2k^2 + 2k + 1 \tag{8}$$

In view of (19) and (2), the corresponding solutions to (1) obtained from set1 are represented as shown below:

$$\left.\begin{aligned} x(k) &= 2k^2 + 4k + 1 \\ y(k) &= 1 - 2k^2 \\ z(k) &= 8(k^2+1)^2 \\ w(k) &= 4(k^2+k)(2k+1) \\ T(k) &= 2k^2 + 4k + 1 \end{aligned}\right\} \tag{9}$$

Properties:

1. $w(k) + z(k) - 2x(k)(t(k) - 1) = 0$

2. $3[x(k) + y(k) + z(k) + T(k) + 6SO_k - 12CP_{k,6} - 3 - 2T_{4,k}]$ is a nasty number

3. The following expressions are cubic integers
   (a) $2[x(k) + y(k) + 2T(k) - 8T_{3,k} - 6T_{4,k} + 2T_{8,k}]$
   (b) $2z(k)w(k) - 128(6P_k^3 + 2T_{3,k} - 2T_{4,k} + 1)T_{4,k}^2$

4. $(x(k) - y(k))^2 - (z(k))^2 - 48F_{4,k,5} + 16P_k^5 = 0$

5. $8\,[T(2^{2n}, 2^{2n}) - 2KY_{2n} + j_{2n+1}]$ is a biquadratic integer

6. $w(k) + y(k) - 16P_k^5 - 7T_{4,k} + T_{8,k} = 1$

7. $12T_{3,k} - 12P_k^4 + 1 - y(k)T(k) - 4(2T_{3,k}.T_{4,k} - CP_{k,6}) \equiv 0 \bmod(2)$

8. $z(k) + T(k) - 192F_{4,k,3} + 64P_k^5 + 92T_{3,k} = 1$

9. $y^2(k)+z(k)-288F_{4,k,3}+336P_k^3-80T_{3,k}=1$

Similarly, the solutions corresponding to set2 can also be found.

Approach3:

Now, rewrite (3) as, $u^2+v^2=T^2*1$ (10)

Let $T=a^2+b^2$ (11)

Also 1 can be written as

$$1=i^n.(-i)^n \tag{12}$$

Substituting (8) in (9) in (7) and using the method of factorisation, define

$$u+iv=i^n(a+ib)^2 \tag{13}$$

Equating real and imaginary parts in (10) we get

$$\left.\begin{aligned} u&=\cos\frac{n\pi}{2}(a^2-b^2)-2ab\sin\frac{n\pi}{2}\\ v&=\sin\frac{n\pi}{2}(a^2-b^2)+2ab\cos\frac{n\pi}{2}\end{aligned}\right\} \tag{14}$$

In view of (2), (8) and (14), the corresponding values of x,y,z,w,p,T are represented as

$$\left.\begin{aligned} x&=\cos\frac{n\pi}{2}(a^2-b^2+2ab)+\sin\frac{n\pi}{2}(a^2-b^2-2ab)\\ y&=\cos\frac{n\pi}{2}(a^2-b^2-2ab)-\sin\frac{n\pi}{2}(a^2-b^2+2ab)\\ w&=2[\cos\frac{n\pi}{2}(a^2-b^2)-\sin\frac{n\pi}{2}2ab][\sin\frac{n\pi}{2}(a^2-b^2)+2ab\cos\frac{n\pi}{2}]\\ z&=2[\sin\frac{n\pi}{2}(a^2-b^2)+2ab\cos\frac{n\pi}{2}]^2\\ T&=a^2+b^2\end{aligned}\right\} \tag{15}$$

Approach4:

Write 1 as $1=\dfrac{(2mn+i(m^2-n^2)(2mn-i(m^2-n^2)}{(m^2+n^2)^2}$

Following the same procedure as above we get the integral solution of (1) as

$$\left.\begin{aligned} x &= (m^2+n^2)[f_1(A,B)+g_1(A,B)] \\ y &= (m^2+n^2)[f_1(A,B)-g_1(A,B)] \\ z &= 2(m^2+n^2)^2 g_1{}^2(A,B) \\ w &= 2(m^2+n^2)^2 f_1(A,B).g_1(A,B) \\ T &= (m^2+n^2)^2(A^2+B^2) \end{aligned}\right\} \quad (16)$$

Where

$$\left.\begin{aligned} f_1(A,B) &= [2mn(A^2-B^2)-2AB(m^2-n^2)] \\ g_1(A,B) &= [(m^2-n^2)(A^2-B^2)+4mnAB] \end{aligned}\right\} \quad (17)$$

The above approaches satisfy the following interesting relations:

1. $w-z-xy+y^2=0$

2. If u,v are the generators of the Pythagorean triangle with sides $(u^2-v^2, 2uv, u^2+v^2)$ then
   (i). $x(u,v).y(u,v).w(u,v)=2(area\ of\ the\ triangle)$
   (ii). $x^2+y^2-z+w = the\ perimeter\ of\ the\ pythagorean\ triangle$

3. The following expressions are nasty numbers:
   (a) $6(z^2+w^2)$
   (b) $3z(T^2+w)$
   (c) $6[x^2(1-z)+y^2(1-z)+z^2+w(4+w)+2T^2]$
   (d) $\dfrac{6(x+y)(2xy+2T^2+2zw)}{2x+(x-y)(z-1)}$

5. The following expressions are cubical integers:
   (a) $T^2(x^2y^2+w^2)$
   (b) $4(x^3+y^3)-6(x+y)z$
   (c) $\dfrac{wz}{2(x+y)}$
   (d) $x(T^2+w)$

6. The following expressions are biquadratic integers:
   (a) $4(x+y)^2T^2+4(w^2+z^2)$
   (b) $16xyT^2+4z^2$

7. $x^2-y^2-2w=0$

8. $z^2-w^2+2xyz=0$

9. 10. $4xyzwT^2 = x^2w^3-x^2z^2w+w^3y^2-wz^2y^2$

10. $x+y+z+w+2T^2-(2x-y)x-y^2 \equiv 0 \pmod 2$

11. $2x^2+4yz+2zw-2T^2=2w+(x-y)(x^2-y^2-2z+xz+yz)$

III. CONCLUSION

IV.

In conclusion, one may search for different patterns of solutions to (1) and their **c**orresponding properties

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