CARDINALITY OF AN ELEMENTARY COUNTABLY INFINITE SET.

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Abstracts.

Using Mathematical Induction to resolved the cardinality of an '*m*' countable infinite sets relating it to a cardinality of natural $'\mathbb{N}'$ and integer numbers $'\mathbb{Z}'$.which is helpful in basic elementary set.

Keyword: Set, Event, Cardinality of set, countable infinite , Natural number and Integer number.

Building blocks

Definition 1.1

A collection of objects which are defined and distinguishable are called set. The objects comprising in a set are generally called element which can be finite or infinite in numbers.

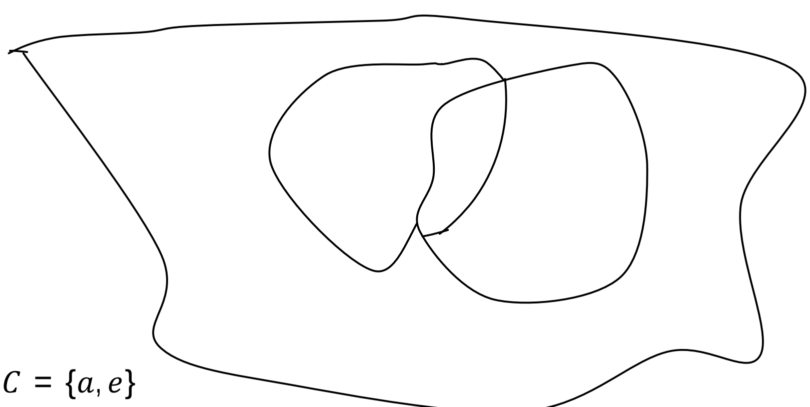
Definition 1.2

Universal set could be denoted by U , is all sets under investigation which are regarded as subsets of fixed set. e.g: in human population studies.

Definition 1.3

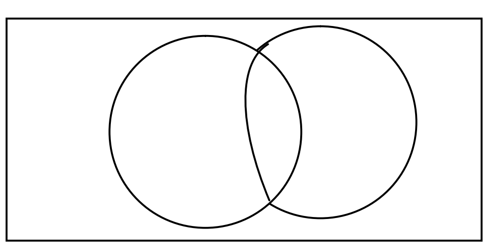
Venn diagram is a diagrammatical representation of set statement which was invented by John Venn (1834 - 1923) an English Mathematician.

If $B = \{x : x \text{ is the irst ive English Alphabet}\}$ and $C = \{x : x \text{ is a vowel English Alphabet}\}$,



$B \cap C = \{a, e\}$

For the sake uniformity, rectangle shape represent universal set U and circle or oval represents the subsets.



For more illustration on Venn diagram using union,intersection,complement and relative complement see [1]

A defined set which has no element is empty or null set, denoted by Ø or { }.

If every element of one set is also a member of a second set then the first set is a subset of the second set.

2.0 PROPERTIES OF SET

Theorem 2.1

let A,B and C be any set then:

I. $A \cup B = B \cup A$, $A \cap B = B \cap A$ } commutative

II. $A \cup A = A$ } idempotent

III. $A \cap A = A$

IV. $A \cup \emptyset = A$

V. $A \cap \emptyset = \emptyset$

VI. $A \cup (B \cup C) = (A \cup B) \cup C$, $A \cap (B \cap C) = (A \cap B) \cap C$ } associative

proof

Let $x \in A \cup (B \cup C)$

- ⇨ $x \in A \; or \; x \in (B \cup C)$
- ⇨ $x \in A \; or \; x \in B \; or \; x \in C$
- ⇨ $x \in (A \cup B) \; or \; x \in C$
- ⇨ $x \in (A \cup B) \cup C$

Let $x \in A \cap (B \cap C)$

- ⇨ $x \in A \; and \; x \in B \cap C$
- ⇨ $x \in A \; and \; x \in B \; and \; x \in C$
- ⇨ $x \in (A \cap B) \; and \; x \in C$
- ⇨ $x \in (A \cap B) \cap C$

VII. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ (distributive)

proof

Let $x \in A \cap (B \cup C)$

- ⇨ $x \in A \; and \; x \in (B \cup C)$
- ⇨ $x \in A \; and \; x \in B \; or \; x \in C$
- ⇨ $x \in A \; and \; x \in B \; or \; x \in A \; and \; x \in C$
- ⇨ $x \in (A \cap B) \; or \; x \in (A \cap C)$
- ⇨ $x \in (A \cap B) \cup (A \cap C)$

VIII. $(A \cup B)^c = A^c \cap B^c$ de morgan's law

proof

Let $x \in (A \cup B)^c$

- ⇨ $x \notin A \cup B, x \notin A \; or \; x \notin B$

⇨ $x \in A^c \; and \; x \in B^c$

⇨ $x \in A^c \cap B^c$

Definition 2.4

A set E is called a subset of set S if every element of E is also an element of S. $E \subseteq S$ or $E \supseteq K$, otherwise $E \subset S$ if there exist at least one element in E that is not in S. for the definition it is trivial to say every set a subset itself.

Theorem 2.2

The empty set Ø is a subset of every set.

proof

Suppose E is a set, then either $E \neq \emptyset \; or \; E = \emptyset$

If $= \phi$, then a set is a subset of itself. if $E \neq \phi$,either $\phi \not\subset K$ or $\phi \subset K$.

Suppose $\phi \not\subset E$, then there exist at least one element in ϕ that is not in E, which contradict the definition of null set .since there is no such element imply that $\phi \subset E$

3.0 COUNTABLE AND UNCOUNTABLE SET

Theorem 3.1: The union of countably many countable sets is a countable set.

Proof: Let the countably many sets be denoted

$$E_1, E_2, E_3, E_4, \dots ..$$

As a “worst case scenario,” we may assume that there is a countably infinite number of sets, and that each set itself is countably infinite. Consider the union

$$E = \bigcup_{i=1}^{\infty} E_i = E_1 \cup E_2 \cup E_3 \cup E_4 \cup \dots \dots$$

of these sets. To show that it is countable, we must put it into bijection with ℵ

Suppose a set E_i has elements

$$e_{i1}, e_{i2}, e_{i3}, e_{i4}, \dots .$$

Then the elements of the union of the sets could be arranged in a doubly-infinite array;

$$\begin{matrix} e_{11} & e_{12} & e_{13} & e_{14} & \cdots \\ e_{21} & e_{22} & e_{23} & e_{24} & \cdots \\ e_{31} & e_{32} & e_{33} & e_{34} & \cdots \\ e_{41} & e_{42} & e_{43} & e_{44} & \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{matrix}$$

Creating a bijection by counting each element as we follow the arrows:

$$\begin{array}{ccccccc}
e_{11} & \rightarrow & e_{12} & & e_{13} & \rightarrow & e_{14} \\
\downarrow & \swarrow & & \nearrow & & \swarrow & \\
e_{21} & & e_{22} & & e_{23} & & \\
\downarrow & \nearrow & & \swarrow & & & \\
e_{31} & & e_{32} & & & & \\
 & \swarrow & & & & & \\
e_{41} & & & & & & \\
\vdots & & & & & &
\end{array}$$

In other words, we have a pairing

$$\begin{array}{ccccccccc}
e_{11} & e_{12} & e_{21} & e_{31} & e_{22} & e_{13} & e_{14} & e_{23} & \cdots \\
\updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \\
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 &
\end{array}$$

Of each element of E with a natural number, and so E is countably infinite.[5]

Corollary 3.1: The set $\mathbb{Z}$ of integers is countably infinite.

Proof.: exhibit $\mathbb{Z}$ as a countable union of countable sets:

$$\mathbb{Z} = \{1,2,3,4, \ldots..\} \cup \{0\} \cup \{-1,-2,-3,-4, \ldots..\}$$

Corollary 3.2.

- The set Q of rational numbers is countably infinite.
- The cartesian product of finitely many countable sets is count-able. see [5] for proofs.

Theorem 3.2. The set R of real numbers is uncountable.

4.0 CARDINALITY OF A SET

Definition4.1

Cardinality of set is the number of the element given in the set.

Two sets A and B have the same cardinality , written $|A| = |B|$ if there exists a bijective function $F : A \rightarrow B$ If no such bijective function exists, then the sets have unequal cardinalities, that is, $|A| \neq |B|$.

Theorem 4.1 :There exists a bijection $F : \mathbb{N} \rightarrow \mathbb{Z}$, Therefor $|\mathbb{N}| = |\mathbb{Z}|$

Theorem 4.2 There exists no bijection $: \mathbb{N} \rightarrow \mathbb{R}$, Therefore $|\mathbb{N}| = |\mathbb{R}|$.

Definition 4.2 The cardinality of the natural numbers is denoted as $\aleph_0$. That is, $|N|=\aleph_0$. Thus any countably infinite set has cardinality $\aleph_0$.

Theorem 4.3 A set E is countably infinite if and only if its elements can be arranged in an infinite list e_1, e_2, e_3, e_4 ....

Let E = be the set of even integers. The function $f: \mathbb{Z} \rightarrow E$ defined as $f(n) = 2n$ is easily seen to be a bijection, so we have $|\mathbb{Z}| = |E|$. Thus, as $|\mathbb{N}| = |\mathbb{Z}| = |E|$. the set E is countably infinite and $|E| = \aleph_0$.

Theorem 4.4 If A and B are both countably infinite, then A∪B is countably infinite.
Proof. Suppose A and B are both countably infinite. By Theorem 4.3, we know we can write A and B in list form as

$$A = \{a_1, a_2, a_3, a_4, \dots.\}$$
$$B = \{ b_1, b_2, b_3, b_4, \dots.\}$$

We can "shuffle" A and B into one infinite list for A∪B as follows.

$$A \cup B = \{ a_1, b_1, a_2, b_2, a_3, b_3, a_4, b_4, \dots.\}$$

(We agree not to list an element twice if it belongs to both A and B .)
Therefore, by Theorem 4.3, it follows that A∪B is countably infinite.

$$|A \cup B| = \aleph_0$$

Then the cardinality of the union of multiple countably infinite sets are as follows:

.**<u>Case 1</u>**

Let $i = 1,2,3, \dots \dots \dots, n$ such that E_i are sequence of sets (event) which are:

countably infinite, non-empty and non-disjoint.

$$|E_1 \cup E_2| = |E_1| + |E_2| - |E_1 \cap E_2|$$

$$|E_1 \cup E_2 \cup E_3| = |E_1| + |E_2| + |E_3| - |E_1 \cap E_2| - |E_1 \cap E_3| - |E_2 \cap E_3| + |E_1 \cap E_2 \cap E_3|$$

For 4th – term

$$|\bigcup_{i=1}^{4} E_i| = \sum_{i=1}^{4} |E_i| - \sum_{1 \le i < j \le 4} |E_i \cap E_j| + \sum_{1 \le i < j < k \le 4} |E_i \cap E_j \cap E_k| - |E_1 \cap E_2 \cap E_3 \cap E_4|$$

s.t $i, j, k \in \mathbb{N}$

For n = m -1 (assumed)

$$\bigcup_{i=1}^{m-1} E_i = \sum_{i=1}^{m-1} |E_i| - \sum_{1 \le i < j \le m-1} |E_i \cap E_j| + \sum_{1 \le i < j < k \le m-1} |E_i \cap E_j \cap E_k| - \cdots. + (-1)^{m-2} |E_1 \cap E_2 \cap \dots. \cap E_{m-1}|$$

For $n = m$

$$= |\bigcup_{i=1}^{m} E_i| = \left|\left(\bigcup_{i=1}^{m-1} E_i\right) \cup E_m\right| = |X \cup E_m|$$

where $X = \bigcup_{i=1}^{m-1} E_i$

$$= |X| + |E_m| - |X \cap E_m| = \sum_{i=1}^{m} |E_i| - \sum_{i=1}^{m-1} |(\cup E_i) \cap E_m|$$

$$= \sum_{i=1}^{m} |E_i| - \sum_{i=1}^{m-1} |\bigcup_{i=1}^{m-1} (E_i \cap E_m)|$$

(Distributive)

$$= \sum_{i=1}^{m} |E_i| - \sum_{1 \le i < j \le m} |E_i \cap E_j| + \sum_{1 \le i < j < k \le m} |E_i \cap E_j \cap E_k| - \cdots + (-1)^{m-1} |E_1 \cap E_2 \cap \ldots \cap E_m|$$

………….. (1)

Case 2

Let $i = 1,2,3, \ldots \ldots n$ such that E_i are sequence of set (event) which are:

countably infinite, non empty but all are disjoint.

Given $(E_i \cap E_j) = \phi, 1 \le i < j \le m$ (all are disjoint), $\phi \cap E_i = \phi$ and cardinality of empty set is zero. Then:

$$|\bigcup_{i=1}^{m} E_i| = \sum_{i=1}^{m} |E_i|$$

……………..(2)

Case 3

Let $i = 1,2,3, \ldots \ldots n$ such that E_i are sequence of set (event) which are:

countably infinite, non empty but some are disjoint

$$|\bigcup_{i=1}^{m} E_i| = \sum_{i=1}^{m} |E_i| - \sum_{1 \le i < j \le m-q}^{m-q} |E_i \cap E_j| + \sum_{1 \le i < j < k \le m-q}^{m-q} |E_i \cap E_j \cap E_k| + - \cdots + (-1)^{m-1} (0)$$

……………(3)

Since

$|E_1 \cap E_2 \cap \ldots \cap E_m| = 0$

Let $m - q$ are set that are disjoint. Where $i < q < m$

Case 4

Let $i = 1,2,3, \ldots \ldots n$ such that E_i are sequence of set (event) which are:

countably infinite and at least one set is empty which is finite. Where $r < m$

$$\left|\bigcup_{i}^{m} E_i\right| = \sum_{i=1}^{m-r} |E_i| + \sum_{1 \le i < j \le m-r} |E_i \cap E_j| + \sum_{1 \le i < j < k \le m-r} |E_1 \cap E_2 \cap E_3| - \cdots + (-1)^{m-r-1} \sum_{1 \le i < j < .. \le m-r-1} |E_1 \cap E_2 \cap \ldots \ldots . \cap E_{m-r-1}| \quad \ldots\ldots\ldots.. [4]$$

m - r is the number of empty set present

Case 5

Cardinality of *m*- multiple Finite set that is not disjoint and non empty tends to case 1 equation 1 which is equal to $\aleph_0$

5.0 CONCLUSION

As m – multiple countable infinite sets tends to infinity ($m \to \infty$),the equation (1,2,3,4) showns the cardinality of union and the intersection of any countably infinite sets , is definitely a member of positive integers and the equations can also be relevant to cardinality of union and intersection of finite sets.

$$X =: \begin{cases} x \in \mathbb{N} \subset \mathbb{Z}^{+} \\ \\ x = \aleph_0 \end{cases}$$

X =:{$X = x$ 'Cardinality of union of countably infinite sets}

6.0 RECOMMENDATION

There should be a computer based program to solve cardinality of countably infinite sets, if possible draw its Venn diagram.

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