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Discounted Generalized Transportation Problem

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Abstract- In generalized transportation problem(GTP), the cost of transportation c_{ij} per unit product from the i^{th} origin to the j^{th} destination is considered as independent of amount of transported commodity x_{ij}. But in real life problems, there are many situations, e.g. quantity discount, price break etc., in which cost of transportation c_{ij} depends upon the amount of transported commodity x_{ij}. Based on these situations, in this paper, we consider a new type of discounted generalized transportation problem in which the cost of transportation c_{ij} per unit product depends on the amount of transported commodity x_{ij}. Thereby, we develop a new algorithm for obtaining the optimum solution of this problem. Finally, a numerical example is illustrated to support the algorithm.

Index Terms- Generalized transportation problem, Step function, Discount, Discounted generalized transportation problem

I. Introduction

Hitchcock [15] was pioneer of the basic transportation problem, Dantzig [13], Charnes and Cooper [11], Appa [1] developed further. Now a days, there are several procedures to solve the transportation problems. Arsham and Khan [2] considered simplex-type algorithm for general transportation problems. Basu et. al. [7,8,9] considered different types of transportation problems.

But in real life, there are many situations, e.g. quantity discount, price breaks etc. where the transportation cost may not be linear. Non linearity depends upon the character of the objective function as well as the character of the constraints. Cooper and Dredes [12] considered an approximate solution method for the fixed charge problem. Bhatia et. al. [10] considered time-cost trade-off in a transportation problem. Klingman and Russel [16] considered solving constrained transportation problems. Thirwani [17] considered fixed charge bi-criterion transportation problem with enhanced flow.

There are many business problems, industrial problems, machine assignment problems, routing problems, etc. that have the characteristics in common with generalized transportation problem that has been studied by several authors. Balas and Ivanescu [4] introduced on the generalized transportation problem. Balachandrana and Thompson [3] considered an operator theory of parametric programming for the generalized transportation Problem. In 1987, Hadley [14] gave the detailed solution procedure for solving generalized transportation problem. Basu and Acharya [5,6] considered different types of generalized transportation problem.

Day by day, the importance of discounted generalized transportation problem is increasing practically in a great deal, but the method for finding the optimum solution of this kind of generalized transportation problems, however, lacks of the desired attention.

There are several differences between classical transportation problem and generalized transportation problem which are given as follows:

1. The rank of the co-efficient matrix of $[x_{ij}]_{mn}$ in generalized transportation problem is (m+n), where as in classical transportation problem it is (m+n – 1)

2. In generalized transportation problem the value of x_{ij} may not be integer though it must be integer in classical transportation problem.

3. The activity vector in generalized transportation problem is $p_{ij} = d_{ij}\, e_i + e_{m+j}$, where d_{ij} = positive constants rather than unity.

Where as in classical transportation problem it is given by $p_{ij} = e_i + e_{m+j}$.

4. In generalized transportation problem, it need not be true that cells corresponding to basic solution form a tree.

5. In generalized transportation problem that $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$ is not necessary.

In this paper we develop a new algorithm to find the solution of discounted generalized transportation problem where the cost function is taken as step function. Thereby we illustrate this problem numerically.

II. PROBLEM FORMULATION

Let the discounted generalized transportation problem consists of 'm' origins and 'n' destinations, where

x_{ij} = the amount of product transported from the i^{th} origin to the j^{th} destination,
c_{ij} = the cost involved in transporting per unit product from the i^{th} origin to the j^{th} destination,
a_i = the number of units available at the origin i,
b_j = the number of units required at the destination j,
d_{ij} = positive constants rather than unity.

Then the cost minimizing discounted generalized transportation problem can be stated as:

$$P_1: \quad Min\, Z = \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij} \tag{1}$$

subject to,

$$\sum_{j=1}^{n} d_{ij}x_{ij} \le a_i\,; \quad \text{for } i=1,2,3,\ldots\ldots\ldots,m. \tag{2}$$

$$\sum_{i=1}^{m} x_{ij} = b_j\,; \quad \text{for } j=1,2,3,\ldots\ldots\ldots,n. \tag{3}$$

and

$$c_{ij} = \begin{cases} c_{ij}^1 & if\ o \le x_{ij} \le x_{ij}^1 \\ c_{ij}^2 & if\ x_{ij}^1 \le x_{ij} \le x_{ij}^2 \\ c_{ij}^3 & if\ x_{ij}^2 \le x_{ij} \le x_{ij}^3 \\ \ldots\ldots\ldots & \\ \ldots\ldots\ldots & \\ \ldots\ldots\ldots & \\ c_{ij}^g & if\ x_{ij}^{g-1} < x_{ij} \end{cases} \tag{4}$$

where $c_{ij}^1 > c_{ij}^2 > c_{ij}^3 > \ldots\ldots\ldots > c_{ij}^g$ and $x_{ij} \ge 0$.

Net Evaluation
Introducing slack variables, the problem P_1 can be written as

$$P_2: \quad Min\, Z = \sum_{i=1}^{m}\sum_{j=1}^{n+1} c_{ij}x_{ij} \tag{5}$$

subject to,

$$a_i - \sum_{j=1}^{n} d_{ij}x_{ij} - x_{i,n+1} = 0\,; \quad \text{for } i=1,2,3,\ldots\ldots\ldots,m. \tag{6}$$

$$b_j - \sum_{i=1}^{m} x_{ij} = 0\,; \quad \text{for } j=1,2,3,\ldots\ldots\ldots,n,n+1. \tag{7}$$

where d_{ij}=positive constants rather than unity, $x_{ij} \ge 0$ for all (i,j), and the values of c_{ij} are given in (4).

Let u_i ($1\le i \le m$) and v_j ($1\le j \le n+1$) be the dual variables.
So that, $d_{ij}u_i + v_j = c_{ij}$ for $1\le I \le m$, $1\le j \le n$.
$u_i = 0$ for $1\le I \le m$, $j = n + 1$. (8)
where u_i, v_j are unrestricted for all (i,j).

Now for any standard Primal L.P.P. with basis B and associated cost vector c_B, the associated solution to its dual problem W_B, is given by $W_B = c_B B^{-1}$. Thus, if p_j is the j^{th} column of the primal constraint matrix, then an expression for evaluating the net-evaluation for minimization problem is given by

$$Z_j - c_j = c_B\,(B^{-1} p_j) - c_j \tag{9}$$
$$= W_B\, p_j - c_j \quad \forall\ j.$$

But, in the present case of rectangular transportation problem, the dual solution can be represented by (u,v) = ($u_1,u_2,\ldots\ldots,u_m,v_1,v_2,\ldots\ldots\ldots,v_n$) and therefore the net evaluations are analogously obtained by simply replacing $c_j \to c_{ij}$, $W_B \to (u,v)$, $p_j \to p_{ij}$ in the above formula. Thus we have the net evaluation as:

$$\begin{aligned} Z_{ij} - c_{ij} &= (u,v)\, p_{ij} - c_{ij} \\ &= (u_1,u_2,\ldots\ldots,u_m,v_1,v_2,\ldots\ldots,v_n)\,[\,d_{ij}e_i + e_{m+j}] - c_{ij} \\ &= (d_{ij}\, u_i + v_j) - c_{ij} \quad \text{for } i=1,2,\ldots\ldots,m;\ j=1,2,\ldots\ldots\ldots,n+1. \end{aligned} \tag{10}$$

where $p_{ij}(= d_{ij}e_i+e_{m+j})$ is the column vector of the constraint matrix associated with the rectangular variable x_{ij}. For simplicity, we shall denote the net evaluation $Z_{ij} - c_{ij}$ by Δ_{ij} in all our further discussion.

Theorem: A solution of the discounted generalized transportation problem will be feasible solution if for any cell (q,r), $c_{qr}^k\, x_{qr} < c_{qr}^l\,(x_{qr} + y_{qr})$, where $1\le k<l \le g$ also $y_{qr}>0$ and $c_{qr}^k > c_{qr}^l$ for $1\le q \le m$, $1 \le r \le n$.
Proof: Let Z_1 be the total cost and F_1 be the total flow where (q,r) be one of its allocated cell and x_{qr} be its allocation where ($x_{qr}^{k-1} < x_{qr} \le x_{qr}^k$).

Then $$Z_1 = \sum_{i=1,i\ne q}^{m}\sum_{j=1,j\ne r}^{n} c_{ij}x_{ij} + c_{qr}^k x_{qr} \tag{11}$$

and $$F_1 = \sum_{i=1,i\ne q}^{m}\sum_{j=1,j\ne r}^{n} x_{ij} + x_{qr} \quad \text{where} \quad x_{qr}^{k-1} < x_{qr} \le x_{qr}^k \tag{12}$$

Also let Z_2 be the total cost and F_2 be the total flow if $(x_{qr} + y_{qr})$ where $x_{qr}^{l-1} < x_{qr} + y_{qr} \leq x_{qr}^{l}$, be allocated at the (q,r) cell and all other allocations are same as in flow F_1. Therefore

$$Z_2 = \sum_{i=1,i\neq q}^{m} \sum_{j=1,j\neq r}^{n} c_{ij} x_{ij} + c_{qr}^{l}(x_{qr} + y_{qr}) \qquad (13)$$

and $$F_2 = \sum_{i=1,i\neq q}^{m} \sum_{j=1,j\neq r}^{n} x_{ij} + (x_{qr} + y_{qr}) \quad \text{where } x_{qr}^{l-1} < (x_{qr} + y_{qr}) \leq x_{qr}^{l} \qquad (14)$$

Obviously $F_1 + y_{qr} > F_1$ where $x_{qr}^{l-1} < (x_{qr} + y_{qr}) \leq x_{qr}^{l}$

Let us assume that the given condition $c_{qr}^{k} x_{qr} > c_{qr}^{l} (x_{qr} + y_{qr})$ does not hold. Then

$Z_1 - Z_2 = c_{qr}^{l} x_{qr} - c_{qr} (x_{qr} + y_{qr}) \geq 0$; => Z_1 > Z_2

But $F_2 > F_1$; So $Z_1 > Z_2$ => $F_1 < F_2$, which is a contradiction.
So, our assumption is wrong. Hence the theorem.

III. ALGORITHM

Step 1. Find the initial solution X_{ij}^{B1} with the associated cost vector c_B^1 and the corresponding cost Z^1 by using North West Corner Rule.

Step 2. Set r=1.

Step 3. Write X_{ij}^{Br} with the associated vector c_B^r and corresponding cost Z^r.

Step 4. Calculate dual variables u_i $(1 \leq i \leq m)$ and v_j $(1 \leq j \leq n)$.

Step 5. Calculate net evaluation Δ_{ij} $\forall$ $(i,j) \notin B$. If $\Delta_{ij} \leq 0$ $\forall$ $(i,j) \notin B$. Then go to step 10.

Step 6. Calculate $\Delta_{st} = \text{Max } \{\Delta_{ij} : \Delta_{ij} > 0\}$. Then (s,t) cell enters into the basis.

Step 7. Let $\sum_{j\in B} d_{ij} y_{st}^{ij} = d_{st} e_s$; $1 \leq i \leq m$.

$\sum_{i\in B} y_{st}^{ij} = e_{m+t}$; $1 \leq j \leq n+1$.

where $p_{st} = d_{st} e_s + e_{m+t}$. After determining y_{st}^{ij} we calculate

$$\theta = \min\left\{\frac{x_{ij}^{B}}{y_{st}^{ij}} : y_{st}^{ij} > 0\right\} = \frac{x_{kl}^{B}}{y_{st}^{kl}}$$

Then (k,l) cell leaves the basis.

Step 8. The improved solution is $\widehat{x_{ij}^{B}} = x_{ij}^{B} - y_{st}^{ij} \times \theta$ for $(i,j) \in B$ and $(i,j) \neq (s,t)$

$= \theta$ for $(i,j) = (s,t)$

Step 9. Set r=r+1, goto step 3.

Step 10. Write the optimum solution is $X^* = X_{ij}^{Br}$ with the associated cost vector $c_B^* = c_B^r$ and the minimum cost is $Z^* = Z^r$.

Step 11. Stop.

IV. NUMERICAL EXAMPLE

We consider the following problem given in Table 1

Table 1

	D_1	D_2	D_3	D_4	Slack	a_i
O_1	c_{11} d_{11}= 0.35	c_{12} d_{12}= 0.5	c_{13} d_{13}=0.35	c_{14} d_{14}=0.5	0 1	200
O_2	c_{21} d_{21}= 0.9	c_{22} d_{22}= 0.84	c_{23} d_{23}=0.3	c_{24} d_{24}=0.4	0 1	500
O_3	c_{31} d_{31}= 0.8	c_{32} d_{22}= 0.4	c_{33} d_{23}=0.74	c_{34} d_{24}=0.9	0 1	400
b_j	200	400	500	1000		

The variable costs are given below:

$c_{11} = 203$ if $0 \le x_{11} \le 100$
$= 201$ if $100 < x_{11} \le 150$
$= 200$ if $x_{11} > 150$

$c_{12} = 401$ if $0 \le x_{12} \le 100$
$= 400$ if $100 < x_{12} \le 300$
$= 399$ if $x_{12} > 300$

$c_{13} = 400$ if $0 \le x_{13} \le 150$
$= 399$ if $150 < x_{13} \le 350$
$= 398$ if $x_{13} > 350$

$c_{14} = 751$ if $0 \le x_{14} \le 400$
$= 750$ if $400 < x_{14} \le 700$
$= 749$ if $x_{14} > 700$

$c_{21} = 502$ if $0 \le x_{21} \le 100$
$= 500$ if $100 < x_{21} \le 150$
$= 498$ if $x_{21} > 150$

$c_{22} = 604$ if $0 \le x_{22} \le 100$
$= 600$ if $100 < x_{22} \le 300$
$= 599$ if $x_{22} > 300$

$c_{23} = 602$ if $0 \le x_{23} \le 200$
$= 600$ if $200 < x_{23} \le 400$
$= 599$ if $x_{23} > 400$

$c_{24} = 752$ if $0 \le x_{24} \le 350$
$= 750$ if $350 < x_{24} \le 600$
$= 749$ if $x_{24} > 600$

$c_{31} = 401$ if $0 \le x_{31} \le 100$
$= 400$ if $100 < x_{31} \le 150$
$= 398$ if $x_{31} > 150$

$c_{32} = 502$ if $0 \le x_{32} \le 150$
$= 500$ if $150 < x_{32} \le 300$
$= 499$ if $x_{32} > 300$

$c_{33} = 602$ if $0 \le x_{33} \le 200$
$= 600$ if $200 < x_{33} \le 350$
$= 599$ if $x_{33} > 350$

$c_{34} = 901$ if $0 \le x_{34} \le 500$
$= 900$ if $500 < x_{34} \le 750$
$= 899$ if $x_{34} > 750$

Applying step 1, we get the following result in Table 2

Table – 2

	D_1	D_2	D_3	D_4	Slack	a_i
O_1	200 $\boxed{200}$ 0.35	400 $\boxed{260}$ 0.5	400 0.35	751 0.5	0 1	200
O_2	502 0.9	600 $\boxed{140}$ 0.84	599 $\boxed{500}$ 0.3	750 $\boxed{581}$ 0.4	0 1	500
O_3	398 0.8	502 0.4	599 0.74	901 $\boxed{419}$ 0.9	0 $\boxed{22.9}$ 1	400
b_j	200	400	500	1000		

Applying step 2, set r=1.

Applying step 3, we get X_{ij}^{B1} = {$x_{11} = 200$, $x_{12} = 260$, $x_{22} = 140$, $x_{23} = 500$, $x_{24} = 581$, $x_{34} = 419$, $x_{3S} = 22.9$} with the associated cost vector C_B^1 = {$c_{11} = 200$, $c_{12} = 400$, $c_{22} = 600$, $c_{23} = 599$, $c_{24} = 750$, $c_{34} = 901$, $c_{3S} = 0$} and the corresponding cost $Z^1 = 1340769$.

Applying step 4, we get dual variables are $u_1 = -1034.2$, $u_2 = -377.5$, $u_3 = 0$, $v_1 = 561.97$, $v_2 = 917.1$, $v_3 = 712.25$ and $v_4 = 901$.

Applying step 5, we get the values of Δ_{ij} $\forall$ $(i,j) \notin B$ are $\Delta_{13} = -49.72$, $\Delta_{14} = -367.1$, $\Delta_{1S} = -1034.2$, $\Delta_{21} = -275.78$, $\Delta_{2S} = -377.5$, $\Delta_{31} = 163.97$, $\Delta_{32} = 415.1$ and $\Delta_{33} = 113.25$.

Applying step 6, we get Δ_{st} = Max {163.97, 415.1, 113.25 } = 415.1 at (3,2) cell. Therefore (3,2) cell enter into the basis.

Applying step 7, we get the values of y_{32}^{ij} are $y_{32}^{11}=0$, $y_{32}^{12}=0$, $y_{32}^{22}=1$, $y_{32}^{23}=0$, $y_{32}^{24}=-2.1$, $y_{32}^{34}=2.1$ and $y_{32}^{3S}=-1.49$.

$\therefore$ θ = min { 140, 199.5} =140 = $\frac{x_{22}^{B}}{y_{32}^{22}}$. Therefore (2,2) cell leaves the basis.

Applying step 8, we get the values of $\overline{x_{ij}^{B}}$ are $\overline{x_{11}^{B}}=200$, $\overline{x_{12}^{B}} = 260$, $\overline{x_{23}^{B}}=500$, $\overline{x_{24}^{B}}=875$, $\overline{x_{32}^{B}}=140$, $\overline{x_{34}^{B}}=125$, $\overline{x_{3S}^{B}}=231.5$ and tabulated in Table 3.

Table – 3

	D_1	D_2	D_3	D_4	Slack	a_i
O_1	200 $\boxed{200}$ 0.35	400 $\boxed{260}$ 0.5	399 0.35	751 0.5	0 1	200
O_2	502 0.9	600 0.84	599 $\boxed{500}$	750 $\boxed{875}$	0 1	500

			0.3	0.4		
O_3	398 0.8	502 $\boxed{140}$ 0.4	602 0.74	901 $\boxed{125}$ 0.9	0 $\boxed{231.5}$ 1	400
b_j	200	400	500	1000		

Applying step 9, set r = r + 1 and go to Step 3. The values of X_{ij}^{B2} with the associated cost vector C_B^2 give in Table 3 and the corresponding cost Z^2=1281780. Proceeding in this way the final table given by Table 4.

Table - 4

	D_1	D_2	D_3	D_4	Slack	a_i
O_1	203 $\boxed{71.5}$ 0.35	401 0.5	398 $\boxed{500}$ 0.35	751 0.5	0 1	200
O_2	500 0.9	600 0.84	602 0.3	749 $\boxed{1000}$ 0.4	0 $\boxed{100}$ 1	500
O_3	400 $\boxed{128.5}$ 0.8	499 $\boxed{400}$ 0.4	602 0.74	901 0.9	0 $\boxed{137.2}$ 1	400
b_j	200	400	500	1000		

Applying step 10, the optimum solution is X^*={ x_{11}= 71.5, x_{13}=500, x_{24}=1000, x_{31}=128.5, x_{32}=400 }, with the associated cost vector C_B^*={c_{11}=203, c_{13}=398, c_{24}=749, c_{31}=400, c_{32}=499 } and the optimum cost Z^*= 1213514.5

REMARK:

If we consider the problem

$$P^1: \quad Min\ Z = \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij}$$

subject to, $\sum_{j=1}^{n} d_{ij}^1 x_{ij} \le a_i$; for i = 1,2,3,……….,m.

$\sum_{i=1}^{m} d_{ij}^2 x_{ij} = b_j$; for j = 1,2,3,……….,n.

where the values of c_{ij} are given in equation (2.4), and $x_{ij} \ge 0$ $(1 \le i \le m, 1 \le j \le n)$.

Then this problem (P^1) can easily be converted to our proposed problem (P_1) by transformation $d_{ij}^2 x_{ij} = w_{ij}$ for $(1 \le i \le m, 1 \le j \le n)$;

References

[1] G. M. Appa, "The transportation problem and its variants," Operational Research Quarterly, 1973, p.p. 79 – 99.

[2] H. Arsham and A. B. Kahn, "A simplex-type algorithm for general transportation problems: An alternating to stepping-stone," Journal of Operational Research Society, 1989, p.p. 581 – 590.

[3] V. Balachandrana and G. L. Thompson, "An Operator Theory of Parametric Programming for the Generalized Transportation Problem-II: Rim, Cost and Bound Operators," Naval Res. Log. Quart, 1975, p.p. 101 – 125.

[4] E. Balas and P. L. Ivanescu, "On the generalized transportation problem," Management Science, 1964, p.p. 188 – 202.

[5] M. Basu and D. Acharya, "An algorithm for the optimum time-cost trade-off in generalized solid transportational problem," International Management System, 2000, p.p. 237 – 250.

[6] M. Basu and D. Acharya, "On quadratic fractional generalized solid bi-criterion transportation problem," An International Journal of Mathematics & Computing, 2002, p.p. 131 – 143.

[7] M. Basu, D. Acharya and A. Das, "The algorithm of finding all paradoxical pair in a linear transportation problem," Discrete Mathematics, Algorithm and Application, 2012, 1250049 (9pages), ©World Scientific Publising Company, DOI: 10.1142/S1793830912500498.

[8] M. Basu, B. B. Pal and A. Kundu, "An algorithm for the optimum time-cost trade-off in three dimensional transportational problem," Optimization, 1993, p.p. 177 – 185.

[9] M. Basu, B. B. Pal and A. Kundu, "An Algorithm For The Optimum Time Cost Trade-off in Fixed Charge Bi-criterion Transportation Problem," Optimization, 1994, p.p. 53 – 68.

[10] H. L. Bhatia, K. Swarup and M. Puri, "Time-cost trade-off in a transportation problem," Opsearch, 1976, p.p. 129 – 142.

[11] A. Charnes and W. W. Cooper, "The stepping stone method of explaning linear programming calculations in transportation problems," Management Science, 1954, p.p 49 – 69.

[12] L. Cooper and C. Dredes, "An approximate solution method for the fixed charge problem," Naval Research Logistics Quarterly, 1967, p.p 101 – 113.

[13] G. B. Dantzig, *Linear Programming and Extensions* (Princeton University Press, NJ, 1963).

[14] G. Hadley, *Linear Programming*, Narosa Publishing House, New Delhi, 1987.

[15] F. L. Hitchcock, "The distribution of a product from several resources to numerous localities," Journal of Mathematical Physics, 1941, p.p 224 – 230.

[16] D. Klingsman and R. Russel, "Solving Constrained Transportation Problems," Operations Research, 1975, p.p. 91 – 105.

[17] D. Thirwani, "A Note on Fixed Charge Bi-criterion Transportation Problem with Enhanced Flow," Indian Journal of Pure and Applied Mathematics, 1998, p.p. 565 – 571.

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