3- Class Association Schemes from Hadamard matrix of Paley type I

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Abstract- In this paper we have constructed Association Schemes from Hadamard matrix of Paley type I. Paley type I Hadamard matrices are skew symmetric in nature. These Association Schemes so obtained are Amorphic. Association schemes.

Index Terms- Hadamard matrices, Association Scheme, Skew Symmetric Hadamard matrix, Amorphic Association Scheme, Strongly Regular Graph, Paley’s Hadamard matrix, Paley’s Hadamard matrix of type I.

I. INTRODUCTION

We begin with the following definitions:

1.1 Hadamard Matrices (Or H-Matrices): Hadamard matrix is a square matrix whose entries are either +1 or −1 and whose rows are mutually orthogonal. If H-matrix of order n exists and n > 2 then n = 4t, where t is an integer. For a brief surveys of H-matrices vide Hall [1], Hedayat and Wallis [2]. For recent constructions vide Horadam [3]. Horton et. al. [4] and Baliga and Horadam [5].

1.2 Association Scheme (AS) vide Brouwer and Haemers [6], Colbourn and Dinitz [7]: Given v treatments 1,2,……,v, a relation satisfying the following conditions are said to be an Association Scheme with m classes: a) Any two treatments are either 1st, 2nd, …………. or mth associates, the relation of association being symmetric. b) Each treatment has ni, ith associates, the number ni being independent of the treatment taken. c) If any two treatments α and β are ith associates, then the number of treatments which are jth associates of α and kth associates of β is pjk and is independent of the pair of ith associates α and β. Let R0, R1,…….,Rm be binary relations on a set X = {1,2,……., v}. Let A = [aij] be the matrix with entries 0 and 1 defined as

\[ a_{jk} = \begin{cases} 1, & \text{if } (j,k) \in R_i \\ 0, & \text{otherwise} \end{cases} \]

Then A is the adjacency matrix of R1.

The set P = (R0, R1,…….,Rm) is called an m–class association scheme if the following conditions are satisfied:

(i) \[ A_0 = I \] (Identity Matrix) and \[ A_i \neq 0, \forall i \]

(ii) \[ \sum_{i=0}^{m} A_i = J \], where J is all-1 matrix

(iii) \[ A_i^T = A_i \] ∀ i \in \{0,1,2,…….,m\}

(iv) There are numbers \[ p_{ij} \] such that

1.3 Skew symmetric Hadamard matrix: A (1, −1) matrix A of order n is said to be of skew type if A−I is skew-symmetric.

1.4 Amorphic Association Scheme: Let a 3-AS be defined by the association matrices I, A1, A2, A3. Then 3-AS is called amorphic if each of A1, A2, A3 is an adjacency matrix of a strongly regular graph.

1.5 Strongly Regular Graph (SRG) (vide Higman [8]): 2-associate Association scheme on a set X is also called Strongly Regular Graph (SRG). The parameters of SRG are (v, k, λ, µ) where v = order of association matrix, k = p11, λ = p11, µ = p11.

1.6 Paley’s method of construction for H-matrices: Paley [9] found two families of Hadamard matrices using the quadratic residues in a finite field GF (q), where q = p3, where p is an odd prime. The quadratic character χ on the cyclic group GF (q)∗ = GF (q)-{0}, defined by

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i) If \( g \) is a quadratic residue in \( \text{GF}(q) \) then \( \chi(g) = 1 \) and

ii) if \( g \) is a quadratic non-residue, \( \chi(g) = -1 \)

Also \( \chi(0) = 0 \). For \( q \), an odd prime power and an ordering \( \{ g_0, g_1, \ldots, g_{q-1} \} \) of \( \text{GF}(q) \), take \( Q = [\chi(g_i - g_j)] \), \( 0 \leq i, j < q \). Let \( S \) be a matrix of order \((q + 1) \times (q + 1)\). Take

\[
S = \begin{bmatrix} 1 & e \\ e^T & Q \end{bmatrix}
\]

where \( e \) is a \( 1 \times q \) array having all entries 1.

### 1.7 Hadamard matrix of Paley type I:

H-matrices of Paley type I are defined for orders \( N = 4m = p + 1 \) (\( m=1,2,3,\ldots \)), where \( p \) is a prime with \( \text{mod}(p,4) = 3 \).

If \( q \) is congruent to 3 (mod 4) then

\[
P_{q+1} = \begin{bmatrix} 1 & e \\ e^T & Q + I_a \end{bmatrix}
\]

is a Hadamard matrix of order \((q + 1)\).

### Construction of Amorphic Association Schemes from Skew Symmetric Hadamard matrix of Paley type I:

Following theorem describes the construction of the Amorphic 3-AS

**Theorem:** If a Paley type I Hadamard matrix \( H \) is\[
\begin{bmatrix} 1 & e \\ e^T & I - \beta_1 + \beta_2 \end{bmatrix}
\]
where \( e \) is a \( 1 \times q \) array with all entries 1 and \( \beta_1, \beta_2 \) are (0,1) matrices, then

\[
A_1 = \beta_1 \times \beta_2 + \beta_2 \times \beta_1
\]

\[
A_2 = \beta_1 \times \beta_1 + \beta_2 \times \beta_2
\]

\[
A_3 = I \times K + K \times I, \quad \text{where} \quad K = \beta_1 + \beta_2
\]
define an amorphic 3-AS.

**Proof:** When \( 4n-1 \) is equal to \( p^t = q \), where \( p \) is a prime and let \( x \) is a primitive element of Galois field \( \text{GF}(p^t) \). Let \( \{ 1, x^2, (x^2)^2, (x^2)^3, \ldots, (x^2)^\frac{q-3}{2} \} \) mod \((4n-1)\) is a difference set. We denote this difference set as \( \{ 1, d_1, d_2, \ldots, d_k \} \)

\[
\alpha = \text{circ}(0100\ldots\ldots0)
\]

Let \( \beta_1 = \alpha + \alpha^d_1 + \alpha^d_2 + \ldots + \alpha^d_k \) and \( \beta_2 = \alpha^{-1} + \alpha^{-d_1} + \alpha^{-d_2} + \ldots + \alpha^{-d_k} \)

Then \( \beta_1 \beta_2 = (k + 1)I + (n - 1)K \) where \( 4n - 1 = q \)

\[
\beta_1 + \beta_2 = K \Rightarrow \beta_2 = K - \beta_1
\]

From (1) we have, \( \beta_1(K - \beta_1) = \left( \frac{q-1}{2} \right)I + \left( \frac{q-3}{4} \right)K \)

\[
\Rightarrow K\beta_1 - \beta_1^2 = \left( \frac{q-1}{2} \right)I + \left( \frac{q-3}{4} \right)K
\]

(2)
As $\beta_i$ is regular $(0, 1)$ matrix, $\beta_i J = J \beta_i = \left(\frac{q-1}{2}\right) J$ and $\beta_2 J = J \beta_2 = \left(\frac{q-1}{2}\right) J$

Also, $\beta_i K = \beta_i (J - I) = \beta_i J - \beta_i$

$= \left(\frac{q-1}{2}\right) J - \beta_i$

$= \left(\frac{q-1}{2}\right) (I + K) - \beta_i$

From (2), we have

$\left(\frac{q-1}{2}\right) I + \left(\frac{q-1}{2}\right) K - \beta_i - \beta_i^2 = \left(\frac{q-1}{2}\right) I + \left(\frac{q-3}{4}\right) K$

$\Rightarrow \beta_i^2 = \left(\frac{q-1}{2}\right) K - \left(\frac{q-1}{2}\right) I - \beta_i$

$= \left(\frac{q+1}{4}\right) K - \beta_i$

$= \left(\frac{q+1}{4}\right)(\beta_1 + \beta_2) - \beta_i$

$= \left(\frac{q-3}{4}\right) \beta_i + \left(\frac{q+1}{4}\right) \beta_2$

Similarly, $\beta_2^2 = \left(\frac{q+1}{4}\right) \beta_1 + \left(\frac{q-3}{4}\right) \beta_2$

If $t = \left(\frac{q-3}{4}\right) \Rightarrow \left(\frac{q+1}{4}\right) = (t+1)$ and $\left(\frac{q-1}{2}\right) = (2t+1)$

$\therefore \beta_i^2 = t \beta_i + (t+1) \beta_2$

$\therefore \beta_2^2 = (t+1) \beta_1 + t \beta_2$

$\therefore \beta_1 \beta_2 = (2t+1) I + t K$

Let $A_1 = \beta_1 \times \beta_2 + \beta_2 \times \beta_1$

$A_2 = \beta_1 \times \beta_1 + \beta_2 \times \beta_2$

$A_3 = I \times K + K \times I$

Then $A_1^2 = \beta_1^2 \times \beta_2^2 + \beta_2^2 \times \beta_1^2 + 2(\beta_1 \beta_2 \times \beta_2 \beta_1)$
\[ \begin{align*}
&= \{t\beta_i + (t+1)\beta_2\} \times \{(t+1)\beta_1 + t\beta_2\} + \{(t+1)\beta_1 + t\beta_2\} \times \{t\beta_i + (t+1)\beta_2\} \\
&+ 2[\{(2t+1)I + tK\} \times \{(2t+1)I + tK\}] \\
&= t(t+1)(\beta_1 \times \beta_1) + t^2(\beta_1 \times \beta_2) + (t+1)^2(\beta_2 \times \beta_1) + t(t+1)(\beta_2 \times \beta_2) \\
&+ t(t+1)(\beta_1 \times \beta_1) + t^2(\beta_2 \times \beta_1) + (t+1)^2(\beta_1 \times \beta_2) + t(t+1)(\beta_2 \times \beta_2) \\
&+ 2[(2t+1)^2 I + (2t+1) t (I \times K) + (2t+1) t (K \times I) + t^2(K \times K)] \\
&= 2(t(t+1)A_2 + t^2 A_1 + (t+1)^2 A_1 + 2(2t+1)^2 I + 2t(2t+1)A_3 + 2t^2(A_1 + A_2) \\
&= 2(2t+1)^2 I + (4t^2 + 2t + 1)A_1 + 2t(2t+1)(J - A_1 - I)
\end{align*} \]

And \( A_2^2 = \beta_1^2 \times \beta_1^2 + \beta_2^2 \times \beta_2^2 + 2(\beta_1 \beta_2 \times \beta_2 \beta_1) \)

\[ \begin{align*}
&= \{t\beta_i + (t+1)\beta_2\} \times \{t\beta_i + (t+1)\beta_2\} + \{(t+1)\beta_1 + t\beta_2\} \times \{(t+1)\beta_1 + t\beta_2\} \\
&+ 2[\{(2t+1)I + tK\} \times \{(2t+1)I + tK\}] \\
&= t(t+1)(\beta_1 \times \beta_2) + t^2(\beta_1 \times \beta_1) + (t+1)^2(\beta_2 \times \beta_2) + t(t+1)\beta_2 \times \beta_1 \\
&+ t(t+1)(\beta_1 \times \beta_2) + t^2(\beta_2 \times \beta_2) + (t+1)^2(\beta_1 \times \beta_1) + t(t+1)\beta_2 \times \beta_1 \\
&+ 2[(2t+1)^2 I + (2t+1) t (I \times K) + (2t+1) t (K \times I) + t^2(K \times K)] \\
&= (2t^2 + 2t + 1)A_2 + 2t(t + 1)A_1 + (t+1)^2 A_1 + 2(2t+1)^2 I + 2t(2t+1)A_3 + 2t^2(A_1 + A_2) \\
&= 2(2t+1)^2 I + (4t^2 + 2t + 1)A_2 + 2t(2t+1)(J - A_2 - I)
\end{align*} \]

And \( A_3^2 = I \times K^2 + K^2 \times I + 2(K \times K) \)

\[ \begin{align*}
&= I \times \{(4t + 2)I + (4t + 1)K\} + \{(4t + 2)I + (4t + 1)K\} \times I + 2(A_1 + A_2) \\
&= (4t + 2)I + (4t + 1)(I \times K) + (4t + 2)I + (4t + 1)(K \times I) + 2(A_1 + A_2) \\
&= 4(2t+1)I + (4t + 1)A_3 + 2(J - A_3 - I)
\end{align*} \]
And \( A_1 A_2 = \beta_1^2 \times \beta_2 \beta_3 + \beta_4 \beta_2 \times \beta_1^2 + \beta_2 \beta_3 \times \beta_1^2 + \beta_2^2 \times \beta_2 \beta_3 \)

\[
= \left\{ (t \beta_1 + (t + 1) \beta_2) \times \{(2t + 1)I + tK \} \right\} + \left\{ (2t + 1)I + tK \times (t + 1) \beta_1 + t \beta_2 \right\} \\
+ \left\{ (2t + 1)I + tK \times (t \beta_1 + (t + 1) \beta_2) \right\} + \left\{ (t + 1) \beta_1 + t \beta_2 \times (2t + 1)I + tK \right\} \\
= (2t + 1)^2 \left\{ K \times I + I \times K \right\} + (4t^2 + 2t) \left\{ K \times K \right\}
\]

And \( A_1 A_3 = \gamma_1 \times K \gamma_2 + K \gamma_1 \times \gamma_2 + \gamma_2 \times K \gamma_1 + K \gamma_2 \times \gamma_1 \)

\[
= \gamma_1 \times \{(2t + 1)I + (2t + 1)K - \gamma_2 \} + \{(2t + 1)I + (2t + 1)K - \gamma_1 \} \times \gamma_2 + \\
\gamma_2 \times \{(2t + 1)I + (2t + 1)K - \gamma_1 \} + \{(2t + 1)I + (2t + 1)K - \gamma_2 \} \times \gamma_1 \\
= (2t + 1)\gamma_1 \times I + (2t + 1)\gamma_1 \times K - \gamma_1 \times \gamma_2 + (2t + 1)I \times \gamma_2 + (2t + 1)K \times \gamma_2 - \gamma_1 \times \gamma_2 + \\
(2t + 1)\gamma_2 \times I + (2t + 1)\gamma_2 \times K - \gamma_2 \times \gamma_1 + (2t + 1)I \times \gamma_1 + (2t + 1)K \times \gamma_1 - \gamma_2 \times \gamma_1 \\
= 4t A_1 + (4t + 2) A_2 + (2t + 1) A_3
\]

And \( A_2 A_3 = \gamma_1 \times \gamma_1 K + K \gamma_1 \times \gamma_1 + \gamma_2 \times \gamma_2 K + K \gamma_2 \times \gamma_2 \)

\[
= \gamma_1 \times \{(2t + 1)I + (2t + 1)K - \gamma_1 \} + \{(2t + 1)I + (2t + 1)K - \gamma_1 \} \times \gamma_1 + \\
\gamma_2 \times \{(2t + 1)I + (2t + 1)K - \gamma_2 \} + \{(2t + 1)I + (2t + 1)K - \gamma_2 \} \times \gamma_2 \\
= (2t + 1)\gamma_1 \times I + (2t + 1)\gamma_1 \times K - \gamma_1 \times \gamma_1 + (2t + 1)I \times \gamma_1 + (2t + 1)K \times \gamma_1 - \gamma_1 \times \gamma_1 + \\
(2t + 1)\gamma_2 \times I + (2t + 1)\gamma_2 \times K - \gamma_2 \times \gamma_2 + (2t + 1)I \times \gamma_2 + (2t + 1)K \times \gamma_2 - \gamma_2 \times \gamma_2 \\
= (4t + 2) A_1 + 4t A_2 + (2t + 1) A_3
\]

Hence \( A_1, A_2 \) and \( A_3 \) define an amorphic 3 - AS.
Conclusion: Association schemes from skew-symmetric Hadamard matrices have also been obtained by Hanaki [10]. However our method is different from that of Hanaki.

REFERENCES

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