On The Reduced Form Of The Adomian Polynomials For The Solutions Of Nonlinear Fractional-Order Volterra Integro-Differential Equations

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Abstract

In this work, we amplify a reliable modification of the Adomian decomposition method as provided in [1, 2, 3] by introducing the reduced form of the Adomian polynomial to replace the original polynomials for handling the nonlinear components and subsequently we applied it to fractional nonlinear integro-differential equation of the Volterra type to observe the results produce by the reduced form of the polynomials as compared to the original polynomials. The proposed polynomials produced some remarkable results when compared with the original Adomian polynomials.

Keywords: Nonlinear fractional-order Volterra Integro-differential equation, modified decomposition Method, Fractional derivative, Adomian polynomials reduced Form of Adomian polynomials.

1. Introduction

Fractional calculus has a long history from 30 September 1695, whilst the derivative of order $\propto = 1/2$ has been described by Leibniz and Newton [3 and 4]. The principle of derivatives and integrals of non-integer order goes back to Leibniz, Liouville, Grùnwald, Letnikov and Riemann; there are many thrilling books about fractional calculus and fractional differential equations [1 and 4]. Our main cognizance in this work is to find out the behavior of the reduced polynomials as compared to the original Adomian polynomials.

During the last decades, researchers have devoted time to work on linear and nonlinear non-integer order differential equations using the Adomian decomposition approach and its modifications [1, 2, 3, 5, 6,7]

2. Preminaries

The mathematical definitions of fractional integrals and fractional derivatives are the problem of several different processes. The maximum frequently used definition of the fractional calculus involves the Riemann-Liouville fractional derivative and the Caputo derivative [8,9]

Definition 1

Riemann – Liouville fractional integral is defined as

$$J^{\propto} f(x) = \frac{1}{\Gamma(\propto)} \int_0^x \frac{f(x)}{(x-t)^{1-\propto}} dt \,, \quad \propto > 0, x > 0$$

$J^{\propto}$ denotes the fractional integral of order $\propto$

Definition 2
Riemann – Liouville fractional derivative denoted $D^{\propto}$ is defined as
$D^{\propto}J^{\propto}f(x) = f(x)$
Definition 3
Riemann-Liouville fractional derivative defined as
$$D^{\propto}f(x) = \frac{1}{\Gamma(m-\propto)}\int_0^x (x-s)^{m-\propto-1}f^m(s)ds\,.$$
m is positive integer with the property that $m-1 < \propto m$.
Definition 4
The Caputo Factional Derivative is defined as
$$D^{\propto}f(x) = \frac{1}{\Gamma(m-\propto)}\int_0^x (x-s)^{m-\propto-1}f^m(s)ds\,.$$
Where m is a positive integer with the property that $m-1 < \propto < m$.
For example if 0 <∝< 1 the caputo fractional derivative is
$$D^{\propto}f(x) = \frac{1}{\Gamma(m-\propto)}\int_0^x (x-s)^{-\propto}f^1(s)ds\,.$$
Hence, we have the following properties:

1. $J^{\propto}J^{v}f = J^{\propto+v}f, \propto, v > 0, f \in C_{\mu}, \mu > 0$
2. $J^{\propto}x^{\gamma} = \frac{\Gamma(\gamma+1)}{\Gamma(\propto+\gamma+1)}x^{\propto+\gamma}\,,\ \propto > 0\,, \gamma > -1\,,\ x > 0$
3. $J^{\propto}D^{\propto}f(x) = f(x) - \sum_{k=0}^{m-1} f^k(0)\frac{x^k}{k!}\ ,\ x > 0\,,\ m-1 < \propto \leq m$
4. $D^{\propto}J^{\propto}\, f(x) = f(x),\ x > 0\,,\ m-1 < \propto \leq m$
5. $D^{\propto}C = 0,\ where\ C\ is\ a\ constant.$
6. $D^{\propto}x^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\propto+1)}x^{\beta-\propto}\,,\ \beta \epsilon N_0\ ,\ \beta \geq \propto$

   Where $\propto$ is an integer and N_0 are natural numbers

2. METHODOLOGY

Consider the nonlinear fractional-order Volterra Integro-Differential equation [3]:
$$D^{\propto}\,u(x) = g(x) + \lambda\int_0^x k(x,t)F\big(u(t)\big)\,dt \qquad \dots(1)$$
With initial conditions
$u'(0) = C_i, i = 0, 1, 2, \dots r-1, r \in N$
The main idea of the new method is replacing the forcing terms $g(x)$ which is either an exponential or trigonometric function by a series of infinite components as presented in [1] but not as a coefficient as was seen in [2, 3].
In [10] $g(x)$ was expresses in terms of Taylors series, owing to this expression and the introduction of the reduced Adomian polynomials in place of the original Adomian polynomials, we establish the following:
From the properties of fractional integral and derivative we know that
$$J^{\propto}D^{\propto}u(x) = u(x) - \sum_{k=0}^{m-1} u^k(0)\frac{x^k}{k!}\ ,$$
Where $D^{\propto}$ is the operator that defines fractional derivative and

$$\sum_{k=0}^{m-1} u^k(0)\frac{x^k}{k!} = c_0 + c_1 x + c_2\frac{x^2}{2!} + c_3\frac{x^3}{3!} + \cdots$$

This was obtained from the given initial conditions [1]

Now, applying the integral operator $J^{\propto}$ to both sides of Eq. (1) we obtain

$$u(x) = \sum_{k=0}^{r-1} u^k(0)\frac{x^k}{k!} + J^{\propto}[\, g_1(x) \ + \ g_2 + g_3(x), \ldots] + J^{\propto}\left(\lambda \int_0^x k(x,t)F\big(u(t)\big)\,dt \right) \quad \ldots (2)$$

The method defines the solution $u(x)$ by the series [2], and the nonlinear function $F\big(u(t)\big)$ is decomposed using the proposed reduced form of Adomian polynomials and the forcing term is expanded in series.

The reduced form of the Adomian polynomials is given as follows:

$X_0 = F(u_0)$

$X_n = u_n F'(u_0)$, Where $n \geq 1$

This gives

$X_0 = F(u_0)$

$X_1 = u_1 F'(u_0)$,

$X_2 = u_2 F'(u_0)$,

$X_3 = u_3 F'(u_0)$ ... (3)

.

.

.

$$\sum_{n=0}^{\infty} u_n(x) = \sum_{k=0}^{m-1} u^k(0)\frac{x^k}{k!} + J^{\propto}[g_1(x) \ + \ g_2 +, \ldots] + J^{\propto}\left(\lambda \int_0^x k(x,t) \sum_{n=0}^{\infty} X_n \, dt \right) \quad \ldots (4)$$

Where

The components $u_0, u_1, u_2, \ldots$ are determined recursively by the following scheme [1,3]:

$u_0(x) = C_0$

$u_1(x) = C_1 x$

$u_2(x) = C_2 \dfrac{x^2}{2}$

$u_3(x) = C_3 \dfrac{x^3}{3!}$

…

$u_{r-1}(x) = \dfrac{C_{r-1}}{(r-1)!} x^{r-1}$

This will be obtained from the given initial condition(s)

And the other terms are obtained from the decomposition of the forcing terms and the reduced polynomials as shown below:

$$u_{n+r}(x) = +J^{\propto}(g_1(x) + g_2(x) + g_3(x) + \cdots) + J^{\propto}\left(\lambda \int_0^x k(x,t) \sum_{n=0}^{\infty} X_n \, dt\right), n \geq 0 \quad \dots (5)$$

$$u_{n+r+1}(x) = J^{\propto}[\ g_1(x)] + J^{\propto}\lambda \int_0^x k(x,t) X_0(t)\, dt$$

$$u_{n+r+2}(x) = J^{\propto}[\ g_2(x)]\ + J^{\propto}\lambda \int_0^x k(x,t) X_1(t)\, dt$$

$$u_{n+r+3}(x) = J^{\propto}[\ g_3(x)] + J^{\propto}\lambda \int_0^x k(x,t) X_2(t)\, dt$$

$$u_{n+r+4}(x) = J^{\propto}\lambda \int_0^x k(x,t) X_3(t)\, dt \qquad \dots (6)$$

…

4 IMPLEMENTATION:

EXAMPLE: 1

We consider the following nonlinear fractional order integro-differential equation of volterra type [3]

$$D^{\propto}u(x) = f(x) - \int_0^x u^2(t)\, dt, \qquad 0 \leq x \leq 1, \qquad 1 \leq \propto \leq 2 \qquad \dots (7)$$

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Where $f(x) = \sinh(x) + \frac{1}{2}\cosh(x)\sinh(x) - \frac{x}{2}$ and subject to the initial conditions

$u(0) = 0, \qquad u'(0) = 1.$

The exact solution for the case $\propto = 2$ is $u(x) = \sinh(x)$.

Applying $J^{\propto}$ on both sides of the equation and introducing the reduced Adomian polynomials for the decomposition of the nonlinear terms as well as expressing the forcing terms in series alongside the application of the initial conditions yields:

$$u(x) = 0 + x + J^{\propto}[f(x)] - J^{\propto}\int_0^x X_n(t)\,dt, n \geq 0$$

The first few reduced Adomian polynomials for $u^2(x)$ are:

$X_0(x) = {u_0}^2(x)$

$X_1(x) = 2u_0(x)u_1(x)$

$X_2(x) = 2u_0(x)u_2(x)$

…

For $\propto = 2$

$$f(x) = \frac{1}{6}x^3 + \frac{1}{40}x^5 + \frac{1}{520}x^7$$

By the improved modified scheme

$u_0(x) = 0$

$u_1(x) = x$

$$u_2(x) = f_0(x) - J^{\propto}\int_0^x X_0(t)\,dt$$

$$u_3(x) = f_1(x) - J^{\propto}\int_0^x X_1(t)\,dt$$

$$u_4(x) = f_2(x) - J^{\propto}\int_0^x X_2(t)\,dt$$

$$u_{n+1}(x) = -J^{\propto}\int_0^x X_{n-1}(t)\,dt, \qquad n \geq 4$$

…

Calculating with the aid of Maple18, we have the

$u_0(x) = 0$

$u_1(x) = x$

$$u_2(x) = \frac{1}{6}x^3 - J^{\propto}\int_0^x X_0(t)\,dt = \frac{1}{6}x^3$$

$$u_3(x) = \frac{1}{40}x^5 - J^{\propto}\int_0^x X_1(t)\,dt = \frac{1}{40}x^5$$

$$u_4(x) = \frac{1}{520}x^7 - J^{\propto}\int_0^x X_2(t)\,dt = \frac{1}{520}x^7$$

$$u_5(x) = -J^{\propto}\int_0^x X_3(t)\,dt = 0$$

…

$u(x) = x + \frac{1}{6}x^3 + \frac{1}{40}x^5 + \frac{1}{520}x^7 + \cdots$

x	EXACT	APPROX.(∝=2)	APPROX.(∝=1.98)	APPROX.(∝=1.9)	APPROX.(∝=1.8)	APPROX.(∝=1.5)
0	0	0	0	0	0	0
0.1	0.100167	0.100166917	0.100179222	0.100237936	0.100338186	0.100953347
0.2	0.201336	0.201341356	0.201420473	0.201784402	0.202366907	0.205423831
0.3	0.30452	0.304561141	0.304791481	0.305828595	0.30742686	0.315089129
0.4	0.410752	0.410925592	0.411412573	0.413572262	0.416812229	0.431389482
0.5	0.521095	0.521628534	0.522494685	0.526291337	0.531870068	0.55577841
0.6	0.636654	0.637993989	0.639377463	0.64538467	0.654064336	0.689835209
0.7	0.758584	0.761515478	0.763570096	0.772421373	0.785032054	0.835345322
0.8	0.888106	0.893899825	0.896796246	0.909190439	0.926638961	0.994371138
0.9	1.026517	1.037116352	1.041044165	1.057754601	1.081038042	1.169321668
1	1.175201	1.193452381	1.198623007	1.220509862	1.250732996	1.363025217

These charts reveal that the depicted numerical results are in good agreement with the exact solution as $\propto$ gets close to 2.

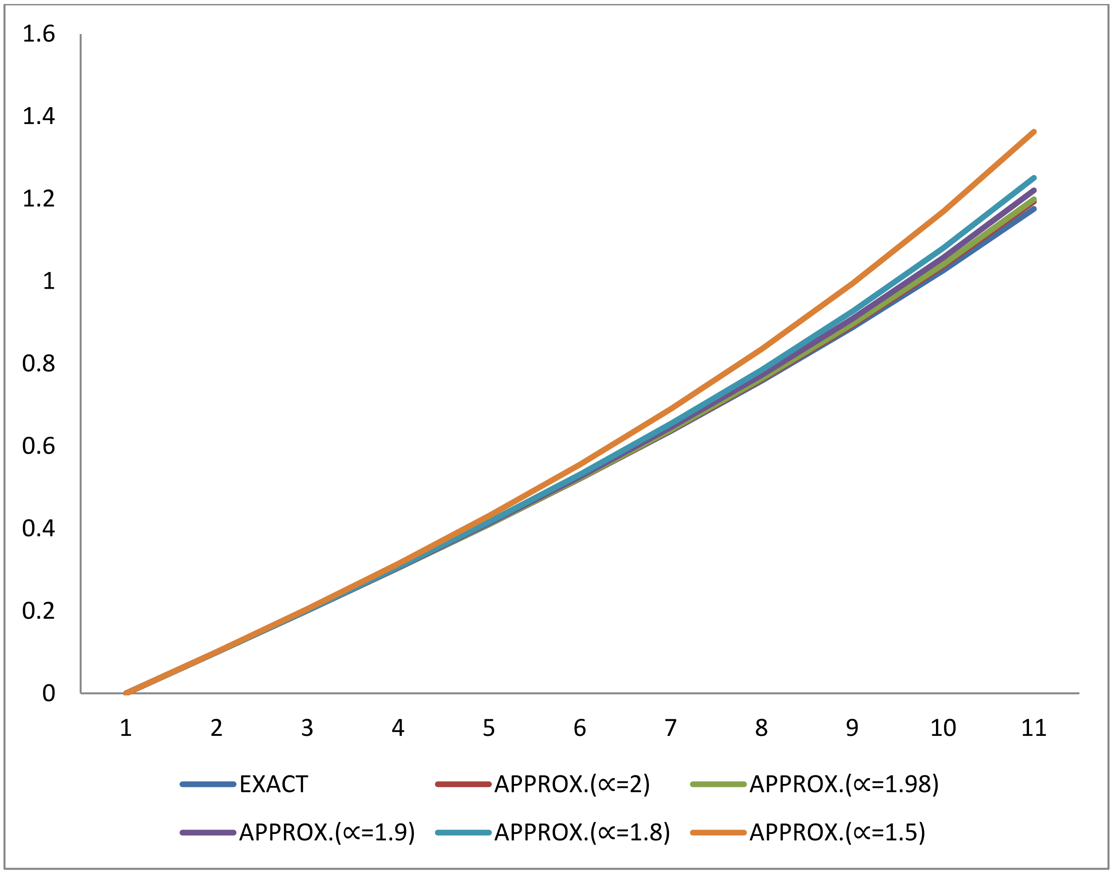

Fig: 1 shows the numerical results for three iterations for various values $1 < \propto \leq 2$. the comparison shows that as $\propto \rightarrow 2$, the approximate solution tends to $\sinh(x)$, which is the exact solution of the equation in the case $\propto$=2.

EXAMPLE: 2

We consider the following nonlinear fractional order integro-differential equation of volterra type [3]

$$D^{\propto}u(x) = f(x) + \int_0^x u^3(t)\,dt, \qquad 0 \leq x < 1, \qquad 0 < \propto \leq 1, \qquad \ldots (8)$$

Where $f(x) = e^x - \frac{1}{3}e^{3x} + \frac{1}{3}$ and subject to the initial conditions

$u(0) = 0.$

The exact solution for the case $\propto = 1$ is $u(x) = e^x$.

Applying the improved modification yields the following:

For $\propto = 1$

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$$u(x) = 1 + J^{\propto}[f(x)] + J^{\propto}\int_0^x X_n(t)\,dt, n \geq 0$$

The first few reduced Adomian polynomials for $u^3(x)$ are:

$$X_0(x) = {u_0}^3(x)$$

$$X_1(x) = 3{u_0}^2(x)u_1(x)$$

$$X_2(x) = 3{u_0}^2(x)u_2(x)$$

…

For $\propto = 1$

$$f(x) = x - \frac{1}{3}x^3 - \frac{1}{3}x^4 - \frac{13}{60}x^5 - \frac{1}{9}x^6$$

By the improved modified scheme

$$u_0(x) = 1$$

$$u_1(x) = f_0(x) + J^{\propto}\int_0^x X_0(t)\,dt$$

$$u_2(x) = f_1(x) + J^{\propto}\int_0^x X_1(t)\,dt$$

$$u_3(x) = f_2(x) + J^{\propto}\int_0^x X_2(t)\,dt$$

$$u_4(x) = f_3(x) + J^{\propto}\int_0^x X_3(t)\,dt$$

$$u_5(x) = f_4(x) + J^{\propto}\int_0^x X_4(t)\,dt$$

$$u_{n+1}(x) = J^{\propto}\int_0^x X_n(t)\,dt, \qquad n \geq 5$$

…

Calculating with the aid of Maple18, we have the

$$u_0(x) = 1$$

$$u_1(x) = x + J^{\propto}\int_0^x X_0(t)\,dt = x + \frac{1}{2}x^2$$

$$u_2(x) = -\frac{1}{3}x^3 + J^{\propto}\int_0^x X_1(t)\,dt = \frac{1}{6}x^3 + \frac{1}{8}x^4$$

$$u_3(x) = -\frac{1}{3}x^4 + J^{\propto}\int_0^x X_2(t)\,dt = -\frac{1}{3}x^4 + \frac{1}{40}x^5 + \frac{1}{80}x^6$$

...

$$u(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{8}x^4 - \frac{1}{3}x^4 + \frac{1}{40}x^5 + \frac{1}{80}x^6 + \cdots$$

EXACT	**APPROX. (∝=1)**	**APPROX. (∝=0.98)**	**APPROX. (∝=0.90)**	**APPROX. (∝=0.8)**	**APPROX. (∝=0.6)**
1	1	1	1	1	1
1.105171	1.105146096	1.111113207	1.138252323	1.180732102	1.303304071
1.221403	1.2210088	1.230595646	1.272788816	1.335128936	1.499795619
1.349859	1.347882363	1.36049273	1.415066827	1.493298381	1.695241745
1.491825	1.485640533	1.501007565	1.566801676	1.659203337	1.903519374
1.648721	1.633789063	1.65178083	1.728194902	1.833713089	2.132931908
1.822119	1.7915272	1.812078127	1.898748081	2.01648618	2.391110307
2.013753	1.957818196	1.980900219	2.077559546	2.206555975	2.6864099
2.225541	2.1314688	2.157079634	2.263495899	2.402606026	3.028551861
2.459603	2.311217763	2.339377548	2.455322324	2.603139786	3.429028952

These charts reveal that the depicted numerical results are in good agreement with the exact solution as ∝ gets close to 1.

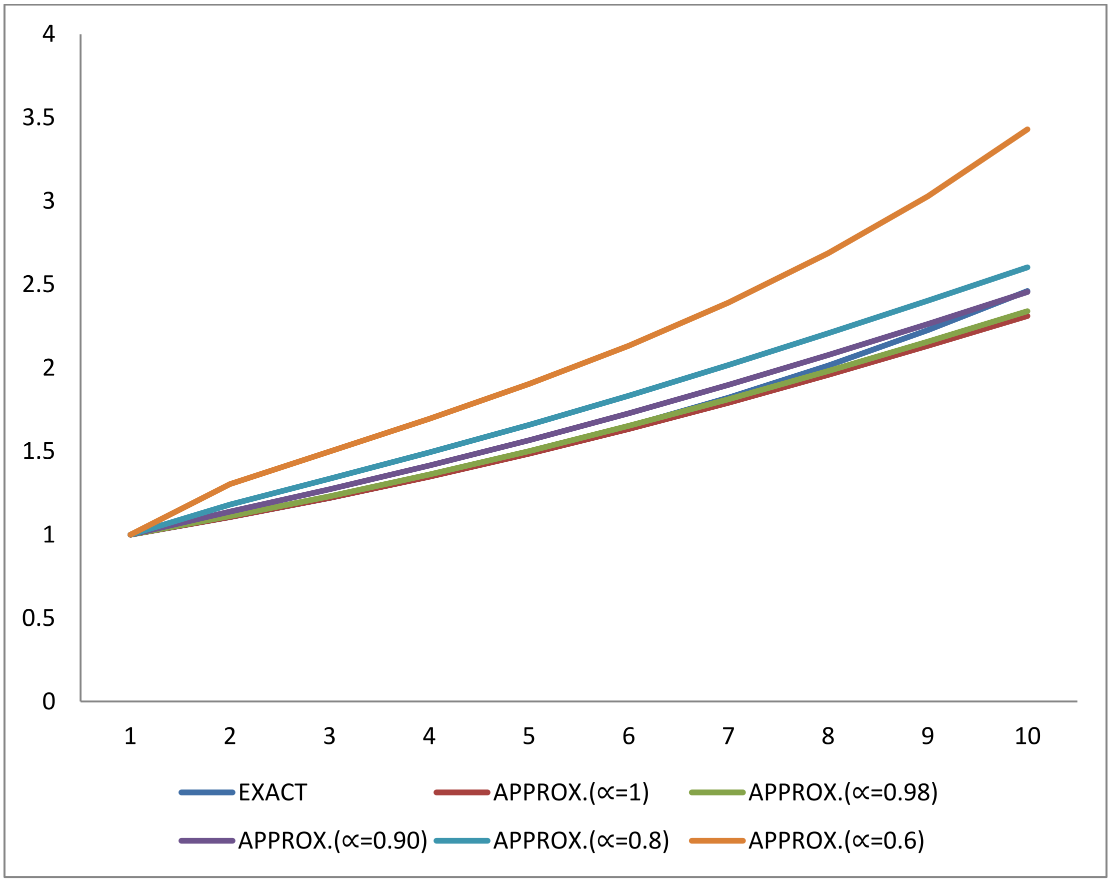

Fig: 2 shows the numerical results for three iterations for various values $0 <\propto\leq 1$. the comparison shows that as $\propto\rightarrow 1$, the approximate solution tends to e^x , which is the exact solution of the equation in the case ∝=1.

5 Conclusion

In this paper, we have successfully used the reduced form of the Adomian polynomials as a replacement for the original Adomian polynomials as presented in [3] and we use it to solve the nonlinear fractional-order Volterra integrodifferential equation. Sometimes it is very difficult to compute the Adomian polynomials. This method overcomes such difficulties by the use of the reduced form of the polynomials. We have discussed two numerical examples of nonlinear fractional-order Volterra integro-differential equations which were earlier presented in [3] to confirm the applicability and the advantages of the proposed Method. We use Maple18 software for computations and MS Excel for graphical representations of the solutions obtained. The achieved results are in good agreement with exact solutions. It is observed that the proposed method is simple and it can be efficiently applied to a large number of similar fractional nonlinear problems.

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