Oscillating Motion in Visco-Elastic Fluid

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Abstract- Here an analysis is made on the high as well as low frequency torsional oscillations in a visco-elastic fluid. The flow is induced induced by a disk oscillating about an axis normal to its plane. The structure of outer boundary layer has been examined matching inner and outer solutions. Kármán-Pohlhausen method has been used for integrating the equations of motion. An effort has been made to analyse the behaviour of different flow parameters involving in the flow of the visco-elastic fluid.

Index Terms- Boundary layer, Oscillation, Rotation, Torsion, Visco-elastic,

I. Introduction

Before starting the discussion it will be pertinent to know what Torsion is. It is periodical movement of a shaft. Such type of flows takes a very important part in the studies of fluid mechanics like power transmission and other systems. Galvagno, Velardocchia and Vigliani [1] recently investigated Torsional oscillationsof an automotive transmission system by means of an experimental test bench. Theodore Von Kármán [2], the Hungarian-American mathematician and Aero-dynamic engineer did some pioneering works on fluid flows due to rotations of disks. This followed by numerous other studies with extension of these ideas to time dependent rotator oscillations of one or two disks including works of Benney[3], Rosenblat[4] and so on. Rao and Kasiviswanath [5] have studied the flow and heat transfer due to torsional oscillations of two disks at different speeds. Puri [6] studied the unsteady flow of an elastic-viscous fluid past an infinite plate. Das, Maji, Jana and Seth [7] studied flow induced by torsional oscillations of a disk in a rotating visco-elastic fluid. Shrivastava [8]considered the torsional oscillations of a second-order fluid when the fluid is of an infinite extent as well as the case when it is bounded by another stationary plate by expanding the velocity components and the pressure in powers of the amplitude of oscillation of the plate.

In this study, we are going to investigate the behaviour of the flow parameters when the flow is caused by the oscillation of a disk undergoing torsion. We have taken the axis of rotation of the disk as normal to the plane of the disk. Both the cases of high as well as low frequency have been worked out. We have taken the visco-elastic fluid with short memories in our investigation. The constitutive equation of the fluid considered in this problem is

$$\sigma^{ij} = -pg_{ij} + 2\eta_0 e^{ij} - 2C\frac{\delta}{\delta t}e^{ij}$$

Where

$e_{ij} = v_{i,j} + v_{j,i}$, η_0 is limiting viscosity of small rate of shear, g_{ij} is the metric tensor with respect to fixed coordinate system x_i , C is the visco-elastic parameter, v_i is the velocity vector and $\frac{\delta}{\delta t}$ denotes the convected derivative of a tensor quantity.

II. EQUATION

Let us consider the geometrical configuration of our problem. Cylindrical polar coordinates (r, θ, z) have been taken here. The axis of rotation is taken as the initial line. The plane of the disk is considered as the plane z plane and the space occupied by the liquid has been considered as $z > 0$. The boundary conditions of the problem will be as

$$u = 0,\ v = rW\cos(\omega t),\ w = 0 \quad at\ z = 0$$

$$u \to 0,\ v \to 0. \quad as\ z \to \infty \ , \qquad (2.i)$$

Here u, v, w are velocity components, ω and W are the frequency and angular speed of the oscillating motion of the disk respectively.

The equations of motion in radial and transverse directions are as follows.

$$\frac{\partial^2 F}{\partial x \partial y} + \epsilon\left[\left(\frac{\partial F}{\partial x}\right)^2 - 2F\frac{\partial^2 F}{\partial x^2} - G^2\right] = \frac{1}{2}\frac{\partial^3 F}{\partial x^3} - K\left[\frac{\partial^4 F}{\partial x^3 \partial y} - \epsilon\left\{2F\frac{\partial^4 F}{\partial x^4} - 4\frac{\partial F}{\partial x}\frac{\partial^3 F}{\partial x^3} + \left(\frac{\partial^2 F}{\partial x^2}\right)^2 - G\frac{\partial^2 G}{\partial x^2} - 3\left(\frac{\partial G}{\partial x}\right)^2\right\}\right] \qquad (2.ii)$$

$$\frac{\partial G}{\partial x} + 2\epsilon\left(\frac{\partial F}{\partial x}G - F\frac{\partial G}{\partial x}\right) = \frac{1}{2}\frac{\partial^2 G}{\partial x^2} - K\left[\frac{\partial^3 G}{\partial x^2 \partial y} - \epsilon\left\{2F\frac{\partial^3 G}{\partial x^3} + \frac{\partial^3 F}{\partial x^3}G - 2\frac{\partial F}{\partial x}\frac{\partial^2 G}{\partial x^2} + 4\frac{\partial^2 F}{\partial x^2}\frac{\partial G}{\partial x}\right\}\right] \qquad (2.iii)$$

Where,

$$u = rW\frac{\partial F}{\partial x},\quad v = rWG(x, y),\quad w = -2W\left(\frac{2\vartheta_1}{\omega}\right)^{\frac{1}{2}} F(x, y) \qquad (2.iv)$$

$$x = z\sqrt{\frac{\omega}{2\vartheta_1}}\ ,\quad y = \omega t,\quad \vartheta_1 = \frac{2\eta_0}{\rho},$$

$$K = \frac{CW}{\eta_0} \quad \text{and} \quad \epsilon = \frac{W}{\omega}$$

The boundary conditions now becomes

$$F = \frac{\partial F}{\partial x} = 0,\ G = \cos y \qquad \text{at} \quad x = 0$$

$$\frac{\partial F}{\partial x}, G \to 0 \qquad \text{at} \quad x \to \infty \qquad (2.v)$$

III. Solution

Let us consider the high frequency case i.e. $\epsilon \ll 1$. We substitute the following series

$F(x,y) = \sum_{i=0}^{\infty} \epsilon^i F_i(x,y)$ (3.i)

$G(x,y) = \sum_{i=0}^{\infty} \epsilon^i G_i(x,y)$ (3.ii)

into the equations (2.ii) and (2.iii).

Equating the coefficients of like powers of ϵ, we obtain the following partial differential equations

$$F_{0_{xy}} = \frac{1}{2} F_{0_{xxx}} - K F_{0_{xxxy}} , \quad (3.iii)$$

$$F_{1_{xy}} + \left(F_{0_x}\right)^2 - 2F_0 F_{0_{xx}} - {G_0}^2 = \frac{1}{2} F_{1_{xxx}} - K\left[F_{1_{xxxy}} - 2F_0 F_{0_{xxxx}} - \left(F_{0_{xx}}\right)^2\right] + 4F_{0_x} F_{0_{xxx}} + G_0 G_{0_{xx}} + 3\left(G_{0_x}\right)^2 , \quad (3.iv)$$

etc.

$$G_{0_y} = \frac{1}{2} G_{0_{xx}} - K G_{0_{xxy}} \quad (3.v)$$

$$G_{1_y} + 2\left[F_{0_x} G_0 - F_0 G_{0_x}\right] = \frac{1}{2} G_{1_{xx}} - K\left[G_{1_{xxy}} - 2F_0 G_{0_{xxx}} + F_0 G_{0_{xx}} - 4F_{0_{xx}} G_{0_x}\right], \quad (3.vi)$$

etc.

In view of the substitutions (3.i) and (3.ii), the boundary conditions (2.v) take the suitable form as

$F_i = F_{i_x} = 0,\ G_0 = \cos y,\ G_{i+1} = 0$ at $x = 0,$ for $i = 0,1,2,\ldots$

$F_{i_x} \to 0,\ G_i \to 0$ as $\to \infty$, for $i = 0,1,2,\ldots$ (3.vii)

Solutions of the equations (3.iii) to (3.vi) have been obtained subject to the boundary conditions (3.vii) as

$$F_0(x,y) = 0 \quad (3.viii)$$

$$G_0(x,y) = e^{-Py} \cos(x - Qy) \quad (3.ix)$$

$$F_1(x,y) = \frac{1}{4P^2}\{1 - 2K(2P^2 + Q^2)\}\left[y + \frac{1}{2P}(e^{-2Py} - 1)\right] + J(y)e^{2ix} \quad (3.x)$$

$$G_1(x,y) = 0 \quad (3.xi)$$

where the fluctuating part $J(y)$ of F_1 is given by

$$J(y) = \left[2(m+in)\sqrt{1+16K^2}\{4K(m+in)^2 - 1\}/(q+is)\right]\left[1 - e^{\frac{-(q+is)y}{\sqrt{1+16K^2}}}\right] - [\{4K(m+in)^2 - 1\}/8\{i(m+in) - (1 - 4iK)(m+in)^3\}]\left[1 - e^{-2(m+in)y}\right] ; \quad (3.xii)$$

where,

$m^2, n^2 = \left[\sqrt{1+4K^2} \mp 2K\right]/(1+4K^2)$

and $q^2, s^2 = 2\left[\sqrt{1+16K^2} \mp 4K\right]$

Using of these solutions in the velocity components we can conclude that as $F_0(x,y) = 0$, the first order solution has no component of radial or axial velocity, only the transverse velocity given by

$$v = rW exp\left(-mz\sqrt{\frac{\omega}{2\vartheta_1}}\right)\cos\left\{\omega x - nz\sqrt{\frac{\omega}{2\vartheta_1}}\right\} \quad (3.xiii)$$

It may be noted that the steady part of the radial component of velocity persists outside a shear wave or inner boundary layer of thickness of order$\left(\frac{\vartheta_1}{m^2\omega}\right)^{\frac{1}{2}}$. The axial velocity persists outside the region thereby conforming continuity. At the edge of the inner boundary layer, the steady parts of the velocity components u and w are

$$u_s = r\epsilon WD\left[1 - exp\left\{-2mz\sqrt{\frac{\omega}{2\vartheta_1}}\right\}\right]$$

$$w_s = \frac{\epsilon WD}{4m^3}\sqrt{\frac{2\vartheta_1}{\omega}}\left[1 - mz\sqrt{\frac{2\omega}{\vartheta_1}} - e^{-mz\sqrt{\frac{\omega}{2\vartheta_1}}}\right] \quad (3.xiv)$$

where $D = \frac{1}{4m^2}\{1 - 2K(2m^2 + n^2)\}$ (3.xv)

Let us study the structure of the outer boundary layer in detail. We are going to match inner and outer expansions. Since there is no oscillatory potential flow in the outer layer, the derivation of the equation is quite obvious. The terms in the inner boundary layer must match at each stage with that of the outer boundary layer. Thus in order to effect a match with inner solution of o(ϵ), the first term of the outer solution for u_s is taken as of (ϵ). Thus for studying the flow in outer layer we write

$F(y) = R(\zeta)$, $G(y) = S(\zeta)$, $y = \epsilon^{-1}\zeta$ (3.xvi)

The transformation expresses the fact that the thickness of the outer layer is of order ϵ^{-1} times that of the inner. Clearly $S = 0$ and the equation for R will be

$$R^{///} = 2\left({R^{/}}^2 - 2RR^{//}\right) + K_1\left(4R^{/}R^{///} - 2RR^{iv} - {R^{//}}^2\right) \quad (3.xvii)$$

with $R^{/}(\infty) = 0$ as evident from the condition that outer solution matches with inner solution as $y \to \infty$.

Here $K_1 = K\epsilon^2$ and the prime denotes differentiation with respect to ζ.

With the help of (3.xiv) and (3.xvi) and using the velocity components we deduce that $R \to D\zeta$ as $\to 0$.

Thus

$$R(0) = 0,\ R^{/}(0) = D \quad (3.xviii)$$

Let us solve the equation (3.xvii) by Kármán-Pohlhausen method. We introduce

$\zeta = \delta\xi,\ R = \delta S$

(3.xix)

Here δ is the non-dimensional boundary layer thickness of the outer boundary layer. With trhis substitution, equation (3.xvii) takes the form

$$S^{///} = 2\left(S^{/2} - 2SS^{//}\right) + K_1\left(4S^{/}S^{///} - 2SS^{iv} - S^{//2}\right)$$

(3.xx)

Where differentiation is with respect to ξ .

Considering the smoothness of the solution over outer boundary layer edge, we assume the form

$S^{/} = D(1 + b\xi)(1 - \xi)^4$

(3.xxi)

where b is a constant to be determined. To determine b and δ we have momentum integral equation as

$$-S^{//}(0) = 6\delta^2 \int_0^1 S^{/2} d\xi + K_1\left[6a^2(4 - b) - 7\int_0^1 S^{//2} d\xi\right]$$

(3.xxii)

The condition that equation (3.xx) is satisfied as $\xi \to 0$ is found as

$$S^{///}(0) = 2\delta^2 S^{/}(o)^2 + K_1\left[4S^{/}(0)S^{///}(0) - S^{//}(0)^2\right]$$

(3.xxiii)

Low Frequency Case ($\epsilon \gg 1$)

Following Riley, let us substitute

$$F(x, y) = \left|\frac{\cos y}{2\epsilon}\right|^{\frac{1}{2}} u(z, y)$$

(3.xxiv)

$$G(x, y) = \cos y\ \ v(z, y)$$

(3.xxv)

where $z = |2\epsilon\cos y|^{\frac{1}{2}} x$, and expand u and v as

$$u = u_0(Z) + \epsilon^{-1}\frac{\tan y}{|\cos y|}u_1(Z) + o(\epsilon^{-2}),$$

$$v = v_0(Z) + \epsilon^{-1}\frac{\tan y}{|\cos y|}v_1(Z) + o(\epsilon^{-2})$$

(3.xxvi)

We substitute (3.xxiv) and (3.xxv) in equations (2.ii) and (2.iii). Equating the coefficients of powers of ϵ^{-1} , we obtain the set of equations as

$$u_0^{///} + \left(2u_0u_0^{//} - u_0^{/2} + v_0^{/2}\right) = K_1\left(2v_0v_0^{//} - 4u_0u_0^{iv}\right)$$

(3.xxvii)

$$v_0^{//} + 2\left(u_0v_0^{/} - u_0^{/}v_0\right) = -2K_1\left(2u_0v_0^{///} + u_0^{///}v_0 - 2u_0^{/}v_0^{//} + 4u_0^{//}v_0^{/}\right)$$ (3.xxviii)

$$u_1^{///} + 2\left(u_0u_1^{//} - u_0^{/}u_1^{/} + u_0^{//}u_1 + v_0v_1\right) = -\left(u_0^{/} + \tfrac{1}{2}xu_0^{//}\right) + K_1\left(4u_0^{///} + xu_0^{iv} + 4u_0u_1^{iv} + 4u_0^{iv}u_1 + 8u_0^{/}u_1^{///} + 8u_0^{///}u_1^{/} - 4u_0^{//}u_1^{/} + 2v_0^{//}v_1 + 2v_0v_1^{//} + 12v_0^{/}v_1^{/}\right)$$

(3.xxix)

$$v_1^{//} + 2\left(u_0v_1^{/} - u_0^{/}v_1 + u_1v_0^{/} - u_1^{/}v_0\right) = -\left(v_0 + \tfrac{1}{2}xv_0^{/}\right) - K_1\left(xv_0^{//} + 4u_0v_1^{///} + 4u_1v_0^{///} + 2u_0^{///}v_1 + 2u_1^{///}v_0 - 4u_0^{/}v_1^{//} - 4u_1^{/}v_0^{//} + 8u_1^{//}v_0^{/} + 8u_0^{//}v_1^{/}\right)$$

(3.xxx)

where $K_1 = |\epsilon\cos y|K$.

With this the accompanying boundary conditions will be

$u_0 = u_0^{/} = 0, v_0 = 1, u_1 = u_1^{/} = v_1 = 0$ at $x = 0$

$u_0^{/} \to 0, v_0 \to 0, u_1^{/} \to 0, v_1 \to 0$ at $x \to \infty$

(3.xxxi)

At this step, we use Kármán-Pohlhausen method with the transformations

$Z = \delta X, u_0 = \delta U_0, u_1 = \delta U_1$

This is subject to the following conditions

$U_0 = U_0^{/} = 0, v_0 = 1, U_1 = U_1^{/} = v_1 = 0$ at $x = 0$

$U_0^{/} \to 0, v_0 \to 0, U_1^{/} \to 0, v_1 \to 0$ at $x \to \infty$

(3.xxxii)

In order to ensure the solutions within the boundary layer passes smoothly to that outside it, we consider the following additional boundary conditions.

$U_l^{//} = U_l^{///} = 0 = v_l^{/} = v_l^{//}\ (l = 0,1)$ at $X = 1$

(3.xxxiii)

Taking into consideration the above two sets of boundary conditions, let us assume the following forms

$U_0^{/} = A_0X(1 + B_0X)(1 - X)^3$, $v_0 = (1 + C_0X)(1 - X)^3$

(3.xxxiv)

$U_1^{/} = A_1X(1 + B_1X)(1 - X)^3$, $v_1 = C_1X(1 + D_1X)(1 - X)^3$

(3.xxxv)

From the expression in (3.xxvi), we note that the solution as given by (3.xxiv), (3.xxv) and (3.xxvi) will not remain uniformly valid in the neighbourhood of a turning point of the disk where $y = 0$ or $(2n + 1)\frac{\pi}{2}$ for integer .

If we take $y = (2n + 1)\frac{\pi}{2} + s, |s| \ll 1$ in the region, the expressions in (3.xxvi) take the forms

$$u = u_0 - \frac{\epsilon^{-1}}{s|s|}u_1 + o(\epsilon^{-2}), \qquad v = v_0 - \frac{\epsilon^{-1}}{s|s|}v_1 + o(\epsilon^{-2})$$

(3.xxxvi)

$s|s|$ being as smaller as ϵ^{-1} , this solution will clearly breakdown. Thus it may be comfortably concluded that $|s| = o\left(\epsilon^{-\frac{1}{2}}\right)$ is the region of non-uniformity. In view of this (3.xxiv) and (3.xxv) suggest us the consideration of

$F = \epsilon^{-3/4}R,\ G = \epsilon^{-1/2}S,\ x = \epsilon^{-1/4}Z,\ s = \epsilon^{-1/2}\sigma$

(3.xxxvii)

Here the new variables introduced of $o(1)$ in the region of uniformity.

IV. Results and Discussion

Using (3.xxi), (3.xxii) and (3.xxiii), we can determine the constants b and δ for different values of K_1. We have calculated the values of, D, δ , $R^{//}(0)$ for $K_1 = 0, .002, .005$. These values of the parameters for different values of visco-elastic parameter are given in the table-1. From the Table-1, we can conclude that absolute vale of $R^{//}(0)$ decreases with the increase in the visco-elastic parameter K_1 . The correctness of the calculations can be guessed by the fact that whereas the exact value of $R^{//}(0)$ is -0.207 for Newtonian fluid i.e. $K_1 = 0$ the same has been calculated as – 0.205.

Let us consider again the inner solution. The term of order $o(\varepsilon)$ in the substituted series (3.i) is given as

$$F_2(x) = c_2\ x^2 \qquad (3.xxiv)$$

Where c_2 is yet unknown. The constant was assumed to be zero by Rosenblat [4] and Srivastava [8]. By matching (3.xxiv) with the appropriate term in the outer solution we get
$c_2 = R^{//}(0) \neq 0$.

Thus the terms of even order in ε do not vanish in the inner solution.

Again the unknown parameters in (3.xxxiv) and (3.xxxv) above are given in Table-2 for different values of the visco-elastic parameterK_1 . It is observed that the absolute values of $v_0{}^{/}(0)$ decreases but the absolute values of $u_0{}^{//}(0)$, $u_1{}^{//}(0)$, $v_1{}^{/}(0)$ increases with the increasing values of K_1 as compared to their values for Newtonian fluid i.e., for the case of $K_1 = 0$.

With these substitutions, equations (2.ii) (2.iii) conclude that in this region the unsteady terms are comparable to in magnitude with the inertia, viscous and visco-elastic terms outside the region where the solution is given by (3.xxiv), (3.xxv) and (3.xxvi), they are of lower order. Thus in the limit, as $\epsilon \to \infty$, the solution as $\epsilon s^2 \to \infty$ is given by (.i) and (.ii).

Table – 1

K_1	0	.002	.005
b	0	-0.0003	-0.0007
D	0.249	0.2413	0.2398
δ	4.971	4.9801	4.9987
$R^{//}(0)$	-0.2052	-0.2041	-0.1998

Table - 2

K_1	0	.004	.01
A_0	1.9289	3.0099	5.0012
B_0	-0.385	-2.0126	-4.1108
C_0	1	0.9014	0.7101
δ	3.5559	5.0364	8.3322
A_1	0.0601	-0.6831	-0.9987
B_1	2.879	3.7712	5.4689
C_1	1.9125	6.0019	11.05
D_1	-3.81	-3.1264	-2.9989
$u_0{}^{//}(0)$	0.5238	0.6127	0.6949
$v_0{}^{/}(0)$	-.5526	0.41117	-.3268
$u_1{}^{//}(0)$	0.021	-0.2201	-0.2373
$v_1{}^{/}(0)$	0.5291	0.9987	1.858

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