Convex and Concave Functions

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Abstract- *In mathematics there is one of the most important properties of real numbers is comparability. In real number system we can compare two distinct real numbers and we can say one of them is smaller or larger than other. The inequalities which we drive are totally dependent on these properties. In this work we study convex and concave functions and use them to drive arithmetic-geometric mean inequality, A-G-H mean inequality, Chebyshey's inequality, Holder inequality, Minkowski's inequality and Jensen's inequality.*

Keywords- Exponential functions, Quadratic functions, Inequalities, A-G mean inequality.

Introduction

A real valued function f(x) defined on an interval is called convex (or convex downward or concave upward), if the line segment between any two points on the graph of the function lies above the graph. In a Euclidean space (or more generally a vector space) of a least two dimensions, a function is convex if its epigraph (the set of points on or above of the graph of the function) is a convex set. Well-known examples of convex function are the quadratic function $f(x) = x^2$ and exponential function $f(x) = e^x$.

Convex function play an important role in many areas of mathematics. They are especially important in the study of optimization problems where they are distinguished by a number of convenient properties. For instance a (strictly) convex function on an open set has no more than one minimum. Even in infinite dimensional spaces, under suitable additional hypothesis, convex function to satisfy such properties and, as a result, they are most well-understood functionals in the calculus of variations. In probability theory, a convex function applied to the expected value of a random variable is always less than or equal to the expected value of the convex function of the random variable. This result known as Jensen's inequality underlies many important inequalities (including for instance, the AM-GM mean inequality and Holder inequality). For more on convex function reader may refer to [6].

Exponential growth is a special case of convexity. Exponential narrowly means \in-creasing at a rate proportional to the current value", while convex growth generally means \increasing at an increasing rate (but not necessarily proportionally to current value)".

In mathematics, an inequality is a relation that holds between two values when they are different. The notation $a \neq b$ means that a is not equal to b: It does not say that one is greater than the other or even that they can be compared in size. If the values in question are element of an ordered set, such as the integers or the real numbers, they can compared in size.

1. The notations $a \leq b$ means that a is less than or equal to b (or, equivalently greater than b, or at most of b).

2. The notation $a \geq b$ means that a is greater than or equal to b (or, equivalently less than b, or at least b).

An additional use of the notation is show that one quantity is much greater than another, normally by several orders of magnitude.

The basic properties of this ordering on **R** is following:

1. Given any two real numbers a and b one of the following relations is true:

2. $a > b$ or $a < b$ or $a=b$;

3. $a > 0$ and $b > 0$ imply $a + b > 0$;

4. $a > 0$ and $b > 0$ imply $ab > 0$.

Some easy consequences of these fundamental properties of ordering is following:

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1. $a < b$ then $a + c < b + c$, for any real c;

2. $a < b$ and $c > 0$ give $ac < bc$; if $c < 0$ we have $ac > bc$;

3. $0 < a < b$ implies $0 < \frac{1}{b} < \frac{1}{a}$;

4. $a < 0$ and $b < 0$, then $ab > 0$; $a < 0$ and $b > 0$ imply $ab < 0$;

5. $a < b$ and $b < c$ together imply $a < c$ (transitivity);

6. If $ac < bc$ and $c > 0$ we have $a < b$;

7. $0 < a < 1$ imply $a^2 < a$, if $a > 1$ we have $a^2 > a$;

8. For any real a, a^2 0;

9. If a and b are positive and $a^2 < b^2$,we have $a < b$.

For more on properties reader may refer to [5].

In this we emphasis here that the subtraction of inequalities is generally not allowed. If $a > b$ and $c > d$, we cannot guarantee that either $ac > bd$ or $ca > db$. For this we can say that :

The reason is obvious : $x > y$ implies $x < y$. Similarly a; b; c; d equal 0; neither $\frac{a}{c} > \frac{b}{d}$ is true nor $\frac{c}{a} > \frac{d}{b}$. Here the reason is simple $x > y$ gives $\frac{1}{x} < \frac{1}{y}$ (x and y not equal to 0). On the other hand we may add two inequalities. If we have all the terms of two inequalities are positive, we can multiply them. If a; b; c; d are positive and $a > b$, $c > d$ holds the inequality $ac > bd$ also holds.

Basic Concepts

1.1 Convex and Concave functions

A plane or solid figure such as polygon or polyhedron is convex if the line segment joining any two point inside it lies wholly inside it. A function f : ***I*** →R is called convex if

$$f((1-\lambda)x + y) \leq (1-\lambda)f(x) + \lambda f(y);$$

for all x and y in ***I*** and all λ **Ɛ** [0; 1]:

If the inequality is reversed i.e

$$f((1-\lambda)x + y) \geq (1-\lambda)f(x) + f(y);$$

for all x and y in I and λ **Ɛ** [0; 1]:

then f is said to be concave. If f is both convex and concave then f is said to be affine.

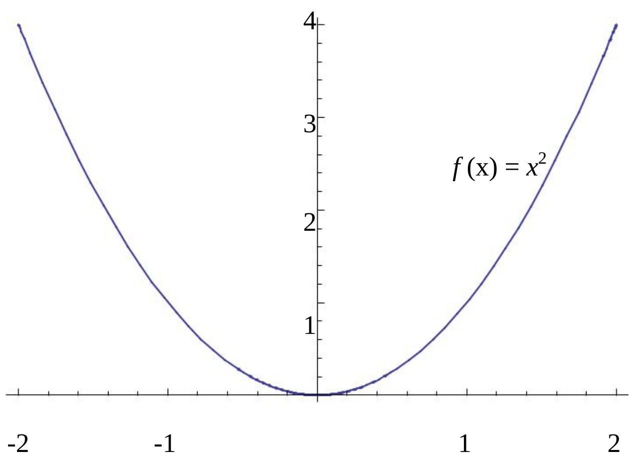
Example 1.1.1. Examine whether the function $f(x) = x^2$ defined on **R** is convex or concave.

Proof. By using the definition of convex function:

$$((1-\lambda)x + y))^2 \leq (1-\lambda)x^2 + y^2$$
$$(1-\lambda)^2x^2 + \lambda^2y^2 + 2(1-\lambda)x{:}\ \lambda y - (1-\lambda)x^2 - \lambda y^2 \leq 0$$
$$(1-\lambda)^2x^2 + \lambda^2y^2 + 2(1-\lambda)x{:}\lambda y - (1-\lambda)x^2 - \lambda y^2 \leq 0$$
$$(1-\lambda)x^2(1-\lambda-1) + \lambda y^2(\lambda-1) + 2(1-\lambda)xy \leq 0$$
$$\lambda(1-\lambda)x^2 + \lambda y^2(\lambda-1) + 2(1-\lambda)xy \leq 0$$
$$\lambda(1-\lambda)(-x^2 - y^2 - 2xy) \leq 0$$
$$-\lambda(1-\lambda)(x^2 + y^2 + 2xy) \leq 0$$
$$-\lambda(1-\lambda)(x+y)^2 \leq 0$$
$$-(\lambda(1-\lambda))(x+y)^2 \leq 0 \tag{1.1.1}$$

We know that

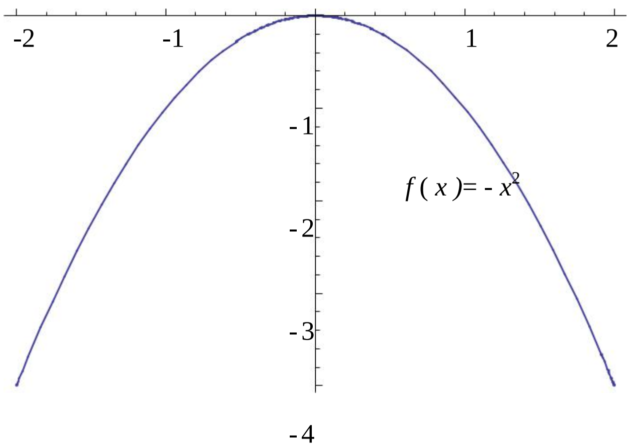
$$0 < \lambda < 1$$
$$-0 > -\lambda > 1$$
$$1 - 0 > 1-\lambda > 1 - 1$$
$$1 > 1-\lambda > 0$$
$$0 < 1-\lambda < 1 \tag{1.1.2}$$



So,$-\lambda((1-\lambda)(x+y)^2) \leq 0$:

So, $f(x) = x^2$ is convex function. □

Remark 1.1.2. $f(x) = -x^2$ is concave function, as shown in the figure below:



Example 1.1.3. Examine whether the function f(x) = a + bx defined on R is convex or concave.

Proof. By using the definition of convex function

$$a + b\{(1-\lambda)x + \lambda y\} \leq (1-\lambda)(a + bx) + \lambda(a + by)$$

$$a + b\{(1-\lambda)x + \lambda y\} - \{(1-\lambda)(a + bx) + \lambda(a + by)\} \leq 0$$

$$a + b(1-\lambda)x + b\lambda y - (1-\lambda)a - (1-\lambda)bx - \lambda a - \lambda by \leq 0$$

$$a - \lambda a + b(1-\lambda)x - (1-\lambda)bx - (1-\lambda)a \leq 0$$

$$a - \lambda a - a + \lambda a = 0: \qquad (1.1.3)$$

So f(x) = a + bx is both convex function and concave function. Hence f(x) = a + bx is affine function. □

The operations with convex functions

(i) Adding two convex functions (defined on some interval) we obtain a convex function ; if one of them is strictly convex, then the sum is also strictly convex.
(ii) Multiplying a (strictly) convex function by a positive scalar we obtain also a (strictly) convex function.
(iii) The restriction of every (strictly) convex function to a subinterval of its domain is also a (strictly) convex function

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