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Tension Quadratic Trigonometric B´Ezier Curve with Two Shape Parameters

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Abstract- In this paper, quadratic trigonometric Bezier´ curve with two shape parameters, with a tension parameter, are presented. The new basis functions share the properties with Bernstein basis functions, so the generated curves inherit many properties of traditional Bezier´ curves. The presence of shape parameters provides a local control on the shape of the curve which enables the designer to control the curve more than the ordinary Bezier´ curve. These type of functions are useful for constructing trigonometric Bezier curves and surfaces, they can be applied to construct continuous shape preserving interpolation spline curves with shape parameters. To better visualize objects and graphics, a tension parameter is included. In this work we constructed the Trigonometric Bezier curves followed by a construction of the shape preserving interpolation spline curves with local shape parameters and a tension parameter.

Index Terms- Blending Functions, Trigonometric Bezier Curves, Tension Parameter, Shape Parameter, Interpolation trigonometric basis functions, shape control.

I. INTRODUCTION

In computer aided geometric design (CAGD), it is often necessary to generate curves and surfaces that approximate shapes with some desired shape features. Designing free form curves and surfaces is a prevalent feature of CAGD. The key problem, simply stated, is to enable the designer to generate curves and surfaces which behave as he wants them to. The parametric cubic is a powerful tool which, when properly defined, is capable of representing most geometric entities of practical interest. In recent years, the trigonometric spline with shape parameters has gained more interest in CAGD, in particular curve design. Han [1] presented a class of quadratic trigonometric polynomial curves with a shape parameter. The shape of the curve was more easily controlled by altering the values of shape parameter than the ordinary quadratic B-Spline curves. Han [2] introduced piecewise quadratic trigonometric polynomial curves with C^1 continuity analogous to the quadratic B-Spline curves which have C^1 continuity. Cubic trigonometric polynomial curves with a shape parameter were discussed by Han [3]. In these papers the authors described the trigonometric polynomial with global shape parameter. Single parameter does not provide local control on the curves. To remedy this, Wu et al presented quadratic trigonometric polynomial curves with multiple shape parameters. Bezier´ technique is one of the methods of analytic representation of curves and surfaces that has won wide acceptance as a valuable tool in CAD/CAM system. They are used to produce curves which appear reasonably smooth at all scales. Today, many CAGD systems feature Bezier´ curves as their major building block, since they are very efficient and attain a number of mathematical properties. Both rational and non-rational forms of Bezier´ curves have been studied by many authors. A cubic trigonometric Bezier´ curve with two shape parameters was presented by Han et al. It enjoyed all the geometric properties of the ordinary cubic Bezier´ curve and was used for spur gear tooth design with S-shaped transition curve Abbas, et al. Liu, et. al presented a study on class of TC-Bezier´ curve with shape parameters. Yang, et. al discussed trigonometric extension of quartic Bezier´ curves attaining G^2 and C^2 continuity. A class of general quartic spline curves with shape parameters were introduced by Han. Yang, et al studied a class of quasi-quartic trigonometric Bezier´ curves and surfaces. It is important to study the spline curve representations that provide local con-trol, that is, the capability of modifying one portion of the curve without altering the remainder. From a practical standpoint, we are interested in constructing the trigonometric polynomial representations which can manipulate a curve e ectually.

The purpose of this paper is to present quadratic trigonometric polynomial blending functions where we include a tension parameter, the latter is mainly important for object visualization. These blending functions are useful for constructing trigonometric Bezier curves and can be applied to construct continuous shape preserving interpolation spline curves with shape parameters. A newly constructed quadratic trigonometric Bezier´ curve with two shape parameters is presented in this paper. The proposed curve inherits all trigonometric properties of the traditional Bezier´ curve and is used to construct open and closed curves.

The remainder of this paper is organized as follows. In section 2, the quadratic Trigonometric polynomial blending functions with shape and tension parameter are described and the properties of these functions are shown. In section 3 Trigonometric Bezier curves are constructed and their properties are discussed. Shape control is discussed in section 4. In section 5, the representation of ellipse and circle are given. The Trigonometric polynomial approximation is discussed in Section 6. Conclusion is given in section 7.

II. QUADRATIC TRIGONOMETRIC BEZIER BASIS FUNCTIONS

Firstly we can define the trigonometric polynomial blending functions with tension parameter and then their properties are given.

2.1 Trigonometric Polynomial Blending Functions with Tension Parameter

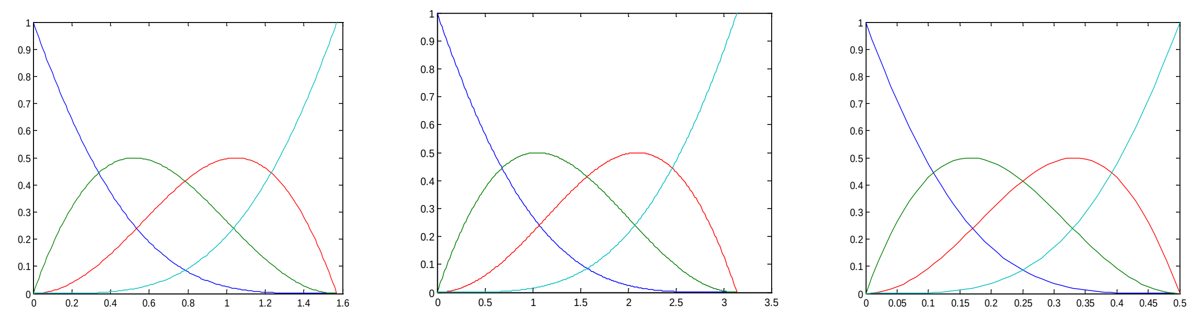
The trigonometric polynomial blending functions are given as follows:

Definition 2.1 for two arbitrarily real value of m and n , β be the tension parameter and $t \in \left[0, \frac{\pi}{2\beta}\right]$. The Quadratic trigonometric Bezier basis functions with tension parameter and two shape parameters $B_{i,\beta}$; i = 0,1, 2,3 are defined as :

$$B_{0,\beta}(t,m,n) = (1-\sin(\beta t))(1+(1-m)\sin(\beta t));$$
$$B_{1,\beta}(t,m,n) = m\sin(\beta t)(1-\sin(\beta t));$$
$$B_{2,\beta}(t,m,n) = n\cos(\beta t)(1-\cos(\beta t));$$
$$B_{3,\beta}(t,m,n) = (1-\cos(\beta t))(1+(1-n)\cos(\beta t)); \quad (1)$$

For m = n = 0, the basis functions are linear trigonometric polynomials. For m, n ≠ 0, the basis functions are quadratic trigonometric polynomials.

Figure 1, plots these basis functions for different values of the tension parameter β in the interval $t \in [0, \pi/2\beta]$



$\beta = 1$ $\beta = .5$ $\beta = \pi$

Figure 1. The curves of blending basis functions (for $t \in [0, \pi/2\beta]$)

The blending functions studied in the present work have the following properties which are analogous to those found for the quadratic trigonometric Bezier basis functions .

2.2 Properties of Trigonometric Polynomial Blending Functions:

Theorem 1. The basis functions $B_{i,\beta}(t,m,n),(i=0,1,2,3)$ defined in (1) have the following properties:

(a) Non-negativity: When $t \in \left[0, \frac{\pi}{2\beta}\right]$ there are $B_{i,\beta}(t,m,n) \geq 0,(i=0,1,2,3)$.

(b) Partition of Unity: One has $\sum_{i=0}^{3} B_{i,\beta}(t,m,n) = 1;$

(c) Symmetry: $B_{i,\beta}(t,m,n) = B_{3-i,\beta}\left(\frac{\pi}{2\beta} - t,m,n\right);$ for i = 0,1,2,3;

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(d) **Maximum** : Each $B_{i,\beta}(t,m,n)$ has one maximum value in $t \in \left[0, \frac{\pi}{2\beta}\right]$

(e) Monotonicity: For a given value of shape parameter mand n , $B_{0,\beta}$ is monotonically decreasing and $B_{3,\beta}$ is monotonically increasing.

III. Trigonometric Bezier Curves with Tension Parameter

This section describes the theory and method of using the tension parameter β to control the form of the interpolating trigonometric Bezier curve $B_{i,\beta}$ (t,m,n). Note that changing the tension factor β does not affect the form of $B_{i,\beta}$ (t,m,n) and the interpolation features at the data points.

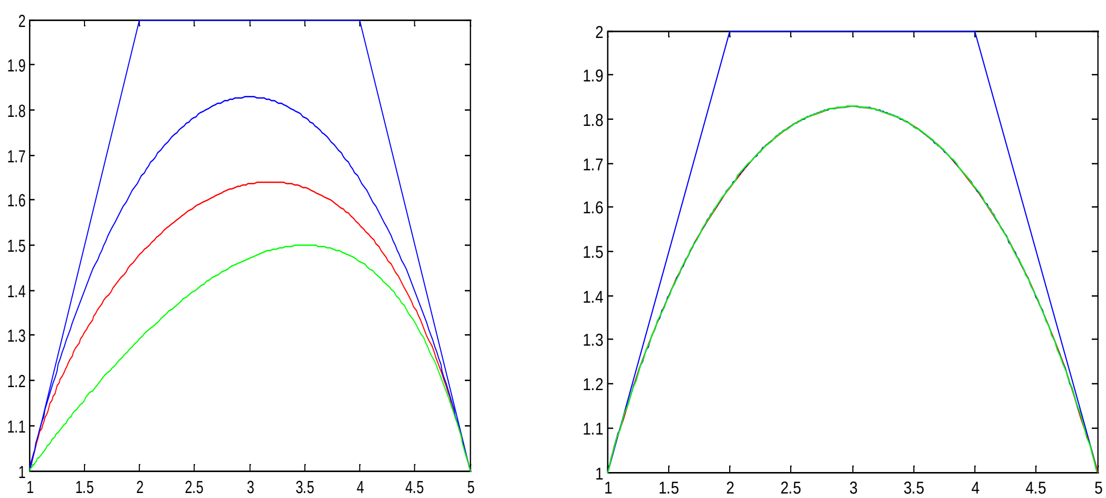
Definition 3.1: Let $P = (P_0, P_1, P_2, P_3)$ be a set of points $P_i \in R^2$ or R^3. The Trigonometric Bezier curves with tension parameter $\beta > 0$ associated with the set P is defined by:

$$R_\beta\ (t,m,n) = \sum_{i=0}^{3} (B_{i,\beta}\ (t,m,n))\ P_i\ ; \quad \text{where } t \in \left[0, \frac{\pi}{2\beta}\right] \text{ and } m,n \in [0,2]; \tag{2}$$

The points P_i ; (i = 0,1,2,3) are called quadratic trigonometric Bezier control points.

Figure 2 shows the quadratic Trigonometric Bezier curves with different tension parameter values. Keeping the same control polygon, as β varies we are not simply changing the domain of a single curve, but defining different curves. It can be seen that quadratic trigonometric Bezier curves with shape and tension parameter are close to the control polygon. Therefore, quadratic trigonometric Bezier curves with shape and tension parameter can nicely preserve the feature of the control polygon. Control polygons provide an important tool in geometric modeling.

The tension like effect of this tension factor β is illustrated in Figures 1 and 2 where the interval changes as a function of β keeping all the properties of the blending functions verified. It is an advantage if the curve being modeled tends to preserve the shape of its control polygon.



m=0,1,2; n=2; $\beta = \pi/4$; **different value of β =1,0.5 and π ;**

Figure 2. The Bezier curves of blending basis functions (for $t \in [0, \pi/2\beta]$)

Theorem 2 The quadratic Bezier curve with shape and tension parameter (2) have the following properties:

(i) End point properties:

$$R_\beta(0,m,n)=P_0; R_\beta\left(\frac{\pi}{2\beta},m,n\right)=P_3;$$
$$R'_\beta(0,m,n)=m\beta(P_1-P_0);$$
$$R'_\beta\left(\frac{\pi}{2\beta},m,n\right)=n\beta(P_3-P_2);$$
$$R''_\beta(0,m,n)=[-2P_0+2m(P_0-P_1)+n(P_2-P_3)+2P_3]\beta^2;$$
$$R''_\beta\left(\frac{\pi}{2\beta},m,n\right)=[2P_0+m(P_1-P_0)+2n(P_3-P_2)-2P_3]\beta^2;$$

(ii) Symmetry: The control points P_i and ; P_{3-i} i =0;1;2;3 define the same quadratic trigonometric B´ezier curve in different parameterizations, i.e. for i = 0; 1; 2; 3.

$$R_\beta(t;\ m;\ n;P_i)=R_\beta\left(\frac{\pi}{2\beta}-t,n,m,P_{3-i}\right)$$ where $t\in\left[0,\frac{\pi}{2\beta}\right]$ and $m,n\in[0,2]$;

$$R_\beta(t;P_0;P_1;P_2;P_3)=R_\beta\left(\frac{\pi}{2\beta}-t;P_0;P_1;P_2;P_3\right)$$

(ii) Geometric invariance: The shape of the curve (2) is independent of the choice of its control points i.e. the curve (2) satisfies the following two equations for i = 0; 1; 2; 3..

$$R_\beta(t;m;n;P_i+q)=R_\beta(t;m;n;P_i)+q;$$
$$R_\beta(t;m;n;P_i*q)=R_\beta(t;m;n;P_i)+q;$$ where $t\in\left[0,\frac{\pi}{2\beta}\right]$ and $m,n\in[0,2]$;

where q is any arbitrary vector in R^2 and T is an arbitrary 2 * 2 matrix.

(iii) Convex hull: The entire curve is contained within the convex hull of its defining control points P_i.

(iv) Variation diminishing property: no straight line intersects a Bezier curve more times than it intersects its control polygon.

IV. SHAPE CONTROL OF THE QUADRATIC TRIGONOMETRIC BEZIER CURVE

For control points $P=(P_0,P_1,P_2,P_3)$ be a set of points $P_i\in R^2$ or R^3. The Trigonometric Bezier curves with tension parameter $\beta>0$ associated with the set P is defined by:

$$R_\beta\ (t,m,n)=\sum_{i=0}^{3}(B_{i,\beta}\ (t,m,n))\ P_i\ ; \qquad (3)$$

where $t \in \left[0, \frac{\pi}{2\beta}\right]$ and $m, n \in [0, 2]$;

We rewrite this equation as follows;

$$R_\beta(t)=(1-\sin^2(\beta t))P_0+(1-\cos^2(\beta t))P_3+m(P_1-P_0)\sin(\beta t)(1-\sin(\beta t))+n(P_2-P_3)\cos(\beta t)(1-\cos(\beta t));$$

Obviously, shape parameter m and n affects the curve on the control edges (P_1-P_0) and (P_2-P_3).The shape parameter m and n serve to effect local control in the curve: m and n as increases, the curve moves in the direction of edges (P_1-P_0) and (P_2-P_3) and as m and n decreases, the curve moves in the opposite direction to the edges (P_1-P_0) and (P_2-P_3). The parameters m and n controls the shape of the curve (3). In figure 2, The quadratic trigonometric Bézier curve with tension parameter gets closer to the control polygon as the values of the parameters m and n increases with tension parameter. In figure 2, the curves are generated by setting the values of m=2,n=2 and $\beta=\pi$ (green lines), m=2,n=1 and $\beta=\pi$ (red lines), m=1,n=2 and $\beta=\pi$ (blue lines), m=2,n=2 and $\beta=1$ (green lines).

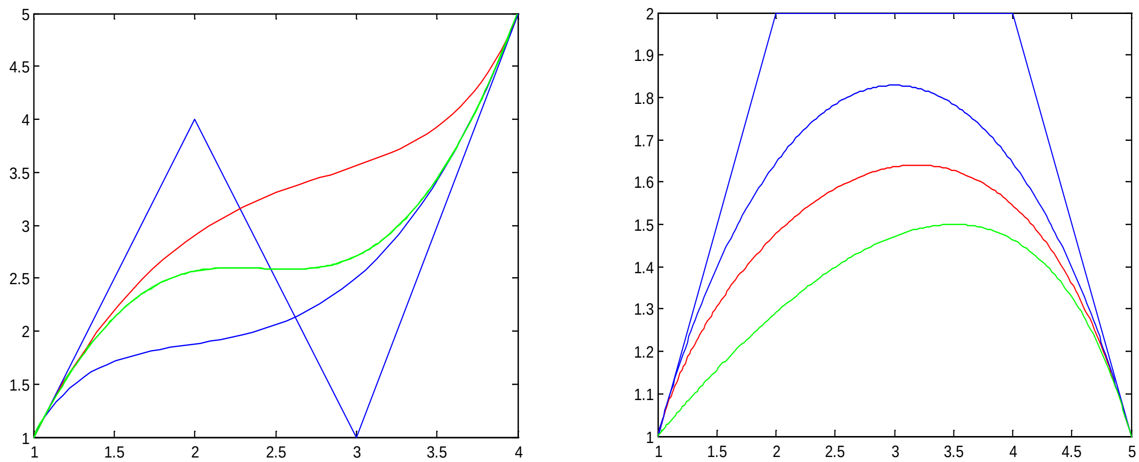
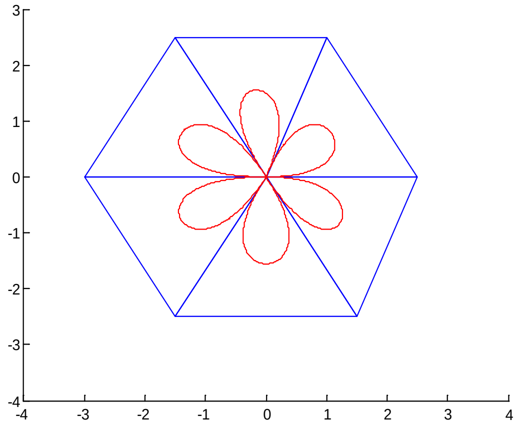


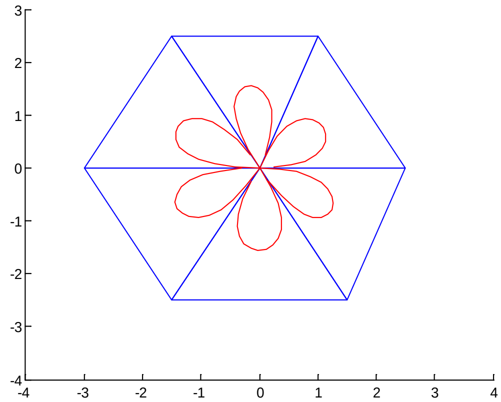
Figure 3: The effect on the shape of quadratic trigonometric Bézier Curves with tension parameter with altering the values of m, n and β.

In order to construct a closed quadratic trigonometric Bézier curves with tension parameter . we can set $P_n = P_0$. In figure 4(a),(b),(c),(d), the closed quadratic trigonometric Bézier curves with tension parameter altering the values of the shape parameters m and n at the same time. The quadratic trigonometric Bézier curves with tension parameter are generated by setting m = 1.5, n = 1.5 and $\beta=\pi/4$ in figure (a), m = 1.5, n = 1.5 and $\beta=10$ in figure (b), m = 1.5, n = 1.5 $\beta=0.8$ in figure (c), m =1.5, n = 1.5 $\beta=30$ in figure (d).

(a)



(b)



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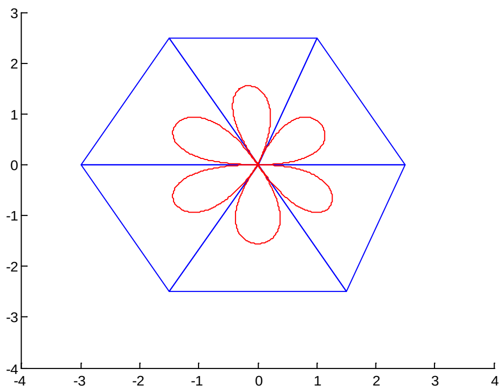
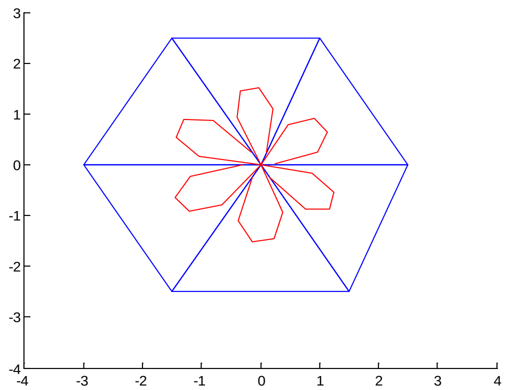


(d)



V. THE REPRESENTATION OF AN ELLIPSE

Theorem 5.1: Let $P = \left(P_0, P_1, P_2, P_3\right)$ be four control points on an ellipse with semi axes ((2√2) a and (4√2) b, by the proper selection of coordinates, their coordinates can be written in the form

$$P_0 = (2a, 0); P_1 = (a, 2b); P_2 = (-a, 2b); P_3 = (-2a, 0);$$

Then the corresponding quadratic trigonometric Bézier curve with tension and the shape parameters m = n = 2 with

β =0.8 and local domain t ∈ [0,4] represents arc of an ellipse with

$$x(t) = 2a(\cos(\beta t) - \sin(\beta t))$$
$$y(t) = 4b(\cos(\beta t) + \sin(\beta t) - 1)$$

Proof: If we take m = n =2 and $P_0 = (2a, 0); P_1 = (a, 2b); P_2 = (-a, 2b); P_3 = (-2a, 0);$

into(3), then the coordinate of quadratic trigonometric Bezier curve with tension and shape parameter are

$$x(t) = 2a(\cos(\beta t) - \sin(\beta t))$$
$$y(t) = 4b(\cos(\beta t) + \sin(\beta t) - 1)$$

This gives the intrinsic equation

$$\left(\frac{x(t)}{(2\sqrt{2})a}\right)^2 + \left(\frac{y(t)+4b}{(4\sqrt{2})b}\right)^2 = 1$$

It is an equation of an ellipse. Figure5. Shows an ellipse.

Corollary 5.1: According to theorem (5.1), if, $a = \frac{1}{2\sqrt{2}}, b = \frac{1}{4\sqrt{2}}$ then the corresponding quadratic trigonometric Bézier curves with tension parameter β =0.8 and the shape parameter $m = n = 2$ and local domain $t \in [0,4]$ represents

arc of an circle. Fig. 6 shows the Circle.

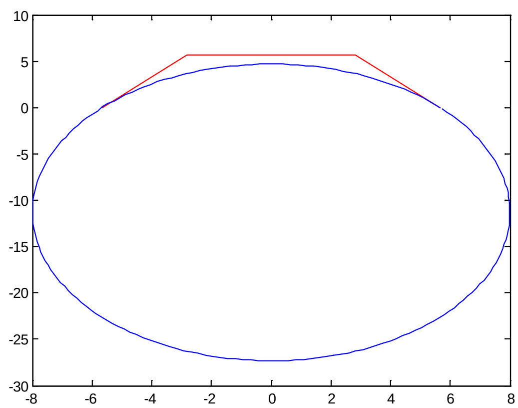


Figure 5: the representation of an ellipse.

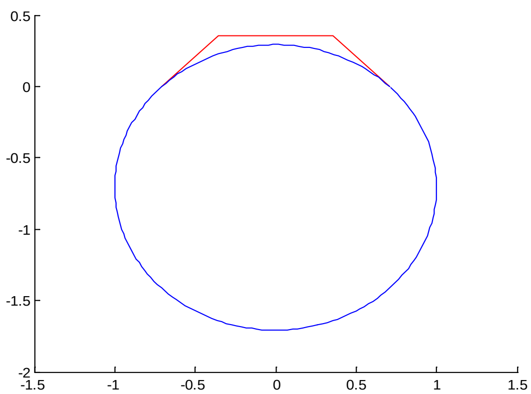


Figure 6: the representation of an circle.

VI. APPROXIMABILITY

Control polygon provides an important tool in geometric modeling. It is an advantage if the curve being modeled tends to preserve the shape of its control polygon. Now we show the relations of the quadratic trigonometric Polynomial B-spline curve and cubic Bézier curves corresponding to their control polygon.

Theorem 3: Suppose P_0, P_1, P_2 and P_3 are not collinear; the relationship between quadratic trigonometric Polynomial B-spline curve $R_\beta(t,m,n)$ and the cubic Bézier curve $B(t) = \sum_{j=0}^{3} P_i (3, j)(1-t)^{3-j} t^j$; $t \in [0, 1]$ with the same control points P_i (i=0,1,2,3) are given by

$$B(0) = R_\beta\ \mathbf{(0,m,n);}$$

$$B(1) = R_\beta\ \mathbf{(}\frac{\pi}{2\beta}\mathbf{,m,n);}$$

$$R_\beta(\frac{\pi}{4\beta}) - P^* = \frac{8\left(\sqrt{2}-1\right)m}{6}(B(\frac{1}{2}) - P^*); \qquad (4)$$

Where $P^* = \frac{P_0 + P_3}{2}$ with the assumption that m = n.

Proof: we assume that m = n then the quadratic trigonometric Polynomial B-spline curve $R_\beta\ (t,m,n) = \sum_{i=0}^{3}(B_{i,\beta}\ (t,m,n))\ P_i$;

where $t \in \left[0, \frac{\pi}{2\beta}\right]$ and $m,n \in [0,2]$;

And

quadratic Bézier curve $B(t) = \sum_{j=0}^{3} P_i(3, j)(1-t)^{3-j}\, t^j$; $t \in [0, 1]$

By simple computation, we have

$$B(0) = R_\beta\ \mathbf{(0,m,n);} \text{ and } B(1) = R_\beta\ \mathbf{(}\frac{\pi}{2\beta}\mathbf{,m,n);}$$

$$B(\frac{1}{2}) = \frac{1}{8}(P_0 + 3P_1 + 3P_2 + P_3)$$

$$B(\frac{1}{2}) - P^* = \frac{-3}{8}(P_0 - P_1 - P_2 + P_3)$$

$$\text{where} P^* = \frac{P_0 + P_3}{2}$$

$$R_\beta(\frac{\pi}{4\beta}) - P^* = \frac{1}{2}\left[m\left(\sqrt{2}-1\right)(P_1 - P_0) + n\left(\sqrt{2}-1\right)(P_2 - P_3)\right]$$

at m = n

$$B(\frac{1}{2}) = \frac{1}{8}(P_0 + 3P_1 + 3P_2 + P_3)$$

$$B(\frac{1}{2}) - P^* = \frac{-3}{8}(P_0 - P_1 - P_2 + P_3)$$

$$\text{where} P^* = \frac{P_0 + P_3}{2}$$

$$R_\beta(\frac{\pi}{4\beta}) - P^* = \frac{-\left(\sqrt{2}-1\right)n}{2}(P_0 - P_1 - P_2 + P_3)$$

$$R_\beta(\frac{\pi}{4\beta}) - P^* = \frac{8\left(\sqrt{2}-1\right)m}{6}(B(\frac{1}{2}) - P^*);$$

Equation (4) holds. These equations show that quadratic trigonometric Bézier Curves with tension parameter can be made closer to the control polygon by altering the values of shape parameters.

Corollary 6.1: The quadratic trigonometric Bézier Curves with tension parameter is closer to the control polygon that the cubic Bezier curve if and only if $\frac{3}{4}(\sqrt{2}+1) \le m$ and $n \le 2$;

Corollary 6.2: when m = n = $\frac{3}{4}(\sqrt{2}+1)$, the quadratic trigonometric Bézier Curves with tension parameter can be closer to the cubic Bézier Curve, i.e. $B(\frac{1}{2}) = R_\beta$ **$(\frac{\pi}{4\beta}, m, n)$;**

Figure shows the relationship among the quadratic trigonometric Bézier Curves with tension parameter (green line), the quadratic trigonometric Bézier Curves with shape parameter (blue line) and the cubic Bézier Curve(red lone).

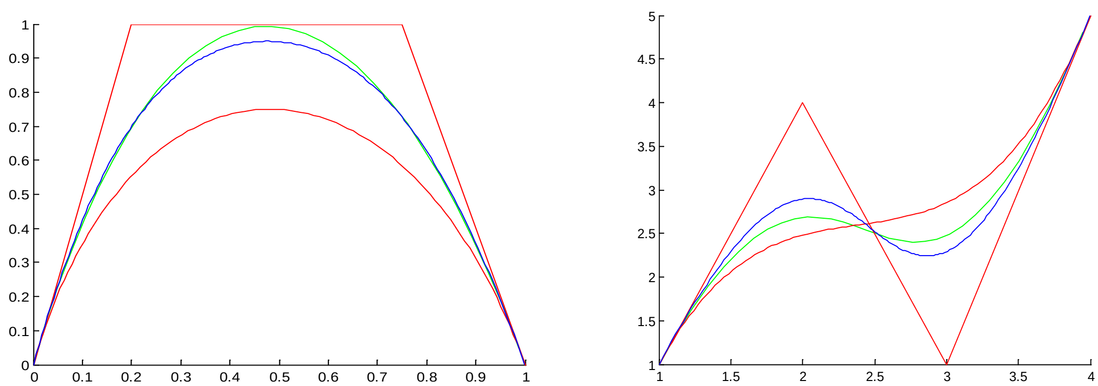


Figure 4. relationship among the quadratic trigonometric Bézier Curves with tension parameter ,the quadratic trigonometric Bézier Curves with shape parameter and the cubic Bézier Curve

VII. CONCLUSION

The trigonometric polynomial blending functions constructed in this paper have the properties analogous to those of the quadratic Bernstein basis functions and the trigonometric Bezier curves are also analogous to the quadratic Bezier ones. In this basis we included the tension parameter which is mainly important for object visualization. The trigonometric Bezier curves are close to the control polygon. Therefore, these trigonometric Bezier curves can preserve the shape of the control polygon. For any shape parameters satisfying the shape preserving conditions, the obtained shape preserving trigonometric interpolation spline curves are all continuous. There is no need to solve a linear system and the changes of a local shape parameter will only affect two curve segments. The shape of the curve can be adjusted using the values of the shape parameters.. Further, it can be extended to tensor product surfaces generalizing this idea to quasi-interpolation with trigonometric spline curve.

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