# Zero-Free Regions and Number of Zeros in Annular Regions of Polynomials with Restricted Coefficients

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***Abstract-*** In this paper we find zero-free regions of a special class of polynomials when their coefficients are restricted to certain conditions. We also find bounds for the number of their zeros in some annular regions.

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***Index Terms***- Bound, Coefficient, Polynomial, Zeros.

## I. INTRODUCTION

The following theorem , called the Enestrom-Kakeya Thorem [2,3] , is well known in the theory of the distribution of zeros of a polynomial:

**Theorem A:** Let $P(z)=\sum_{j=0}^{n}a_j z^j$ be a polynomial of degree $n$ such that

$$a_n \ge a_{n-1} \ge ...... \ge a_1 \ge a_0 > 0.$$

Then all the zeros of P(z) lie in $|z| \le 1$.

Many generalizations ,extensions and refinements of the above result are available in the literature..

Recently Ramulu , Gangadhar and Reddy [4] proved the following results :

**Theorem B:** Let $P(z)=\sum_{j=0}^{n}a_j z^j$ be a polynomial of degree $n \ge m \ge 2$ with real coefficients such that for some $k \ge 1, \delta > 0$,

$$ka_n \ge a_{n-1} \le a_{n-2} \ge a_{n-3} \le a_{n-4} \ge ... \ge a_{n-m+1} \le a_{n-m} \ge a_{n-m-1} \ge ... \ge a_4 \ge a_3 \ge a_2 \ge a_1 \ge a_0 - \delta$$

if both n and n-m are even or odd

or

$$ka_n \ge a_{n-1} \le a_{n-2} \ge a_{n-3} \le a_{n-4} \ge ... \le a_{n-m+1} \ge a_{n-m} \ge a_{n-m-1} \ge ... \ge a_4 \ge a_3 \ge a_2 \ge a_1 \ge a_0 - \delta$$

if n is even and n-m is odd or if n is odd and n-m is even.

Then all the zeros of P(z) lie in

$$|z+k-1| \le \frac{1}{|a_n|}[2\delta + ka_n + |a_0| - a_0 + 2\{(a_{n-2}+a_{n-4}+...+a_{n-m+2}+a_{n-m}) - (a_{n-1}+a_{n-3}+... + a_{n-m+3}+a_{n-m+1})\}]$$

if both n and n-m are even or odd

or

$$|z+k-1| \le \frac{1}{|a_n|}[2\delta + ka_n + |a_0| - a_0 + 2\{(a_{n-2}+a_{n-4}+...+a_{n-m+3}+a_{n-m+1}) - (a_{n-1}+a_{n-3}+... + a_{n-m+4}+a_{n-m+2})\}]$$

if n is even and n-m is odd or if n is odd and n-m is even.

**Theorem C:** Let $P(z)=\sum_{j=0}^{n}a_j z^j$ be a polynomial of degree $n \ge m \ge 2$ with real coefficients such that for some $0 < r \leq 1, \delta > 0$,

$$ra_n \leq a_{n-1} \geq a_{n-2} \leq a_{n-3} \geq a_{n-4} \leq \ldots \leq a_{n-m+1} \geq a_{n-m} \geq a_{n-m-1} \geq \ldots \geq a_4 \geq a_3 \geq a_2 \geq a_1 \geq a_0 - \delta$$

if both n and n-m are even or odd

or

$$ra_n \leq a_{n-1} \geq a_{n-2} \leq a_{n-3} \geq a_{n-4} \leq \ldots \geq a_{n-m+1} \leq a_{n-m} \geq a_{n-m-1} \geq \ldots \geq a_4 \geq a_3 \geq a_2 \geq a_1 \geq a_0 - \delta$$

if n is even and n-m is odd or if n is odd and n-m is even.

Then all the zeros of P(z) lie in

$$|z| \leq \frac{1}{|a_n|}[|a_0| - a_0 - a_n + 2\{(a_{n-1} + a_{n-3} + \ldots + a_{n-m+3} + a_{n-m+1}) - (a_{n-2} + a_{n-4} + \ldots + a_{n-m+4} + a_{n-m+2})\}]$$

if both n and n-m are even or odd

or

$$|z| \leq \frac{1}{|a_n|}[|a_0| - a_0 - a_n + 2\{(a_{n-1} + a_{n-3} + \ldots + a_{n-m+2} + a_{n-m}) - (a_{n-2} + a_{n-4} + \ldots + a_{n-m+3} + a_{n-m+1})\}]$$

if n is even and n-m is odd or if n is odd and n-m is even.

**Theorem D:** Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree $n \geq m \geq 2$ with real coefficients such that for some $k \geq 1, \delta > 0$,

$$ka_n \geq a_{n-1} \leq a_{n-2} \geq a_{n-3} \leq a_{n-4} \geq \ldots \geq a_{n-m+1} \leq a_{n-m} \leq a_{n-m-1} \leq \ldots \leq a_4 \leq a_3 \leq a_2 \leq a_1 \leq a_0 + \delta$$

if both n and n-m are even or odd

or

$$ka_n \geq a_{n-1} \leq a_{n-2} \geq a_{n-3} \leq a_{n-4} \geq \ldots \leq a_{n-m+1} \geq a_{n-m} \leq a_{n-m-1} \leq \ldots \leq a_4 \leq a_3 \leq a_2 \leq a_1 \leq a_0 + \delta$$

if n is even and n-m is odd or if n is odd and n-m is even.

Then all the zeros of P(z) lie in

$$|z + k - 1| \leq \frac{1}{|a_n|}[2\delta + ka_n + |a_0| + a_0 + 2\{(a_{n-2} + a_{n-4} + \ldots + a_{n-m+4} + a_{n-m+2}) - (a_{n-1} + a_{n-3} + \ldots + a_{n-m+3} + a_{n-m+1})\}]$$

if both n and n-m are even or odd

or

$$|z + k - 1| \leq \frac{1}{|a_n|}[2\delta + ka_n + |a_0| + a_0 + 2\{(a_{n-2} + a_{n-4} + \ldots + a_{n-m+3} + a_{n-m+1}) - (a_{n-1} + a_{n-3} + \ldots + a_{n-m+2} + a_{n-m})\}]$$

if n is even and n-m is odd or if n is odd and n-m is even.

**Theorem E:** Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree $n \geq m \geq 2$ with real coefficients such thatfor some $0 < r \leq 1, \delta > 0$,

$$ra_n \leq a_{n-1} \geq a_{n-2} \leq a_{n-3} \geq a_{n-4} \leq \ldots \leq a_{n-m+1} \geq a_{n-m} \leq a_{n-m-1} \leq \ldots \leq a_4 \leq a_3 \leq a_2 \leq a_1 \leq a_0 + \delta$$

if both n and n-m are even or odd

or

$$ra_n \leq a_{n-1} \geq a_{n-2} \leq a_{n-3} \geq a_{n-4} \leq \ldots \geq a_{n-m+1} \leq a_{n-m} \leq a_{n-m-1} \leq \ldots \leq a_4 \leq a_3 \leq a_2 \leq a_1 \leq a_0 + \delta$$

if n is even and n-m is odd or if n is odd and n-m is even.

Then all the zeros of P(z) lie in

$$|z| \le \frac{1}{|a_n|}[|a_n| + |a_0| + a_0 - r(|a_n| + a_n) + 2\delta + 2\{(a_{n-1} + a_{n-3} + ... + a_{n-m+3} + a_{n-m+1}) - (a_{n-2} + a_{n-4} + ... + a_{n-m+2} + a_{n-m})\}]$$

if both n and n-m are even or odd

or

$$|z| \le \frac{1}{|a_n|}[|a_n| + |a_0| + a_0 - r(|a_n| + a_n) + 2\delta + 2\{(a_{n-1} + a_{n-3} + ... + a_{n-m+4} + a_{n-m+2}) - (a_{n-2} + a_{n-4} + ... + a_{n-m+3} + a_{n-m+1})\}]$$

if n is even and n-m is odd or if n is odd and n-m is even.

**2. Main Results**

In this paper we find zero-free regions and bounds for the number of zeros of the polynomials considered in the above theorems and prove

**Theorem 1.** Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree $n \ge m \ge 2$ with real coefficients such that for some $k \ge 1, \delta > 0$,

$$ka_n \ge a_{n-1} \le a_{n-2} \ge a_{n-3} \le a_{n-4} \ge ... \ge a_{n-m+1} \le a_{n-m} \ge a_{n-m-1} \ge ... \ge a_4 \ge a_3 \ge a_2 \ge a_1 \ge a_0 - \delta$$

if both n and n-m are even or odd

or

$$ka_n \ge a_{n-1} \le a_{n-2} \ge a_{n-3} \le a_{n-4} \ge ... \le a_{n-m+1} \ge a_{n-m} \ge a_{n-m-1} \ge ... \ge a_4 \ge a_3 \ge a_2 \ge a_1 \ge a_0 - \delta$$

if n is even and n-m is odd or if n is odd and n-m is even.

Then P(z) has no zero in $|z| < \frac{|a_0|}{M}$, where

$$M = |a_n|R^{n+1} + R^n\{k(|a_n| + a_n) - |a_n| + 2\delta - a_0 + 2(a_{n-2} + a_{n-4} + ...... + a_{n-m+2} + a_{n-m}) - 2(a_{n-1} + a_{n-3} + ..... + a_{n-m+3} + a_{n-m+1})\}, \text{ for } R \ge 1$$

$$= |a_n|R^{n+1} + R\{k(|a_n| + a_n) - |a_n| + 2\delta - a_0 + 2(a_{n-2} + a_{n-4} + ...... + a_{n-m+2} + a_{n-m}) - 2(a_{n-1} + a_{n-3} + ..... + a_{n-m+3} + a_{n-m+1})\}, \text{ for } R \le 1,$$

if n and n-m are both even or odd.

If n is even and n-m is odd or if n is odd and n-m is even, then P(z) has no zero in in $|z| < \frac{|a_0|}{M'}$, where

$$M' = |a_n|R^{n+1} + R^n\{k(|a_n| + a_n) - |a_n| + 2\delta - a_0 + 2(a_{n-2} + a_{n-4} + ...... + a_{n-m+3} + a_{n-m+1}) - 2(a_{n-1} + a_{n-3} + ..... + a_{n-m+4} + a_{n-m+2})\} \text{ for } R \ge 1,$$

$$= |a_n|R^{n+1} + R\{k(|a_n| + a_n) - |a_n| + 2\delta - a_0 + 2(a_{n-2} + a_{n-4} + ...... + a_{n-m+3} + a_{n-m+1}) - 2(a_{n-1} + a_{n-3} + ..... + a_{n-m+4} + a_{n-m+2})\}, \text{ for } R \le 1.$$

Further, the number of zeros of P(z) in $\frac{|a_0|}{M} \le |z| \le \frac{R}{c}, c > 1$ does not exceed $\frac{1}{\log c}\log\frac{M_1}{|a_0|}$ in case n and n-m are both even or odd and if n is even and n-m is odd or if n is odd and n-m is even , then the number of zeros of P(z) in $\frac{|a_0|}{M'} \le |z| \le \frac{R}{c}, c > 1$ does not exceed $\frac{1}{\log c}\log\frac{M_1'}{|a_0|}$, where

$$M_1 = |a_n|R^{n+1} + |a_0| + R^n\{k(|a_n| + a_n) - |a_n| + 2\delta - a_0 + 2(a_{n-2} + a_{n-4} + ...... + a_{n-m+2} + a_{n-m}) - 2(a_{n-1} + a_{n-3} + ..... + a_{n-m+3} + a_{n-m+1})\} \text{ for } R \ge 1,$$

$$= |a_n|R^{n+1} + |a_0| + R\{k(|a_n| + a_n) - |a_n| + 2\delta - a_0 + 2(a_{n-2} + a_{n-4} + ...... + a_{n-m+2} + a_{n-m})$$
$$-2(a_{n-1} + a_{n-3} + ..... + a_{n-m+3} + a_{n-m+1})\} \text{ for } R \le 1,$$
$$M_1' = |a_n|R^{n+1} + |a_0| + R^n\{k(|a_n| + a_n) - |a_n| + 2\delta - a_0 + 2(a_{n-2} + a_{n-4} + ...... + a_{n-m+3} + a_{n-m+1})$$
$$-2(a_{n-1} + a_{n-3} + ..... + a_{n-m+4} + a_{n-m+2})\} \text{ for } R \ge 1,$$
$$= |a_n|R^{n+1} + |a_0| + R\{k(|a_n| + a_n) - |a_n| + 2\delta - a_0 + 2(a_{n-2} + a_{n-4} + ...... + a_{n-m+3} + a_{n-m+1})$$
$$-2(a_{n-1} + a_{n-3} + ..... + a_{n-m+4} + a_{n-m+2})\} \text{ for } R \le 1.$$

Taking R=1 in Theorem 1, we get the following

**Corollary 1:** Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree $n \ge m \ge 2$ with real coefficients such that for some $k \ge 1, \delta > 0$,

$$ka_n \ge a_{n-1} \le a_{n-2} \ge a_{n-3} \le a_{n-4} \ge ... \ge a_{n-m+1} \le a_{n-m} \ge a_{n-m-1} \ge ... \ge a_4 \ge a_3 \ge a_2 \ge a_1 \ge a_0 - \delta$$

if both n and n-m are even or odd
or

$$ka_n \ge a_{n-1} \le a_{n-2} \ge a_{n-3} \le a_{n-4} \ge ... \le a_{n-m+1} \ge a_{n-m} \ge a_{n-m-1} \ge ... \ge a_4 \ge a_3 \ge a_2 \ge a_1 \ge a_0 - \delta$$

if n is even and n-m is odd or if n is odd and n-m is even.

Then P(z) has no zero in $|z| < \frac{|a_0|}{M}$, where

$$M = k(|a_n| + a_n) + 2\delta - a_0 + 2(a_{n-2} + a_{n-4} + ...... + a_{n-m+2} + a_{n-m})$$
$$-2(a_{n-1} + a_{n-3} + ..... + a_{n-m+3} + a_{n-m+1})\},$$

if n and n-m are both even or odd .

If n is even and n-m is odd or if n is odd and n-m is even, then P(z) has no zero in in $|z| < \frac{|a_0|}{M'}$, where

$$M' = k(|a_n| + a_n) + 2\delta - a_0 + 2(a_{n-2} + a_{n-4} + ...... + a_{n-m+3} + a_{n-m+1})$$
$$-2(a_{n-1} + a_{n-3} + ..... + a_{n-m+4} + a_{n-m+2})\}.$$

Further, the number of zeros of P(z) in $\frac{|a_0|}{M} \le |z| \le \frac{1}{c}, c > 1$ does not exceed $\frac{1}{\log c}\log\frac{M_1}{|a_0|}$ in case n and n-m are both even or odd , and if n is even and n-m is odd or if n is odd and n-m is even , then the number of zeros of P(z) in $\frac{|a_0|}{M'} \le |z| \le \frac{1}{c}, c > 1$ does not exceed $\frac{1}{\log c}\log\frac{M_1'}{|a_0|}$, where

$$M_1 = k(|a_n| + a_n) + 2\delta + |a_0| - a_0 + 2(a_{n-2} + a_{n-4} + ...... + a_{n-m+2} + a_{n-m})$$
$$-2(a_{n-1} + a_{n-3} + ..... + a_{n-m+3} + a_{n-m+1})\},$$
$$M_1' = k(|a_n| + a_n) + 2\delta + |a_0| - a_0 + 2(a_{n-2} + a_{n-4} + ...... + a_{n-m+3} + a_{n-m+1})$$
$$-2(a_{n-1} + a_{n-3} + ..... + a_{n-m+4} + a_{n-m+2})\}.$$

**Theorem 2:** Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree $n \ge m \ge 2$ with real coefficients such that for some $0 < k \le 1, \delta > 0$,

$$ka_n \le a_{n-1} \ge a_{n-2} \le a_{n-3} \ge a_{n-4} \le ... \le a_{n-m+1} \ge a_{n-m} \ge a_{n-m-1} \ge ... \ge a_4 \ge a_3 \ge a_2 \ge a_1 \ge a_0 - \delta$$

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if both n and n-m are even or odd
or

$$ka_n \le a_{n-1} \ge a_{n-2} \le a_{n-3} \ge a_{n-4} \le ... \ge a_{n-m+1} \le a_{n-m} \ge a_{n-m-1} \ge ... \ge a_4 \ge a_3 \ge a_2 \ge a_1 \ge a_0 - \delta$$

if n is even and n-m is odd or if n is odd and n-m is even.

Then P(z) h as no zero in $|z| < \frac{|a_0|}{N}$, if n and n-m are both even or odd.,where

$$N = |a_n|R^{n+1} + R^n\{|a_n| - k(|a_n| + a_n) + 2\delta - a_0 + 2(a_{n-1} + a_{n-3} + ...... + a_{n-m+3} + a_{n-m+1})$$
$$-2(a_{n-2} + a_{n-4} + ..... + a_{n-m+4} + a_{n-m+2})\} \text{ for } R \ge 1$$
$$= |a_n|R^{n+1} + R\{|a_n| - k(|a_n| + a_n) + 2\delta - a_0 + 2(a_{n-1} + a_{n-3} + ...... + a_{n-m+3} + a_{n-m+1})$$
$$-2(a_{n-2} + a_{n-4} + ..... + a_{n-m+4} + a_{n-m+2})\} \text{ for } R \le 1.$$

If n is even and n-m is odd or if n is odd and n-m is even , then P(z) has no zero in $|z| < \frac{|a_0|}{N'}$ ,.where

$$N' = |a_n|R^{n+1} + R^n\{|a_n| - k(|a_n| + a_n) + 2\delta - a_0 + 2(a_{n-1} + a_{n-3} + ...... + a_{n-m+2} + a_{n-m})$$
$$-2(a_{n-2} + a_{n-4} + ..... + a_{n-m+4} + a_{n-m+2})\} \text{ for } R \ge 1$$
$$= |a_n|R^{n+1} + R\{|a_n| - k(|a_n| + a_n) + 2\delta - a_0 + 2(a_{n-1} + a_{n-3} + ...... + a_{n-m+2} + a_{n-m})$$
$$-2(a_{n-2} + a_{n-4} + ..... + a_{n-m+4} + a_{n-m+2})\} \text{ for } R \le 1.$$

Further, the number of zeros of P(z) in $\frac{|a_0|}{N} \le |z| \le \frac{R}{c}, c > 1$ does not exceed $\frac{1}{\log c}\log\frac{N_1}{|a_0|}$ in case n and n-m are both even or odd and if n is even and n-m is odd or if n is odd and n-m is even , then the number of zeros of P(z) in $\frac{|a_0|}{N'} \le |z| \le \frac{R}{c}, c > 1$ does not exceed $\frac{1}{\log c}\log\frac{N_1'}{|a_0|}$ , where

$$N_1 = |a_n|R^{n+1} + |a_0| + R^n\{k(|a_n| + a_n) - |a_n| + 2\delta - a_0 + 2(a_{n-1} + a_{n-3} + ...... + a_{n-m+3} + a_{n-m+1})$$
$$-2(a_{n-2} + a_{n-4} + ..... + a_{n-m+4} + a_{n-m+2})\} \text{ for } R \ge 1,$$
$$= |a_n|R^{n+1} + |a_0| + R\{k(|a_n| + a_n) - |a_n| + 2\delta - a_0 + 2(a_{n-1} + a_{n-3} + ...... + a_{n-m+3} + a_{n-m+1})$$
$$-2(a_{n-2} + a_{n-4} + ..... + a_{n-m+4} + a_{n-m+2})\} \text{ for } R \le 1,$$
$$N_1' = |a_n|R^{n+1} + |a_0| + R^n\{k(|a_n| + a_n) - |a_n| + 2\delta - a_0 + 2(a_{n-1} + a_{n-3} + ...... + a_{n-m+2} + a_{n-m})$$
$$-2(a_{n-2} + a_{n-4} + ..... + a_{n-m+3} + a_{n-m+1})\} \text{ for } R \ge 1,$$
$$= |a_n|R^{n+1} + |a_0| + R\{k(|a_n| + a_n) - |a_n| + 2\delta - a_0 + 2(a_{n-1} + a_{n-3} + ...... + a_{n-m+2} + a_{n-m})$$
$$-2(a_{n-2} + a_{n-4} + ..... + a_{n-m+3} + a_{n-m+1})\} \text{ for } R \le 1.$$

Taking R=1 $c = \frac{1}{\delta}, 0 < \delta < 1$ in Theorem 2, we get the following

**Corollary 2.** Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree $n \ge m \ge 2$ with real coefficients such that for some $0 < k \le 1, \delta > 0$,

$$ka_n \le a_{n-1} \ge a_{n-2} \le a_{n-3} \ge a_{n-4} \le ... \le a_{n-m+1} \ge a_{n-m} \ge a_{n-m-1} \ge ... \ge a_4 \ge a_3 \ge a_2 \ge a_1 \ge a_0 - \delta$$

if both n and n-m are even or odd
or

$ka_n \le a_{n-1} \ge a_{n-2} \le a_{n-3} \ge a_{n-4} \le ... \ge a_{n-m+1} \le a_{n-m} \ge a_{n-m-1} \ge ... \ge a_4 \ge a_3 \ge a_2 \ge a_1 \ge a_0 - \delta$
if n is even and n-m is odd or if n is odd and n-m is even.

Then P(z) h as no zero in $|z| < \frac{|a_0|}{N}$, if n and n-m are both even or odd.,where

$$N = 2|a_n| - k(|a_n| + a_n) + 2\delta - a_0 + 2(a_{n-1} + a_{n-3} + ...... + a_{n-m+3} + a_{n-m+1})$$
$$- 2(a_{n-2} + a_{n-4} + ..... + a_{n-m+4} + a_{n-m+2})\}.$$

If n is even and n-m is odd or if n is odd and n-m is even , then P(z) has no zero in $|z| < \frac{|a_0|}{N'}$,.where

$$N' = 2|a_n| - k(|a_n| + a_n) + 2\delta - a_0 + 2(a_{n-1} + a_{n-3} + ...... + a_{n-m+2} + a_{n-m})$$
$$- 2(a_{n-2} + a_{n-4} + ..... + a_{n-m+3} + a_{n-m+1})\}$$

Further, the number of zeros of P(z) in $\frac{|a_0|}{N} \le |z| \le \delta, 0 < \delta < 1$ does not exceed $\frac{1}{\log \frac{1}{\delta}} \log \frac{N_1}{|a_0|}$ in case n and n-m are both even or odd and if n is even and n-m is odd or if n is odd and n-m is even , then

the number of zeros of P(z) in $\frac{|a_0|}{N'} \le |z| \le \delta, 0 < \delta < 1$ does not exceed $\frac{1}{\log \frac{1}{\delta}} \log \frac{N_1'}{|a_0|}$, where

$$N_1 = k(|a_n| + a_n) + 2\delta + |a_0| - a_0 + 2(a_{n-1} + a_{n-3} + ...... + a_{n-m+3} + a_{n-m+1})$$
$$- 2(a_{n-2} + a_{n-4} + ..... + a_{n-m+4} + a_{n-m+2})\}$$
$$N_1' = k(|a_n| + a_n) + 2\delta + |a_0| - a_0 + 2(a_{n-1} + a_{n-3} + ...... + a_{n-m+2} + a_{n-m})$$
$$- 2(a_{n-2} + a_{n-4} + ..... + a_{n-m+3} + a_{n-m+1})\}.$$

**Theorem 3:** Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree $n \ge m \ge 2$ with real coefficients such that for some $k \ge 1, \delta > 0$,

$ka_n \ge a_{n-1} \le a_{n-2} \ge a_{n-3} \le a_{n-4} \ge ... \ge a_{n-m+1} \le a_{n-m} \le a_{n-m-1} \le ... \le a_4 \le a_3 \le a_2 \le a_1 \le a_0 + \delta$
if both n and n-m are even or odd
or
$ka_n \ge a_{n-1} \le a_{n-2} \ge a_{n-3} \le a_{n-4} \ge ... \le a_{n-m+1} \ge a_{n-m} \le a_{n-m-1} \le ... \le a_4 \le a_3 \le a_2 \le a_1 \le a_0 + \delta$
if n is even and n-m is odd or if n is odd and n-m is even.

Then P(z) h as no zero in $|z| < \frac{|a_0|}{L}$, if n and n-m are both even or odd.,where

$$L = |a_n| R^{n+1} + R^n \{k(|a_n| + a_n) - |a_n| + 2\delta + a_0 + 2(a_{n-2} + a_{n-4} + ...... + a_{n-m+4} + a_{n-m+2})$$
$$- 2(a_{n-1} + a_{n-3} + ..... + a_{n-m+3} + a_{n-m+1})\} \text{ for } R \ge 1$$
$$= |a_n| R^{n+1} + R\{k(|a_n| + a_n) - |a_n| + 2\delta + a_0 + 2(a_{n-2} + a_{n-4} + ...... + a_{n-m+2})$$
$$- 2(a_{n-1} + a_{n-3} + ..... + a_{n-m+3} + a_{n-m+1})\} \text{ for } R \le 1.$$

If n is even and n-m is odd or if n is odd and n-m is even , then P(z) has no zero in $|z| < \frac{|a_0|}{L'}$,.where

$$L' = |a_n| R^{n+1} + R^n \{k(|a_n| + a_n) - |a_n| + 2\delta + a_0 + 2(a_{n-2} + a_{n-4} + ..... + a_{n-m+4} + a_{n-m+2})$$

$-2(a_{n-1}+a_{n-3}+......+a_{n-m+3}+a_{n-m+1})\}$ for $R \geq 1$

$=|a_n|R^{n+1}+R\{k(|a_n|+a_n)-|a_n|+2\delta+a_0+2(a_{n-2}+a_{n-4}+.....+a_{n-m+4}+a_{n-m+2})$

$-2(a_{n-1}+a_{n-3}+......+a_{n-m+3}+a_{n-m+1})\}$ for $R \leq 1$.

Further, the number of zeros of P(z) in $\frac{|a_0|}{L} \leq |z| \leq \frac{R}{c}, c>1$ does not exceed $\frac{1}{\log c}\log\frac{L_1}{|a_0|}$ in case n and n-m are both even or odd and if n is even and n-m is odd or if n is odd and n-m is even , then the number of zeros of P(z) in $\frac{|a_0|}{L'} \leq |z| \leq \frac{R}{c}, c>1$ does not exceed $\frac{1}{\log c}\log\frac{L_1'}{|a_0|}$, where

$L_1=|a_n|R^{n+1}+|a_0|+R^n\{k(|a_n|+a_n)-|a_n|+2\delta+a_0+2(a_{n-2}+a_{n-4}+......+a_{n-m+4}+a_{n-m+2})$

$-2(a_{n-1}+a_{n-3}+.....+a_{n-m+3}+a_{n-m+1})\}$ for $R \geq 1$,

$=|a_n|R^{n+1}+|a_0|+R\{k(|a_n|+a_n)-|a_n|+2\delta+a_0+2(a_{n-2}+a_{n-4}+......+a_{n-m+2}+a_{n-m+2})$

$-2(a_{n-1}+a_{n-3}+.....+a_{n-m+3}+a_{n-m+1})\}$ for $R \leq 1$,

$L_1'=|a_n|R^{n+1}+|a_0|+R^n\{k(|a_n|+a_n)-|a_n|+2\delta+a_0+2(a_{n-2}+a_{n-4}+......+a_{n-m+3}+a_{n-m+1})$

$-2(a_{n-1}+a_{n-3}+.....+a_{n-m+2}+a_{n-m})\}$ for $R \geq 1$,

$=|a_n|R^{n+1}+|a_0|+R\{k(|a_n|+a_n)-|a_n|+2\delta+a_0+2(a_{n-2}+a_{n-4}+......+a_{n-m+3}+a_{n-m+1})$

$-2(a_{n-1}+a_{n-3}+.....+a_{n-m+4}+a_{n-m+2})\}$ for $R \leq 1$.

Taking k=1, $\delta=0$ in Theorem 3, we get the following

**Corollary 3 :** Let $P(z)=\sum_{j=0}^{n}a_j z^j$ be a polynomial of degree $n \geq m \geq 2$ with real coefficients such that $a_n \geq a_{n-1} \leq a_{n-2} \geq a_{n-3} \leq a_{n-4} \geq ... \geq a_{n-m+1} \leq a_{n-m} \leq a_{n-m-1} \leq ... \leq a_4 \leq a_3 \leq a_2 \leq a_1 \leq a_0$
if both n and n-m are even or odd
or
$a_n \geq a_{n-1} \leq a_{n-2} \geq a_{n-3} \leq a_{n-4} \geq ... \leq a_{n-m+1} \geq a_{n-m} \leq a_{n-m-1} \leq ... \leq a_4 \leq a_3 \leq a_2 \leq a_1 \leq a_0$
if n is even and n-m is odd or if n is odd and n-m is even.

Then P(z) h as no zero in $|z| < \frac{|a_0|}{L}$, if n and n-m are both even or odd.,where

$L=|a_n|R^{n+1}+R^n\{a_n+a_0+2(a_{n-1}+a_{n-3}+......+a_{n-m+3}+a_{n-m+1})$

$-2(a_{n-1}+a_{n-3}+......+a_{n-m+3}+a_{n-m+1})\}$ for $R \geq 1$

$=|a_n|R^{n+1}+R\{a_n+a_0+2(a_{n-2}+a_{n-4}+.....+a_{n-m+4}+a_{n-m+2})$

$-2(a_{n-1}+a_{n-3}+......+a_{n-m+3}+a_{n-m+1})\}$ for $R \leq 1$.

If n is even and n-m is odd or if n is odd and n-m is even , then P(z) has no zero in $|z| < \frac{|a_0|}{L'}$ ,.where

$L'=|a_n|R^{n+1}+R^n\{a_n+a_0+2(a_{n-2}+a_{n-4}+.....+a_{n-m+3}+a_{n-m+1})$

$-2(a_{n-1}+a_{n-3}+......+a_{n-m+2}+a_{n-m})\}$ for $R \geq 1$

$=|a_n|R^{n+1}+R\{a_n+a_0+2(a_{n-1}+a_{n-3}+......+a_{n-m+3}+a_{n-m+1})$

$-2(a_{n-1}+a_{n-3}+......+a_{n-m+2}+a_{n-m})\}$ for $R \leq 1$.

Further, the number of zeros of P(z) in $\frac{|a_0|}{L} \le |z| \le \frac{R}{c}, c > 1$ does not exceed $\frac{1}{\log c}\log\frac{L_1}{|a_0|}$ in case n and n-m are both even or odd and if n is even and n-m is odd or if n is odd and n-m is even , then the number of zeros of P(z) in $\frac{|a_0|}{L'} \le |z| \le \frac{R}{c}, c > 1$ does not exceed $\frac{1}{\log c}\log\frac{L_1'}{|a_0|}$, where

$$L_1 = |a_n|R^{n+1} + |a_0| + R^n\{a_n + a_0 + 2(a_{n-2} + a_{n-4} + ...... + a_{n-m+4} + a_{n-m+2})$$
$$-2(a_{n-1} + a_{n-3} + ..... + a_{n-m+3} + a_{n-m+1})\} \text{ for } R \ge 1,$$
$$= |a_n|R^{n+1} + |a_0| + R\{a_n + a_0 + 2(a_{n-2} + a_{n-4} + ...... + a_{n-m+4} + a_{n-m+2})$$
$$-2(a_{n-1} + a_{n-3} + ..... + a_{n-m+3} + a_{n-m+1})\} \text{ for } R \le 1,$$
$$L_1' = |a_n|R^{n+1} + |a_0| + R^n\{a_n + a_0 + 2(a_{n-2} + a_{n-4} + ...... + a_{n-m+3} + a_{n-m+1})$$
$$-2(a_{n-1} + a_{n-3} + ..... + a_{n-m+2} + a_{n-m})\} \text{ for } R \ge 1,$$
$$= |a_n|R^{n+1} + |a_0| + R\{a_n + a_0 + 2(a_{n-2} + a_{n-4} + ...... + a_{n-m+3} + a_{n-m+1})$$
$$-2(a_{n-1} + a_{n-3} + ..... + a_{n-m+2} + a_{n-m})\} \text{ for } R \le 1.$$

**Theorem 4:** Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree $n \ge m \ge 2$ with real coefficients such that for some $0 < k \le 1, \delta > 0$,

$$ka_n \le a_{n-1} \ge a_{n-2} \le a_{n-3} \ge a_{n-4} \le ... \le a_{n-m+1} \ge a_{n-m} \le a_{n-m-1} \le ... \le a_4 \le a_3 \le a_2 \le a_1 \le a_0 + \delta$$

if both n and n-m are even or odd
or

$$ka_n \le a_{n-1} \ge a_{n-2} \le a_{n-3} \ge a_{n-4} \le ... \ge a_{n-m+1} \le a_{n-m} \le a_{n-m-1} \le ... \le a_4 \le a_3 \le a_2 \le a_1 \le a_0 + \delta$$

if n is even and n-m is odd or if n is odd and n-m is even.

Then P(z) h as no zero in $|z| < \frac{|a_0|}{D}$, if n and n-m are both even or odd.,where

$$D = |a_n|R^{n+1} + R^n\{|a_n| - k(|a_n| + a_n) + 2\delta + a_0 + 2(a_{n-1} + a_{n-3} + ...... + a_{n-m+3} + a_{n-m+1})$$
$$-2(a_{n-2} + a_{n-4} + ..... + a_{n-m+2} + a_{n-m})\} \text{ for } R \ge 1$$
$$= |a_n|R^{n+1} + R\{|a_n| - k(|a_n| + a_n) + 2\delta + a_0 + 2(a_{n-1} + a_{n-3} + ...... + a_{n-m+3} + a_{n-m+1})$$
$$-2(a_{n-2} + a_{n-4} + ..... + a_{n-m+2} + a_{n-m})\} \text{ for } R \le 1.$$

If n is even and n-m is odd or if n is odd and n-m is even , then P(z) has no zero in $|z| < \frac{|a_0|}{D'}$,.where

$$D' = |a_n|R^{n+1} + R^n\{|a_n| - k(|a_n| + a_n) + 2\delta + a_0 + 2(a_{n-1} + a_{n-3} + ...... + a_{n-m+4} + a_{n-m+2})$$
$$-2(a_{n-2} + a_{n-4} + ..... + a_{n-m+3} + a_{n-m+1})\} \text{ for } R \ge 1$$
$$= |a_n|R^{n+1} + R\{|a_n| - k(|a_n| + a_n) + 2\delta + a_0 + 2(a_{n-1} + a_{n-3} + ...... + a_{n-m+4} + a_{n-m+2})$$
$$-2(a_{n-2} + a_{n-4} + ..... + a_{n-m+3} + a_{n-m+1})\} \text{ for } R \le 1.$$

Further, the number of zeros of P(z) in $\frac{|a_0|}{D} \le |z| \le \frac{R}{c}, c > 1$ does not exceed $\frac{1}{\log c}\log\frac{D_1}{|a_0|}$ in case n and n-m are both even or odd and if n is even and n-m is odd or if n is odd and n-m is even , then the number of zeros of P(z) in $\frac{|a_0|}{D'} \le |z| \le \frac{R}{c}, c > 1$ does not exceed $\frac{1}{\log c}\log\frac{D_1'}{|a_0|}$, where

$$D_1 = |a_n|R^{n+1} + |a_0| + R^n\{|a_n| - k(|a_n| + a_n) + 2\delta + a_0 + 2(a_{n-1} + a_{n-3} + ...... + a_{n-m+3} + a_{n-m+1})$$
$$- 2(a_{n-2} + a_{n-4} + ..... + a_{n-m+2} + a_{n-m})\} \text{ for } R \ge 1$$
$$= |a_n|R^{n+1} + |a_0| + R\{|a_n| - k(|a_n| + a_n) + 2\delta + a_0 + 2(a_{n-1} + a_{n-3} + ...... + a_{n-m+3} + a_{n-m+1})$$
$$- 2(a_{n-2} + a_{n-4} + ..... + a_{n-m+2} + a_{n-m})\} \text{ for } R \le 1.$$
$$D_1' = |a_n|R^{n+1} + |a_0| + R^n\{|a_n| - k(|a_n| + a_n) + 2\delta + a_0 + 2(a_{n-1} + a_{n-3} + ...... + a_{n-m+4} + a_{n-m+2})$$
$$- 2(a_{n-2} + a_{n-4} + ..... + a_{n-m+3} + a_{n-m+1})\} \text{ for } R \ge 1$$
$$= |a_n|R^{n+1} + |a_0| + R\{|a_n| - k(|a_n| + a_n) + 2\delta + a_0 + 2(a_{n-1} + a_{n-3} + ...... + a_{n-m+4} + a_{n-m+2})$$
$$- 2(a_{n-2} + a_{n-4} + ..... + a_{n-m+3} + a_{n-m+1})\} \text{ for } R \le 1.$$

Taking k=1, $\delta = 0$ in Theorem 4, we get the following

**Corollary 4:** Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree $n \ge m \ge 2$ with real coefficients such that

$$a_n \le a_{n-1} \ge a_{n-2} \le a_{n-3} \ge a_{n-4} \le ... \le a_{n-m+1} \ge a_{n-m} \le a_{n-m-1} \le ... \le a_4 \le a_3 \le a_2 \le a_1 \le a_0$$

if both n and n-m are even or odd

or

$$a_n \le a_{n-1} \ge a_{n-2} \le a_{n-3} \ge a_{n-4} \le ... \ge a_{n-m+1} \le a_{n-m} \le a_{n-m-1} \le ... \le a_4 \le a_3 \le a_2 \le a_1 \le a_0$$

if n is even and n-m is odd or if n is odd and n-m is even.

Then P(z) h as no zero in $|z| < \frac{|a_0|}{D}$, if n and n-m are both even or odd., where

$$D = |a_n|R^{n+1} + R^n\{-a_n + a_0 + 2(a_{n-1} + a_{n-3} + ...... + a_{n-m+3} + a_{n-m+1})$$
$$- 2(a_{n-2} + a_{n-4} + ..... + a_{n-m+2} + a_{n-m})\} \text{ for } R \ge 1$$
$$= |a_n|R^{n+1} + R\{-a_n + a_0 + 2(a_{n-1} + a_{n-3} + ...... + a_{n-m+3} + a_{n-m+1})$$
$$- 2(a_{n-2} + a_{n-4} + ..... + a_{n-m+2} + a_{n-m})\} \text{ for } R \le 1.$$

If n is even and n-m is odd or if n is odd and n-m is even , then P(z) has no zero in $|z| < \frac{|a_0|}{D'}$ ,.where

$$D' = |a_n|R^{n+1} + R^n\{-a_n + a_0 + 2(a_{n-1} + a_{n-3} + ...... + a_{n-m+4} + a_{n-m+2})$$
$$- 2(a_{n-2} + a_{n-4} + ..... + a_{n-m+3} + a_{n-m+1})\} \text{ for } R \ge 1$$
$$= |a_n|R^{n+1} + R\{-a_n + a_0 + 2(a_{n-1} + a_{n-3} + ...... + a_{n-m+4} + a_{n-m+2})$$
$$- 2(a_{n-2} + a_{n-4} + ..... + a_{n-m+3} + a_{n-m+1})\} \text{ for } R \le 1.$$

Further, the number of zeros of P(z) in $\frac{|a_0|}{D} \le |z| \le \frac{R}{c}, c > 1$ does not exceed $\frac{1}{\log c}\log\frac{D_1}{|a_0|}$ in case n and n-m are both even or odd and if n is even and n-m is odd or if n is odd and n-m is even , then the number

of zeros of P(z) in $\frac{|a_0|}{D'} \le |z| \le \frac{R}{c}, c > 1$ does not exceed $\frac{1}{\log c}\log\frac{D_1'}{|a_0|}$, where

$$D_1 = |a_n|R^{n+1} + |a_0| + R^n\{-a_n + a_0 + 2(a_{n-1} + a_{n-3} + ...... + a_{n-m+3} + a_{n-m+1})$$

$-2(a_{n-2}+a_{n-4}+.....+a_{n-m+2}+a_{n-m})\}$ for $R\geq 1$,

$=|a_n|R^{n+1}+|a_0|+R\{-a_n+a_0+2(a_{n-1}+a_{n-3}+......+a_{n-m+3}+a_{n-m+1})$

$-2(a_{n-2}+a_{n-4}+.....+a_{n-m+2}+a_{n-m})\}$ for $R\leq 1$,

$D_1' =|a_n|R^{n+1}+|a_0|+R^n\{-a_n+a_0+2(a_{n-1}+a_{n-3}+......+a_{n-m+4}+a_{n-m+2})$

$-2(a_{n-2}+a_{n-4}+.....+a_{n-m+3}+a_{n-m+1})\}$ for $R\geq 1$,

$=|a_n|R^{n+1}+|a_0|+R\{-a_n+a_0+2(a_{n-1}+a_{n-3}+......+a_{n-m+4}+a_{n-m+2})$

$-2(a_{n-2}+a_{n-4}+.....+a_{n-m+3}+a_{n-m+1})\}$ for $R\leq 1$.

Many other results can also be obtained from the above results for different values of the parameters.

## 3. lemmas

**Lemma 1:** Let f(z) (not identically zero) be analytic for $|z|\leq R,\ f(0)\neq 0$ and $f(a_k)=0,$ $k=1,2,......,n$. Then

$$\frac{1}{2\pi}\int_0^{2\pi}\log\left|f(\mathrm{Re}^{i\theta}\right|d\theta-\log|f(0)|=\sum_{j=1}^{n}\log\frac{R}{|a_j|}.$$

Lemma 1 is the famous Jensen's Theorem (see page 208 of [1]).

**Lemma 2:** Let f (z) be analytic for $|z|\leq R,\ f(0)\neq 0$ and $|f(z)|\leq M$ for $|z|\leq R$. Then the number of zeros of f(z) in $|z|\leq \frac{R}{c}, c>1$ does not exceed $\frac{1}{\log c}\log\frac{M}{|f(0)|}$.

Lemma 2 is a simple deduction from Lemma 2.

## 4. Proofs of Theorems

**Proof of Theorem 1.**

Consider the polynomial

F(z)=(1-z)P(z)

$=(1-z)(a_nz^n+a_{n-1}z^{n-1}+a_{n-2}z^{n-2}+.......+a_{n-m}z^{n-m}+......+a_3z^3+a_2z^2+a_1z+a_0)$

$=-a_nz^{n+1}-(k-1)a_nz^n+(ka_n-a_{n-1})z^n+(a_{n-1}-a_{n-2})z^{n-1}+......+(a_{n-m+1}-a_{n-m})z^{n-m+1}$

$+(a_{n-m}-a_{n-m-1})z^{n-m}+(a_{n-m-1}-a_{n-m-2})z^{n-m-1}+.......+(a_3-a_2)z^3+(a_2-a_1)z^2$

$+(a_1-a_0)z+a_0$

$=G(z)+a_0$, where

$G(z)=-a_nz^{n+1}-(k-1)a_nz^n+(ka_n-a_{n-1})z^n+(a_{n-1}-a_{n-2})z^{n-1}+......+(a_{n-m+1}-a_{n-m})z^{n-m+1}$

$+(a_{n-m}-a_{n-m-1})z^{n-m}+(a_{n-m-1}-a_{n-m-2})z^{n-m-1}+.......+(a_3-a_2)z^3+(a_2-a_1)z^2$

$+(a_1-a_0)z.$

For $|z|\leq R$, we have , by using the hypothesis, for both n and n-m even or odd,

$|G(z)|\leq|a_n||z|^{n+1}+(k-1)|a_n||z|^n+|ka_n-a_{n-1}||z|^n+|a_{n-1}-a_{n-2}||z|^{n-1}+......+|a_{n-m+1}-a_{n-m}||z|^{n-m+1}$

$+|a_{n-m}-a_{n-m-1}||z|^{n-m}+|a_{n-m-1}-a_{n-m-2}||z|^{n-m-1}+.......+|a_3-a_2||z|^3+|a_2-a_1||z|^2$

$+|a_1-a_0||z|$

$\leq|a_n|R^{n+1}+(k-1)|a_n|R^n+|ka_n-a_{n-1}|R^n+|a_{n-1}-a_{n-2}|R^{n-1}+......+|a_{n-m+1}-a_{n-m}|R^{n-m+1}$

$$+|a_{n-m}-a_{n-m-1}|R^{n-m}+|a_{n-m-1}-a_{n-m-2}|R^{n-m-1}+.......+|a_3-a_2|R^3+|a_2-a_1|R^2$$
$$+|a_1-a_0|R$$
$$\leq |a_n|R^{n+1}+R^n\{(k-1)|a_n|+|ka_n-a_{n-1}|+|a_{n-1}-a_{n-2}|+......+|a_{n-m+1}-a_{n-m}|$$
$$+|a_{n-m}-a_{n-m-1}|+|a_{n-m-1}-a_{n-m-2}|+.......+|a_3-a_2|+|a_2-a_1|$$
$$+|a_1-\delta-a_0+\delta|\}$$
$$\leq |a_n|R^{n+1}+R^n\{(k-1)|a_n|+(ka_n-a_{n-1})+(a_{n-2}-a_{n-1})+......+(a_{n-m}-a_{n-m+1})$$
$$+(a_{n-m}-a_{n-m-1})+......+(a_3-a_2)+(a_2-a_1)+(a_1+\delta-a_0)+\delta\}$$
$$\leq |a_n|R^{n+1}+R^n\{k(|a_n|+a_n)-|a_n|+2\delta-a_0+2(a_{n-2}+a_{n-4}+......a_{n-m+2}+a_{n-m})$$
$$-2(a_{n-1}+a_{n-3}+.....+a_{n-m+3}+a_{n-m+1})\}$$

for $R\geq 1$ and

$$|G(z)|\leq |a_n|R^{n+1}+R\{k(|a_n|+a_n)-|a_n|+2\delta-a_0+2(a_{n-2}+a_{n-4}+......+a_{n-m+2}+a_{n-m})$$
$$-2(a_{n-1}+a_{n-3}+.....+a_{n-m+3}+a_{n-m+1})\}$$

for $R\leq 1$.

Since G(z) is analytic for $|z|\leq R, G(0)=0$, it follows by Schwarz lemma that

$|G(z)|\leq M|z|$ for $|z|\leq R$, where

$$M=|a_n|R^{n+1}+R^n\{k(|a_n|+a_n)-|a_n|+2\delta-a_0+2(a_{n-2}+a_{n-4}+......+a_{n-m+2}+a_{n-m})$$
$$-2(a_{n-1}+a_{n-3}+.....+a_{n-m+3}+a_{n-m+1})\}, \text{ for } R\geq 1$$
$$=|a_n|R^{n+1}+R\{k(|a_n|+a_n)-|a_n|+2\delta-a_0+2(a_{n-2}+a_{n-4}+......+a_{n-m+2}+a_{n-m})$$
$$-2(a_{n-1}+a_{n-3}+.....+a_{n-m+3}+a_{n-m+1})\}, \text{ for } R\leq 1.$$

Therefore , for $|z|\leq R$,

$$|F(z)|=|G(z)+a_0|$$
$$\geq |a_0|-|G(z)|$$
$$\geq |a_0|-M|z|$$
$$>0$$

if $|z|<\dfrac{|a_0|}{M}$

In other words, F(z) h as no zero in $|z|<\dfrac{|a_0|}{M}$.

Since the zeros of P(z) are also the zeros of F(z), it follows that P(z) h as no zero in $|z|<\dfrac{|a_0|}{M}$ if n and n-m are both even or odd.

Similarly, if n is even and n-m is odd or if n is odd and n-m is even , then rearranging the terms we can show that P(z) has no zero in $|z|<\dfrac{|a_0|}{M'}$.

Again , for $|z|\leq R$ and for n and n-m both even or odd,

$$|F(z)|\leq |a_n|R^{n+1}+|a_0|+R^n\{k(|a_n|+a_n)-|a_n|+2\delta-a_0+2(a_{n-2}+a_{n-4}+......+a_{n-m+2}+a_{n-m})$$

$-2(a_{n-1}+a_{n-3}+.....+a_{n-m+3}+a_{n-m+1})\}$

for $R\geq 1$ and

$|F(z)|\leq |a_n|R^{n+1}+|a_0|+R\{k(|a_n|+a_n)-|a_n|+2\delta-a_0+2(a_{n-2}+a_{n-4}+......+a_{n-m+2}+a_{n-m})$

$-2(a_{n-1}+a_{n-3}+.....+a_{n-m+3}+a_{n-m+1})\}$

for $R\leq 1$.

Hence, by Lemma 2, the number of zeros of F(z) and hence P(z) in $|z|\leq \frac{R}{c}, c>1$ does not exceed

$\frac{1}{\log c}\log\frac{M_1}{|a_0|}$ in case n and n-m are both even or odd, where

$M_1=|a_n|R^{n+1}+|a_0|+R^n\{k(|a_n|+a_n)-|a_n|+2\delta-a_0+2(a_{n-2}+a_{n-4}+......+a_{n-m+2}+a_{n-m})$

$-2(a_{n-1}+a_{n-3}+.....+a_{n-m+3}+a_{n-m+1})\}$ for $R\geq 1$,

$=|a_n|R^{n+1}+|a_0|+R\{k(|a_n|+a_n)-|a_n|+2\delta-a_0+2(a_{n-2}+a_{n-4}+......+a_{n-m+2}+a_{n-m})$

$-2(a_{n-1}+a_{n-3}+.....+a_{n-m+3}+a_{n-m+1})\}$ for $R\leq 1$,

Since P(z) has no zero in $|z|<\frac{|a_0|}{M}$ if n and n-m are both even or odd, it follows that the number of zeros of P(z) in $\frac{|a_0|}{M}\leq |z|\leq \frac{R}{c}, c>1$ does not exceed $\frac{1}{\log c}\log\frac{M_1}{|a_0|}$ in case n and n-m are both even or odd.

The case of n even and n-m odd or n odd and n-m even is similar.

**Proof of Theorem 2.**

As in the proof of Theorem 1,we have, for the functions F(z) and G(z), for both n and n-m even or odd and for $|z|\leq R, R\geq 1$,

$|G(z)|\leq |a_n||z|^{n+1}+(1-k)|a_n||z|^n+|ka_n-a_{n-1}||z|^n+|a_{n-1}-a_{n-2}||z|^{n-1}+......+|a_{n-m+1}-a_{n-m}||z|^{n-m+1}$

$+|a_{n-m}-a_{n-m-1}||z|^{n-m}+|a_{n-m-1}-a_{n-m-2}||z|^{n-m-1}+.......+|a_3-a_2||z|^3+|a_2-a_1||z|^2$

$+|a_1-a_0||z|$

$\leq |a_n|R^{n+1}+(1-k)|a_n|R^n+|ka_n-a_{n-1}|R^n+|a_{n-1}-a_{n-2}|R^{n-1}+......+|a_{n-m+1}-a_{n-m}|R^{n-m+1}$

$+|a_{n-m}-a_{n-m-1}|R^{n-m}+|a_{n-m-1}-a_{n-m-2}|R^{n-m-1}+.......+|a_3-a_2|R^3+|a_2-a_1|R^2$

$+|a_1-a_0|R$

$\leq |a_n|R^{n+1}+R^n\{(1-k)|a_n|+|ka_n-a_{n-1}|+|a_{n-1}-a_{n-2}|+......+|a_{n-m+1}-a_{n-m}|$

$+|a_{n-m}-a_{n-m-1}|+|a_{n-m-1}-a_{n-m-2}|+.......+|a_3-a_2|+|a_2-a_1|$

$+|a_1-\delta-a_0+\delta|\}$

$\leq |a_n|R^{n+1}+R^n\{(1-k)|a_n|+(a_{n-1}-ka_n)+(a_{n-1}-a_{n-2})+......+(a_{n-m+1}-a_{n-m})$

$+(a_{n-m}-a_{n-m-1})+......(a_3-a_2)+(a_2-a_1)+(a_1+\delta-a_0)+\delta\}$

$\leq |a_n|R^{n+1}+R^n\{|a_n|-k(|a_n|+a_n)+2\delta-a_0+2(a_{n-1}+a_{n-3}+......+a_{n-m+3}+a_{n-m+1})$

$-2(a_{n-2}+a_{n-4}+.....+a_{n-m+4}+a_{n-m+2})\}$

and for $|z|\leq R, R\geq 1$,

$|G(z)|\leq |a_n|R^{n+1}+R\{|a_n|-k(|a_n|+a_n)+2\delta-a_0+2(a_{n-1}+a_{n-3}+......+a_{n-m+3}+a_{n-m+1})$

$-2(a_{n-2}+a_{n-4}+.....+a_{n-m+4}+a_{n-m+2})\}$

Since G(z) is analytic for $|z|\le R, G(0)=0$, it follows by Schwarz lemma that

$|G(z)|\le N|z|$ for $|z|\le R$, where

$$N=|a_n|R^{n+1}+R^n\{|a_n|-k(|a_n|+a_n)+2\delta-a_0+2(a_{n-1}+a_{n-3}+......+a_{n-m+3}+a_{n-m+1})$$
$$-2(a_{n-2}+a_{n-4}+.....+a_{n-m+4}+a_{n-m+2})\} \text{ for } R\ge 1$$
$$=|a_n|R^{n+1}+R^n\{|a_n|-k(|a_n|+a_n)+2\delta-a_0+2(a_{n-1}+a_{n-3}+......+a_{n-m+3}+a_{n-m+1})$$
$$-2(a_{n-2}+a_{n-4}+.....+a_{n-m+4}+a_{n-m+2})\} \text{ for } R\le 1.$$

Therefore, for $|z|\le R$,

$$|F(z)|=|G(z)+a_0|$$
$$\ge |a_0|-|G(z)|$$
$$\ge |a_0|-N|z|$$
$$>0$$

if $|z|<\dfrac{|a_0|}{N}$.

In other words, F(z) h as no zero in $|z|<\dfrac{|a_0|}{N}$

Since the zeros of P(z) are also the zeros of F(z), it follows that P(z) h as no zero in $|z|<\dfrac{|a_0|}{N}$ if n and n-m are both even or odd.

Similarly, if n is even and n-m is odd or if n is odd and n-m is even , then rearranging the terms we can show that P(z) has no zero in $|z|<\dfrac{|a_0|}{N'}$.

Again , for $|z|\le R$ and for n and n-m both even or odd,

$$|F(z)|\le |a_n|R^{n+1}+|a_0|+R^n\{k(|a_n|+a_n)-|a_n|+2\delta-a_0+2(a_{n-2}+a_{n-4}+......+a_{n-m+2})$$
$$-2(a_{n-1}+a_{n-3}+.....+a_{n-m+3}+a_{n-m+1})\}$$

for $R\ge 1$ and

$$|F(z)|\le |a_n|R^{n+1}+|a_0|+R\{k(|a_n|+a_n)-|a_n|+2\delta-a_0+2(a_{n-2}+a_{n-4}+......+a_{n-m+2})$$
$$-2(a_{n-1}+a_{n-3}+.....+a_{n-m+3}+a_{n-m+1})\}$$

for $R\le 1$

Hence, by Lemma 2, the number of zeros of F(z) and hence P(z) in $|z|\le \dfrac{R}{c}, c>1$ does not exceed

$\dfrac{1}{\log c}\log\dfrac{N_1}{|a_0|}$ ,where

$$N_1=|a_n|R^{n+1}+|a_0|+R^n\{k(|a_n|+a_n)-|a_n|+2\delta-a_0+2(a_{n-2}+a_{n-4}+......+a_{n-m+2})$$
$$-2(a_{n-1}+a_{n-3}+.....+a_{n-m+3}+a_{n-m+1})\} \text{ for } R\ge 1,$$
$$=|a_n|R^{n+1}+|a_0|+R\{k(|a_n|+a_n)-|a_n|+2\delta-a_0+2(a_{n-2}+a_{n-4}+......+a_{n-m+2})$$

$-2(a_{n-1}+a_{n-3}+.....+a_{n-m+3}+a_{n-m+1})\}$ for $R\leq 1$,

Since P(z) has no zero in $|z|<\frac{|a_0|}{N}$ if n and n-m are both even or odd, it follows that the number of zeros of P(z) in $\frac{|a_0|}{N}\leq |z|\leq \frac{R}{c}, c>1$ does not exceed $\frac{1}{\log c}\log\frac{N_1}{|a_0|}$ if n and n-m are both even or odd.

The case of n even and n-m odd or n odd and n-m even is similar.

**Proof of Theorem 3.**

As in the proof of Theorem 1,we have, for the functions F(z) and G(z), for both n and n-m even or odd and for $|z|\leq R, R\geq 1$,

$$|G(z)|\leq |a_n||z|^{n+1}+(k-1)|a_n||z|^n+|ka_n-a_{n-1}||z|^n+|a_{n-1}-a_{n-2}||z|^{n-1}+......+|a_{n-m+1}-a_{n-m}||z|^{n-m+1}$$
$$+|a_{n-m}-a_{n-m-1}||z|^{n-m}+|a_{n-m-1}-a_{n-m-2}||z|^{n-m-1}+.......+|a_3-a_2||z|^3+|a_2-a_1||z|^2$$
$$+|a_1-a_0||z|$$
$$\leq |a_n|R^{n+1}+(k-1)|a_n|R^n+|ka_n-a_{n-1}|R^n+|a_{n-1}-a_{n-2}|R^{n-1}+......+|a_{n-m+1}-a_{n-m}|R^{n-m+1}$$
$$+|a_{n-m}-a_{n-m-1}|R^{n-m}+|a_{n-m-1}-a_{n-m-2}|R^{n-m-1}+.......+|a_3-a_2|R^3+|a_2-a_1|R^2$$
$$+|a_1-a_0|R$$
$$\leq |a_n|R^{n+1}+R^n\{(k-1)|a_n|+|ka_n-a_{n-1}|+|a_{n-1}-a_{n-2}|+......+|a_{n-m+1}-a_{n-m}|$$
$$+|a_{n-m}-a_{n-m-1}|+|a_{n-m-1}-a_{n-m-2}|+.......+|a_3-a_2|+|a_2-a_1|$$
$$+|a_1-\delta-a_0+\delta|\}$$
$$\leq |a_n|R^{n+1}+R^n\{(k-1)|a_n|+(ka_n-a_{n-1})+(a_{n-2}-a_{n-1})+......+(a_{n-m}-a_{n-m+1})$$
$$+(a_{n-m-1}-a_{n-m})+......+(a_2-a_3)+(a_1-a_2)+(a_0-a_1+\delta)+\delta\}$$
$$\leq |a_n|R^{n+1}+R^n\{k(|a_n|+a_n)-|a_n|+2\delta+a_0+2(a_{n-2}+a_{n-4}+......+a_{n-m+4}+a_{n-m+2})$$
$$-2(a_{n-1}+a_{n-3}+.....+a_{n-m+3}+a_{n-m+1})\}$$

and for $|z|\leq R, R\leq 1$,

$$|G(z)|\leq |a_n|R^{n+1}+R\{k(|a_n|+a_n)-|a_n|+2\delta+a_0+2(a_{n-2}+a_{n-4}+......+a_{n-m+4}+a_{n-m+2})$$
$$-2(a_{n-1}+a_{n-3}+.....+a_{n-m+3}+a_{n-m+1})\}.$$

Since G(z) is analytic for $|z|\leq R, G(0)=0$, it follows by Schwarz lemma that

$|G(z)|\leq L|z|$ for $|z|\leq R$, where

$$L=|a_n|R^{n+1}+R^n\{k(|a_n|+a_n)-|a_n|+2\delta+a_0+2(a_{n-2}+a_{n-4}+......+a_{n-m+4}+a_{n-m+2})$$
$$-2(a_{n-1}+a_{n-3}+.....+a_{n-m+3}+a_{n-m+1})\} \text{ for } R\geq 1$$
$$=|a_n|R^{n+1}+R\{k(|a_n|+a_n)-|a_n|+2\delta+a_0+2(a_{n-2}+a_{n-4}+......+a_{n-m+2})$$
$$-2(a_{n-1}+a_{n-3}+.....+a_{n-m+3}+a_{n-m+1})\} \text{ for } R\leq 1.$$

Therefore, for $|z|\leq R$,

$$|F(z)|=|G(z)+a_0|$$

$$\geq |a_0| - |G(z)|$$
$$\geq |a_0| - L|z|$$
$$> 0$$

if $|z| < \frac{|a_0|}{L}$.

In other words, F(z) h as no zero in $|z| < \frac{|a_0|}{L}$.

Since the zeros of P(z) are also the zeros of F(z), it follows that P(z) h as no zero in $|z| < \frac{|a_0|}{L}$ if n and n-m are both even or odd.
Similarly, if n is even and n-m is odd or if n is odd and n-m is even , then rearranging the terms we can show that P(z) has no zero in $|z| < \frac{|a_0|}{L'}$.

Again , for $|z| \leq R$ and for n and n-m both even or odd,

$$|F(z)| \leq |a_n| R^{n+1} + |a_0| + R^n \{k(|a_n| + a_n) - |a_n| + 2\delta + a_0 + 2(a_{n-2} + a_{n-4} + ...... + a_{n-m+4} + a_{n-m+2}) - 2(a_{n-1} + a_{n-3} + ..... + a_{n-m+3} + a_{n-m+1})\} \text{ for } R \geq 1$$

and

$$|F(z)| \leq |a_n| R^{n+1} + |a_0| + R\{k(|a_n| + a_n) - |a_n| + 2\delta + a_0 + 2(a_{n-2} + a_{n-4} + ...... + a_{n-m+4} + a_{n-m+2}) - 2(a_{n-1} + a_{n-3} + ..... + a_{n-m+3} + a_{n-m+1})\} \text{ for } R \leq 1.$$

Hence, by Lemma 2, the number of zeros of F(z) and hence P(z) in $|z| \leq \frac{R}{c}, c > 1$ does not exceed

$\frac{1}{\log c} \log \frac{L_1}{|a_0|}$ ,where

$$L_1 = |a_n| R^{n+1} + |a_0| + R^n \{k(|a_n| + a_n) - |a_n| + 2\delta + a_0 + 2(a_{n-2} + a_{n-4} + ...... + a_{n-m+4} + a_{n-m+2}) - 2(a_{n-1} + a_{n-3} + ..... + a_{n-m+3} + a_{n-m+1})\} \text{ for } R \geq 1,$$
$$= |a_n| R^{n+1} + |a_0| + R\{k(|a_n| + a_n) - |a_n| + 2\delta + a_0 + 2(a_{n-2} + a_{n-4} + ...... + a_{n-m+2}) - 2(a_{n-1} + a_{n-3} + ..... + a_{n-m+3} + a_{n-m+1})\} \text{ for } R \leq 1.$$

Since P(z) has no zero in $|z| < \frac{|a_0|}{L}$ if n and n-m are both even or odd, it follows that the number of zeros of P(z) in $\frac{|a_0|}{L} \leq |z| \leq \frac{R}{c}, c > 1$ does not exceed $\frac{1}{\log c} \log \frac{L_1}{|a_0|}$ if n and n-m are both even or odd.

The case of n even and n-m odd or n odd and n-m even is similar.

**Proof of Theorem 4.**

As in the proof of Theorem 1,we have, for the functions F(z) and G(z), for both n and n-m even or odd and for $|z| \leq R, R \geq 1$,

$$|G(z)| \leq |a_n||z|^{n+1} + (1-k)|a_n||z|^n + |ka_n - a_{n-1}||z|^n + |a_{n-1} - a_{n-2}||z|^{n-1} + ...... + |a_{n-m+1} - a_{n-m}||z|^{n-m+1}$$
$$+ |a_{n-m} - a_{n-m-1}||z|^{n-m} + |a_{n-m-1} - a_{n-m-2}||z|^{n-m-1} + ....... + |a_3 - a_2||z|^3 + |a_2 - a_1||z|^2$$
$$+ |a_1 - a_0||z|$$

$$\leq |a_n|R^{n+1} + (1-k)|a_n|R^n + |ka_n - a_{n-1}|R^n + |a_{n-1} - a_{n-2}|R^{n-1} + ...... + |a_{n-m+1} - a_{n-m}|R^{n-m+1}$$
$$+|a_{n-m} - a_{n-m-1}|R^{n-m} + |a_{n-m-1} - a_{n-m-2}|R^{n-m-1} + ....... + |a_3 - a_2|R^3 + |a_2 - a_1|R^2$$
$$+|a_1 - a_0|R$$
$$\leq |a_n|R^{n+1} + R^n\{(1-k)|a_n| + |ka_n - a_{n-1}| + |a_{n-1} - a_{n-2}| + ...... + |a_{n-m+1} - a_{n-m}|$$
$$+|a_{n-m} - a_{n-m-1}| + |a_{n-m-1} - a_{n-m-2}| + ....... + |a_3 - a_2| + |a_2 - a_1|$$
$$+|a_1 - \delta - a_0 + \delta|\}$$
$$\leq |a_n|R^{n+1} + R^n\{(1-k)|a_n| + (a_{n-1} - ka_n) + (a_{n-1} - a_{n-2}) + ...... + (a_{n-m+1} - a_{n-m})$$
$$+(a_{n-m-1} - a_{n-m}) + ...... + (a_2 - a_3) + (a_1 - a_2) + (a_0 - a_1 + \delta) + \delta\}$$
$$\leq |a_n|R^{n+1} + R^n\{|a_n| - k(|a_n| + a_n) + 2\delta + a_0 + 2(a_{n-1} + a_{n-3} + ...... + a_{n-m+3} + a_{n-m+1})$$
$$-2(a_{n-2} + a_{n-4} + ..... + a_{n-m+2} + a_{n-m})\}$$

and for $|z| \leq R, R \leq 1$,

$$|G(z)| \leq |a_n|R^{n+1} + R\{|a_n| - k(|a_n| + a_n) + 2\delta + a_0 + 2(a_{n-1} + a_{n-3} + ...... + a_{n-m+3} + a_{n-m+1})$$
$$-2(a_{n-2} + a_{n-4} + ..... + a_{n-m+2} + a_{n-m})\}.$$

Since G(z) is analytic for $|z| \leq R, G(0) = 0$, it follows by Schwarz lemma that

$|G(z)| \leq D|z|$ for $|z| \leq R$, where

$$D = |a_n|R^{n+1} + R^n\{|a_n| - k(|a_n| + a_n) + 2\delta + a_0 + 2(a_{n-1} + a_{n-3} + ...... + a_{n-m+3} + a_{n-m+1})$$
$$-2(a_{n-2} + a_{n-4} + ..... + a_{n-m+2} + a_{n-m})\} \quad \text{for } R \geq 1$$
$$= |a_n|R^{n+1} + R\{|a_n| - k(|a_n| + a_n) + 2\delta + a_0 + 2(a_{n-1} + a_{n-3} + ...... + a_{n-m+3} + a_{n-m+1})$$
$$-2(a_{n-2} + a_{n-4} + ..... + a_{n-m+2} + a_{n-m})\} \quad \text{for } R \leq 1.$$

Therefore, for $|z| \leq R$,

$$|F(z)| = |G(z) + a_0|$$
$$\geq |a_0| - |G(z)|$$
$$\geq |a_0| - D|z|$$
$$> 0$$

if $|z| < \frac{|a_0|}{D}$.

In other words, F(z) h as no zero in $|z| < \frac{|a_0|}{D}$.

Since the zeros of P(z) are also the zeros of F(z), it follows that P(z) h as no zero in $|z| < \frac{|a_0|}{D}$ if n and n-m are both even or odd. Similarly, if n is even and n-m is odd or if n is odd and n-m is even , then rearranging the terms we can show that P(z) has no zero in $|z| < \frac{|a_0|}{D'}$.

Again , for $|z| \leq R$ and for n and n-m both even or odd,

$$|F(z)| \leq |a_n|R^{n+1} + |a_0| + R^n\{|a_n| - k(|a_n| + a_n) + 2\delta + a_0 + 2(a_{n-1} + a_{n-3} + ...... + a_{n-m+3} + a_{n-m+1})$$

$-2(a_{n-2}+a_{n-4}+.....+a_{n-m+2}+a_{n-m})\}$ for $R\geq 1$

and

$$|F(z)|\leq |a_n|R^{n+1}+|a_0|+R\{|a_n|-k(|a_n|+a_n)+2\delta+a_0+2(a_{n-1}+a_{n-3}+......+a_{n-m+3}+a_{n-m+1})$$
$$-2(a_{n-2}+a_{n-4}+.....+a_{n-m+2}+a_{n-m})\} \text{ for } R\leq 1.$$

Hence, by Lemma 2, the number of zeros of F(z) and hence P(z) in $|z|\leq \frac{R}{c}, c>1$ does not exceed

$\frac{1}{\log c}\log\frac{D_1}{|a_0|}$ ,where

$$D_1=|a_n|R^{n+1}+|a_0|+R^n\{|a_n|-k(|a_n|+a_n)+2\delta+a_0+2(a_{n-1}+a_{n-3}+......+a_{n-m+3}+a_{n-m+1})$$
$$-2(a_{n-2}+a_{n-4}+.....+a_{n-m+2}+a_{n-m})\} \text{ for } R\geq 1$$
$$=|a_n|R^{n+1}+|a_0|+R\{|a_n|-k(|a_n|+a_n)+2\delta+a_0+2(a_{n-1}+a_{n-3}+......+a_{n-m+3}+a_{n-m+1})$$
$$-2(a_{n-2}+a_{n-4}+.....+a_{n-m+2}+a_{n-m})\} \text{ for } R\leq 1,$$

Since P(z) has no zero in $|z|<\frac{|a_0|}{D}$ if n and n-m are both even or odd, it follows that the number of zeros of P(z) in $\frac{|a_0|}{L}\leq |z|\leq \frac{R}{c}, c>1$ does not exceed $\frac{1}{\log c}\log\frac{L_1}{|a_0|}$ if n and n-m are both even or odd.

The case of n even and n-m odd or n odd and n-m even is similar.

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