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Existence of solutions of nonlinear fractional impulsive delay integro differential equation of Sobolev type

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Abstract- In this paper we prove the existence of solutions of nonlinear fractional delay integrodifferential equations with impulsive conditions.

Index Terms- Delay Integrodifferential, Fractional Impulse, Sobolev.

I. Introduction

Fractional Calculus deals with the generalization of integrals and derivatives of noninteger order. Fractional Calculus involves a wide area of applications by bringing into a broader paradigm concept of physics, mathematics and engineering. Though the concepts and the calculus of fractional derivatives are few centuries old, it is realized only recently that these derivatives form an excellent framework for modeling experiments for fractional models on population dynamics are discussed in [7]. In [5, 6], the authors have proved the existence of solutions of abstract fractional differential equations by using fixed point techniques.

II. Preliminaries

We need some basic definitions and properties of fractional calculus which are used in this paper. Let X be a Banach space and $\mathbb{R}_+ = [0,\infty)$. Suppose $f \in L_1(\mathbb{R}_+)$. Let $C(J,X)$ be the Banach space of continuous functions $x(t)$ with $x(t) \in X$ for $t \in J = [0,T]$and $\|x\|_{C(J,X)} = max_{t\in J}\|x(t)\|$. Let $B(X)$ denote the bounded linear operators from X into $X$with the norm $\|A\|_{B(X)} = \sup\{\|A(y)\|: \|y\| = 1\}$. Also consider the Banach space

$$PC(J,X) = \begin{Bmatrix} u: J \to X : u \in C\big((t_k, t_{k+1}], X\big), k = 0,1,\dots,m \text{and there exist } u(t_k^-) \text{and } u(t_k^+), \\ k = 1,2,\dots,m \text{with } u(t_k^-) = u(t_k) \end{Bmatrix},$$

with the norm $\|u\|_{PC} = sup_{t\in J}\|u(t)\|$. Set $J' = [0,T]\backslash\{t_1,t_2,\dots,t_m\}$.

Definition 2.1:[3]

The Riemann – Liouville fractional integral operator of order $\alpha > 0$, of function $f \in L_1(\mathbb{R}_+)$ is defined as

$$I_{0+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s)ds,$$

where is theEuler gamma function.

Definition 2.2:

The Riemann – Liouvillefractional derivative of order $\alpha > 0$, $n-1 < \alpha < n, n \in N$, is defined as

$$^{(R\text{-}L)}D_{0+}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{n-\alpha-1} f(s)ds,$$

where the function *f(t)* has absolutely continuous derivatives up to order (n - 1).

Definition 2.3:

The Caputo fractional derivative of order $\alpha > 0, n-1 < \alpha < n$, is defined as

$$^{C}D_{0+}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)}\int_0^t (t-s)^{n-\alpha-1} f^n(s)ds,$$

where the function *f(t)* has absolutely continuous derivatives up to order (n – 1). If $0 < \alpha < 1$, then

$$^{C}D_{0+}^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{f'(s)}{(t-s)^{\alpha}} ds,$$

where$f'(s) = Df(s) = \frac{df(s)}{ds}$ and f is an abstract function with values in *X*.

We shall state some properties of the operators I_{0+}^{α} and $^{C}D_{0+}^{\alpha}$.

Proposition 2.4.*For* $\alpha, \beta > 0$ *and f as a suitable function we have*

(i) $I_{0+}^{\alpha} I_{0+}^{\beta} f(t) = I_{0+}^{\alpha+\beta} f(t)$
(ii) $I_{0+}^{\alpha} I_{0+}^{\beta} f(t) = I_{0+}^{\beta} I_{0+}^{\alpha} f(t)$
(iii) $I_{0+}^{\alpha}\big(f(t) + g(t)\big) = I_{0+}^{\alpha} f(t) + I_{0+}^{\alpha} g(t)$
(iv) $I_{0+}^{\alpha}\, {}^{C}D_{0+}^{\alpha} f(t) = f(t) - f(0), 0 < \alpha < 1$
(v) ${}^{C}D_{0+}^{\alpha} I_{0+}^{\alpha} f(t) = f(t)$
(vi) ${}^{C}D_{0+}^{\alpha} f(t) = I_{0+}^{1-\alpha} Df(t) = I_{0+}^{1-\alpha} f'(t), 0 < \alpha < 1$
(vii) ${}^{C}D_{0+}^{\alpha}\, {}^{C}D_{0+}^{\beta} f(t) \neq CD_{0+}^{\alpha+\beta} f(t)$
(viii) ${}^{C}D_{0+}^{\alpha}\, {}^{C}D_{0+}^{\beta} f(t) \neq {}^{C}D_{0+}^{\beta}\, {}^{C}D_{0+}^{\alpha} f(t)$.

We observe from the above that both the Riemann – Liouville and the Caputo fractional differential operators do possess neither semigroup nor commutative properties, which are inherent to the derivatives on integer order. For our convenience, let us take ${}^{C}D_{0+}^{\alpha} as\, {}^{C}D^{\alpha}$.

Consider the nonlinear fractional impulsive delay integrodifferential equation of Sobolev type of the form

$${}^{C}D^{q}\big(Bu(t)\big) + Au(t) = f\left(t, u(\alpha(t), \int_0^t h(t,s,u(\beta(s))ds\right), t \in J = [0,T], t = t_k \qquad (2.1)$$

$$\Delta u|_{t=t_k} = I_k\big(u(t_k^-)\big), \qquad (2.2)$$

$$u(0) = u_0 \qquad (2.3)$$

where $0 < q < 1$, $f: J \times X \times X \longrightarrow X, h: \Delta \times X \longrightarrow X, \alpha, \beta: J \longrightarrow J$ are continuous, $I_k: X \longrightarrow X, k = 1,2,\dots,m. u_0 \in X, 0 = t_0 < t_1 < t_2 < \cdots < t_m < t_{m+1} = T$.
$\Delta u|_{t=t_k} = u(t_k^+) - u(t_k^-)$, $\Delta = \{(t,s): 0 \le s \le t \le T\}$.

A and*B* are linear operators with domain and ranges contained in a Banach space *X*.

It is easy to prove that the Equation 2.1 is equivalent to the integral equation

$$u(t) = \begin{cases} u_0 - \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} B^{-1} Au(s)ds \\ + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} B^{-1}\left(f\left(s, u(\alpha(s)), \int_0^s h\left(s,\tau,u(\beta(\tau))\right)ds\right)\right), if\ t \in [0,t_1] \\ \\ u_0 - \frac{1}{\Gamma(q)}\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}(t_i - s)^{q-1} B^{-1}Au(s)ds - \frac{1}{\Gamma(q)}\int_{t_k}^{t}(t-s)^{q-1}B^{-1}Au(s)ds \\ + \frac{1}{\Gamma(q)}\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}(t_i - s)^{q-1}B^{-1}\left(f\left(s,u(\alpha(s)), \int_0^s h\left(s,\tau,u(\beta(\tau))\right)d\tau\right)\right)ds \\ + \frac{1}{\Gamma(q)}\int_{t_k}^{t}(t-s)^{q-1}B^{-1}\left(f\left(s,u(\alpha(s)), \int_0^s h\left(s,\tau,u(\beta(\tau))\right)d\tau\right)\right)ds \\ + \sum_{i-1}^{k} B^{-1} I_i\big(u(t_i^-)\big), if\ t \in (t_k, t_{k+1}] \end{cases} \qquad (2.4)$$

By a local solution of the abstract Cauchy problem (2.1), we mean an abstract function *u* such that the following conditions are satisfied:

(i) $u \in PC(J,X)$and $u \in D(A)$ on J';
(ii) $\frac{d^q u}{dt^q}$exists and continuous on J', where $0 < q < 1$;
(iii) *u* satisfies Equation (2.1) on J' and satisfies the conditions

$\Delta u|_{t=t_k} = I_k\big(u(t_k^-)\big), u(0) = u_0 \in X$or that is equivalent *u* satisfying the integral equation (2.4)
We assume the following hypotheses to prove the existence of solutions
(H1) $A: D(A) \subset X \longrightarrow X$and $B: D(B) \subset X \longrightarrow X$are closed linear operators
(H2) $D(B) \subset D(A)$and B is bijective
(H3) $B^{-1}: X \longrightarrow D(B)$ is compact
(H4) $B^{-1}A : X \longrightarrow D(B)$ is continuous. [4]
(H5) $f: J \times X \times X \to X$ is continuous and there exists a constant $L > 0$, such that

$$\|f(t,s,u) - f(t,s,v)\| \le L[\|x-y\| + \|u-v\|], \text{ for all } u,v \in X.$$

Let us denote$B_1 u(t) = \int_0^t h\left(t, u(\alpha(s)), u(\beta(s))\right) ds$.
(H6) $h: \Delta \times X \to X$ is continuous and there exists a constant $L_1 > 0$, such that
$\left\|h\big(t, u(\alpha(t)), B_1 u(t)\big) - h\big(t, v(\alpha(t)), B_1 v(t)\big)\right\| \le L_1\|u - v\|$for all $u, v \in X$.
(H7) $A: D(A) \subset X \longrightarrow X$ is a continuous bounded linear operator and there exists a constant *M* > 0, such that

$\|A(u)-A(v)\| \le M\|u-v\|$for all $u, v \in X$.
(H8) The functions $I_k: X \to X$are continuous and there exists a constant $L_2 > 0$, such that
$\|I_k(u)-I_k(v)\| \le L_2\|u-v\|$, for each $u, v \in X$ and $k = 1,2,\dots,m$.
Let $B_r = \{u \in X: \|u\| \le r\}$ for some $r > 0$. For brevity let us take $\gamma = \frac{T^q}{\Gamma(q+1)}$ and $K = \|B^{-1}\| = sup_{x\in X}\{\|B^{-1}x\|: \|x\| \le 1\}$, $N = \max_{t\in J}\|f(t,0,0)\|$, $N_1 = \max_{(t,s)\in\Delta}\|h(t,s,0)\|$

Further we assume that
(H9) $\|u_0\| + (m+1)\gamma K M_0 + mKL_2 r \le r$, where $M_0 = Mr + Lr + LL_1Tr + LN_1T$
(H10) Let $p = (m+1)\gamma KM + (m+1)\gamma KL_1 + mKL_2$ be such that $0 \le p < 1$.

III. Main Results

Theorem 3.1:
If the hypotheses (H1) – (H10) *are satisfied, then the nonlinear fractional impulsive delay integrodifferential equation* (2.1) - (2.3) *has a unique solution.*
Proof:
Let $Z = C(J, B_r)$. Define the mapping $\varphi: Z \to Z$ by

$$\varphi u(t) = u_0 - \frac{1}{\Gamma(q)}\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}(t_i-s)^{q-1}B^{-1}Au(s)ds - \frac{1}{\Gamma(q)}\int_{t_k}^{t}(t-s)^{q-1}B^{-1}Au(s)ds$$

$$+\frac{1}{\Gamma(q)}\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}(t_i-s)^{q-1}B^{-1}\left(f\left(s,u(\alpha(s)),B_1u(s)\right)\right)ds$$

$$+\frac{1}{\Gamma(q)}\int_{t_k}^{t}(t-s)^{q-1}B^{-1}\left(f\left(s,u(\alpha(s)),B_1u(s)\right)\right)ds$$

$$+\sum_{i-1}^{k}B^{-1}I_i\left(u(t_i^-)\right) \qquad (3.1)$$

and we have to show that φ has a fixed point, and this fixed point is a solution of the Equation (2.1). First we show that $\varphi B_r \subset B_r$.
From the assumptions we have

$$\|\varphi u(t)\| \le \|u_0\| + \frac{1}{\Gamma(q)}\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}(t_i-s)^{q-1}\|B^{-1}Au(s)\|ds + \frac{1}{\Gamma(q)}\int_{t_k}^{t}(t-s)^{q-1}\|B^{-1}Au(s)\|ds$$

$$+\frac{1}{\Gamma(q)}\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}(t_i-s)^{q-1}\left\|B^{-1}\left(f\left(s,u(\alpha(s)),B_1u(s)\right)\right)\right\|ds$$

$$+\frac{1}{\Gamma(q)}\int_{t_k}^{t}(t-s)^{q-1}\left\|B^{-1}\left(f\left(s,u(\alpha(s)),B_1u(s)\right)\right)\right\|ds$$

$$+\sum_{i-1}^{k}\left\|B^{-1}I_i\left(u(t_i^-)\right)\right\|.$$

$$\le \|u_0\| + \frac{1}{\Gamma(q)}\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}(t_i-s)^{q-1}\|B^{-1}\|\|Au(s)\|ds + \frac{1}{\Gamma(q)}\int_{t_k}^{t}(t-s)^{q-1}\|B^{-1}\|\|Au(s)\|ds$$

$$+\frac{1}{\Gamma(q)}\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}(t_i-s)^{q-1}\|B^{-1}\|\left\|\left(f\left(s,u(\alpha(s)),B_1u(s)\right)\right)\right\|ds$$

$$+\frac{1}{\Gamma(q)}\int_{t_k}^{t}(t-s)^{q-1}\|B^{-1}\|\left\|\left(f\left(s,u(\alpha(s)),B_1u(s)\right)\right)\right\|ds$$

$$+\sum_{i-1}^{k}\|B^{-1}\|\left\|I_i\left(u(t_i^-)\right)\right\|.$$

$$\le \|u_0\| + (m+1)\frac{T^q}{\Gamma(q+1)}KM\|u\| + (m+1)\frac{T^q}{\Gamma(q+1)}K[L\|u\| + L\|B_1u(s)\|] + mKL_2\|u\|$$

$$\le \|u_0\| + (m+1)\gamma K[r + Lr + LL_1rT + LN_1T] + mKL_2r$$

$$\le r.$$

Thus, φ maps B_r maps into itself.
For $u_1, u_2 \in Z$, we have

$$\|\varphi u_1(t) - \varphi u_2(t)\| \leq \frac{1}{\Gamma(q)} \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} (t_i - s)^{q-1} \|B^{-1}A[u_1(s) - u_2(s)\| ds$$

$$+ \frac{1}{\Gamma(q)} \int_{t_k}^{t} (t - s)^{q-1} \|B^{-1}A[u_1(s) - u_2(s)]\| ds$$

$$+ \frac{1}{\Gamma(q)} \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} (t_i - s)^{q-1} \left\| B^{-1} \left[\left(f\left(s, u_1(\alpha(s)), B_1 u_1(s)\right)\right) - \left(f\left(s, u_2(\alpha(s)), B_1 u_2(s)\right)\right)\right] \right\| ds$$

$$+ \frac{1}{\Gamma(q)} \int_{t_k}^{t} (t - s)^{q-1} \left\| B^{-1} \left[\left(f\left(s, u_1(\alpha(s)), B_1 u_1(s)\right)\right) - \left(f\left(s, u_2(\alpha(s)), B_1 u_2(s)\right)\right)\right] \right\| ds$$

$$+ \textstyle\sum_{i-1}^{k} \|B^{-1}[I_i(u_1(t_i^-)) - I_i(u_2(t_i^-))]\|.$$

$$\leq (m+1)\frac{T^q}{\Gamma(q+1)} KM\|u_1 - u_2\| + (m+1)\frac{T^q}{\Gamma(q+1)} KL_1\|u_1 - u_2\| + mKL_2\|u_1 - u_2\|$$

$$\leq [\gamma(m+1)KM + \gamma(m+1)KL_1 + mKL_2]\|u_1 - u_2\|$$

$$\leq p\|u_1 - u_2\|.$$

Since $0 \leq p < 1$, φ is a contraction mapping and therefore there exists an unique fixed point $u \in Z$ such that $\varphi u(t) = u(t)$. Any fixed point of φ is the solution of (2.1). □

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