ISSN 2250-3153

Inventory Models with Weibull Deterioration and Time-Varying Holding Cost

Ritu Raj*, **Naresh Kumar Kaliraman***, **Dr Shalini Chandra****, **Dr Harish Chaudhary*****

* Research Scholar, Centre for Mathematical Sciences, Banasthali University, P.O. Banasthali Vidyapith, Banasthali – 304022, Rajasthan, India
** Associate Professor, Centre for Mathematical Sciences, Banasthali University, P.O. Banasthali Vidyapith, Banasthali – 304022, Rajasthan,India
*** Assistant Professor, Department of Management Studies, Indian Institute of Technology, Delhi, New Delhi-110016, India

Abstract- This paper develops inventory models for deteriorating item; the rate of deterioration is Weibull distribution deterioration with two parameters. It considers ramp-type demand. Shortage is allowed and it is completely backlogged. In these models, we consider time dependent holding cost. These models are developed under two different replenishment policies; (i) without shortage (ii) with shortage. The aim of these models is to find the optimal solution for minimizing the total inventory costs for both the above mentioned policies. To optimize the models numerical illustrations have been carried out and sensitivity analysis occurred to evaluate the result of parameters on assessment variables and the entire cost of these models.

Index Terms- Holding Cost, Ramp-Type Demand, Shortage, Weibull Deterioration.

I. Introduction

Inventory is a fundamental part of manufacturing, distribution, retail infrastructure and demand plays an important role in choosing the beneficial inventory strategy. Researchers were developed the inventory models assuming the demand of the objects to be constant, stock dependent, linearly increasing, linearly decreasing, exponentially increasing and exponentially decreasing with time etc. after that , it has been noticed that the above mentioned demand policies do not accurately describe the demand of assured objects such as newly launched cosmetics, garments, fashion items, electronics etc., for which the demand increases with time as they are launched into the market and after some time it becomes stable. Ramp-type demand pattern is introduced to consider the demand of such types of materials. Mostly the ramp- type demand rate is increasing for some time when new variety of consumer goods comes to the marketplace. In case of ramp-type demand pattern, the demand increases linearly at the beginning and when the market comes into a steady state, the demand becomes stable until the end of the inventory cycle. For example, the demand of festival greetings likes sweets, gods and goddess's pictures, phatakhes, paintings etc., follows ramp-type demand rate. At the beginning demands of these items increases linearly from starting of October to the end of November month, after that demands of these items becomes constant.

The inventory of deteriorating items is a big problem for any organization in the supply chain system. The finished goods are deteriorating with time in inventory. Some items used in daily life like fruits, vegetables, milk products, meat etc., are deteriorating at higher rate instead of other items. It is necessary that consume these items within time limit to avoid deterioration. The inventory model with ramp-type demand rate was projected by Mandal and Pal [9]. Wu and Ouyang [2] developed a replenishment inventory model for deteriorating items with ramp-type demand rate. Wu [3] developed an economic order quantity inventory model for deteriorating items. He considered weibull distribution deterioration, partial backlogging and ramp-type demand. The inventory model with Weibull deterioration and related issues have been studied by many researchers. Silver [6] presented a heuristic for deteriorating items inventory model with time dependent linear demand. Goyal and Giri [5] extended review on deterioration inventory model. Covert and Philip [1] presented an inventory model where the time to deterioration is described with two parameters Weibull distribution deterioration. Ghosh and Chaudhuri [7] developed an inventory model for deteriorating items with two parameters Weibull distribution deterioration, shortages and quadratic demand rate. Sanni [8] developed an inventory model for three parameters Weibull deteriorating items, quadratic demand rate and shortages are allowed. Shortages are permitted and customers will wait till the next order arrives i.e. called a backorder model. All demands are fulfilled instantly. Ghare and Schrader [4] developed an exponentially deteriorating inventory model. In this paper, they considered stable rate of deterioration with no shortage. There after a lot of research work has been done. Holding cost is considered as known and constant in most of the models. Now, holding cost may not be considered as constant, it is varying with regular change in time value of money and change in price index. There is lot of competitions in globalization, so holding cost may not remain constant over time. Naddor[10], Van Deer Veen [13], Weiss [14] assumed holding cost as a non-linear function over time for which the items are held in inventory and the amount of the on-hand inventory. Roy [11] proposed an EOQ model for deteriorating items in which deterioration rate and holding cost are considered as linearly increasing function of time, selling price is dependent on demand rate and shortage is allowed and completely backlogged. Tripathi [12] presented an inventory model for non-deteriorating items under permissible delay in payments where holding cost is a function of time.

The aim of this paper is developing inventory models where rate of deterioration is two parameters weibull distribution deterioration. Holding cost as a linearly increasing function of time. These models have been studied under two different inventory policies such as without shortage and with shortage. The main purpose of this paper is to show that there exist a unique optimal cycle time to minimize the total inventory cost

ISSN 2250-3153

per unit time. To optimize these models numerical illustrations have been carried out and sensitivity analysis occurred to study the result of parameters on assessment variables and the entire cost of these models.

II. Assumptions

We consider the following assumptions for developing mathematical model:
1. The inventory system consider single item.
2. T is the fixed unit of time for each ordering cycle.
3. The replenishment rate is infinite.
4. The demand rate $F(t)$ is a ramp type function:

$$F(t)=\begin{cases} Dt, & t<\mu, \\ D\mu, & t\geq\mu, \end{cases}$$

Dt is positive and continuous for $t\in[0,T]$.

5. The lead time is zero.
6. Shortage is allowed and completely backlogged.
7. Holding cost H_c per unit is time dependent:
   $H_c(t)=a+bt$, where $a>0$ and $b>0$.
8. The rate of deterioration is two parameters weibull deterioration denoted by
   $\theta=\alpha\beta t^{\beta-1}$, where $\alpha>0$, $\beta>0$ and $0\leq t\leq t_1$.
9. Q is the ordering quantity.

III. Notations

We consider the following notations for developing mathematical model:
1. D_c is the deterioration cost per unit per item.
2. μ is the parameter of the ramp type demand function.
3. t_1 is the time when inventory level reaches zero.
4. t_1^* is the optimal point for the replenishment policy.
5. S_c is the shortage cost per unit per item.
6. $I(t)$ is the on hand inventory level at time t over the ordering cycle $[0,T]$.
7. T is considered annually.

IV. Mathematical Model

These models are developed for the following two different replenishment policies:
(i) Without Shortage.
(ii) With Shortage.

V. Models without Shortage

The inventory model is starting with no shortage. At this stage, the inventory level reaches its maximum level and then production is stopped. The inventory depletes to zero due to demand and deterioration during the time interval $[0,t_1]$ and falls zero at $t=t_1$. Thus, shortage occurs during $[t_1, T]$, which is completely backlogged. Therefore, the inventory is described by the system of differential equations:

$$\frac{dI_a(t)}{dt} = -\theta I_a(t) - F(t), 0 \le t \le t_1 \text{ with boundary condition } I_a(t_1) = 0 \quad ...(1)$$

$$\frac{dI_b(t)}{dt} = -F(t), t_1 \le t \le T \text{ with boundary condition } I_b(t_1) = 0 \quad ...(2)$$

We considered two cases, $(i)\ t < \mu$ and $(ii)\ t \ge \mu$ for solving these differential equations:

5.1Case (i): $t < \mu$

From equation (1), we get

$$I_a(t) = De^{-\alpha t^{\beta}}\left[\frac{1}{2}\left(t_1^{2} - t^{2}\right) + \frac{\alpha}{\beta+2}\left(t_1^{\beta+2} - t^{\beta+2}\right)\right] \quad ...(3)$$

Equation (2), solve by the following two ways:

$$\frac{dI_b(t)}{dt} = -Dt, \qquad t_1 \le t \le \mu \qquad ...(i)$$

$$\frac{dI_b(t)}{dt} = -D\mu, \qquad \mu \le t \le T,\ \mathrm{I}(\mu_-) = \mathrm{I}(\mu_+) \qquad ...(ii)$$

We have

$$I_b(t) = -\frac{D}{2}\left(t^2 - t_1^{2}\right) \quad ...(4)$$

and $$I_b(t) = \frac{D\mu^2}{2} - D\mu t + \frac{D}{2}t_1^{2} \quad ...(5)$$

Total deterioration amount during $[0, t_1]$

$$D_c = \int_0^{t_1} Dte^{\alpha t^{\beta}}dt - \int_0^{t_1} Dtdt = \frac{\alpha D}{\beta+2}t_1^{\beta+2} \quad ...(6)$$

Total holding cost during $[0, t_1]$ is

$$HC = \int_0^{t_1} H_c(t) I_a(t)dt = \int_0^{t_1}(a+bt)De^{-\alpha t^{\beta}}\left[\frac{1}{2}\left(t_1^{2} - t^{2}\right) + \frac{\alpha}{\beta+2}\left(t_1^{\beta+2} - t^{\beta+2}\right)\right]dt$$

$$HC = D\left[Pt_1^{2\beta+4} + Qt_1^{2\beta+3} + Rt_1^{\beta+4} + St_1^{\beta+3} + \frac{b}{8}t_1^{4} + \frac{t_1^{3}}{3}\right] \quad ...(7)$$

See Appendix-A

Lost sale amount during $\left[t_1,T\right]$ is

$$S_c=\int_{t_1}^{T}-I_b\left(t\right)dt=\int_{t_1}^{\mu}-I_b\left(t\right)dt+\int_{\mu}^{T}-I_b\left(t\right)dt=\int_{t_1}^{\mu}\frac{D}{2}\left(t^2-t_1^{\,2}\right)dt+\int_{\mu}^{T}\left(D\mu t-\frac{D\mu^2}{2}-\frac{D}{2}t_1^{\,2}\right)dt$$

$$S_c=\frac{D}{6}\left(A_1-3Tt_1^2+2t_1^3\right) \qquad ...(8)$$

See Appendix-A

Order quantity is

$$OQ=-\int_{0}^{t_1}e^{\alpha t^{\beta}}Dtdt-D\int_{t_1}^{\mu}tdt-\int_{\mu}^{T}D\mu dt=D\int_{0}^{t_1}\left(1+\alpha t^{\beta}\right)tdt+D\int_{t_1}^{\mu}tdt+\int_{\mu}^{T}D\mu dt$$

$$OQ=D\left(\frac{t_1^{\,2}}{2}+\frac{\alpha}{\beta+2}t_1^{\,\beta+2}-\frac{1}{2}t_1^{\,2}\right)+A_2 \qquad ...(9)$$

See Appendix-A

Total cost during $\left[0,T\right]$ = holding cost + lost sale amount + deterioration cost $TC_1\left(t_1\right)=HC+S_c+D_c$

$$TC_1\left(t_1\right)=D\left[Pt_1^{\,2\beta+4}+Qt_1^{\,2\beta+3}+Rt_1^{\,\beta+4}+St_1^{\,\beta+3}+\frac{\alpha}{\beta+2}t_1^{\,\beta+2}+\frac{b}{8}t_1^{\,4}+\frac{2}{3}t_1^{\,3}-\frac{T}{2}t_1^2+\frac{A_1}{6}\right]...(10)$$

5.1.1 Solution:

$$\frac{\partial TC_1}{\partial t_1}=D\left[\begin{array}{l}P\left(2\beta+4\right)t_1^{\,2\beta+3}+Q\left(2\beta+3\right)t_1^{\,2\beta+2}+R\left(\beta+4\right)t_1^{\,\beta+3}+S\left(\beta+3\right)t_1^{\,\beta+2}\\+\alpha t_1^{\,\beta+1}+\frac{b}{2}t_1^{\,3}+2t_1^{\,2}-Tt_1\end{array}\right] \qquad ...(11)$$

$$\frac{\partial^2 TC_1}{\partial t_1^{\,2}}=D\left[\begin{array}{l}P\left(2\beta+4\right)\left(2\beta+3\right)t_1^{\,2\beta+2}+Q\left(2\beta+3\right)\left(2\beta+2\right)t_1^{\,2\beta+1}+R\left(\beta+4\right)\\\left(\beta+3\right)t_1^{\,\beta+2}+S\left(\beta+3\right)\left(\beta+2\right)t_1^{\,\beta+1}+\alpha\left(\beta+1\right)t_1^{\,\beta}+\frac{3b}{2}t_1^{\,2}+4t_1-T\end{array}\right] \qquad ...(12)$$

Main objective to minimize the total relevant cost for the model starting without shortage, the necessary condition to minimize the total relevant cost is

$\frac{\partial TC_1}{\partial t_1}=0$, We get

$$DP\left(2\beta+4\right)t_1^{\,2\beta+3}+DQ\left(2\beta+3\right)t_1^{\,2\beta+2}+DR\left(\beta+4\right)t_1^{\,\beta+3}+DS\left(\beta+3\right)t_1^{\,\beta+2}$$

$$+D\alpha t_1^{\,\beta+1}+\frac{Db}{2}t_1^{\,3}+2Dt_1^{\,2}-DTt_1=0 \qquad ...(13)$$

Using the software Mathematica, we can calculate the optimal value of t_1 by equation (13) and the optimal value $TC_1(t_1)$ of the total relevant cost is determined by equation (10). The optimal value of t_1 satisfy the sufficient condition for minimizing total relevant cost $TC_1(t_1)$ is $\frac{\partial^2 TC_1}{\partial t_1^{\,2}} > 0$ ...(14)

The sufficient condition is satisfied.

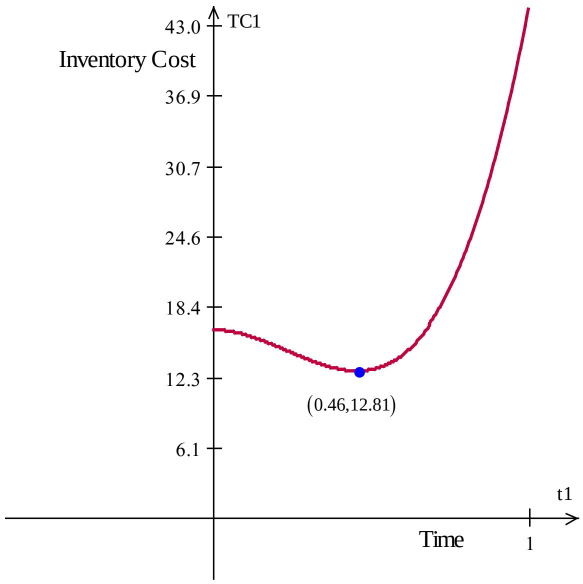

Figure 1: Graphical representation of total relevant cost without shortage.

5.1.2 Numerical Example:

Let us consider

$$a = \$1/unit, b = \$0.5/unit, S_c = \$1/unit/year, d_c = \$5/unit, \alpha = 0.02, \beta = 2, T = 1year,$$

$$D = 100, \mu = 0.8\,year.$$

$$P = -0.00000625, Q = -0.00002, R = 0.0004,\ S = 0.003, A_1 = 0.99, A_2 = 16$$

Thus, the optimal value of t_1 is $t_1^* \to 0.46 < \mu$. The optimal ordering quantity is $OQ = 16.02$. The minimum relevant cost is $TC_1 = 12.81$

5.1.3 Sensitivity Analysis:

To know, how the optimal solution is affected by the values of parameters, we derive the sensitivity analysis for some parameters. The particular values of some parameters are increased or decreased by $+25\%$, -25% and $+50\%$, -50%. After that, we derive the value of t_1 and TC_1 with the help of increased or decreased values of S_c and d_c. The result of the minimum relevant cost is existing in the following table 1.

Table: 1

Parameters	Actual Values	50% Increased	50% Decreased	25% Increased	25% Decreased
S_c	1	1.5	0.5	1.25	0.75
d_c	5	7.5	2.5	6.25	3.75

ISSN 2250-3153

t_1^*	0.46	0.55	0.31	0.51	0.4
TC_1	12.81	13.28	7.4	14.96	10.32
OQ	16.02	16.045	16.005	16.034	16.013

From the result of above table, we observe that total relevant cost and ordering quantity is much affected by deterioration cost and shortage cost, other parameters are less sensitive.

5.2 Case II: $t \geq \mu$

Equation (1), solve by the following two ways:

$$\frac{dI_a(t)}{dt} + \theta I_a(t) = -Dt, \qquad 0 \leq t \leq \mu,\ \mathrm{I}(\mu_-) = \mathrm{I}(\mu_+) \qquad \ldots(i)$$

$$\frac{dI_a(t)}{dt} + \theta I_a(t) = -D\mu, \qquad \mu \leq t \leq t_1, \quad \mathrm{I}(t_1) = 0 \qquad \ldots(ii)$$

We have

$$\frac{dI_a(t)}{dt} + \alpha\beta t^{\beta-1} I_a(t) = -Dt, \qquad 0 \leq t \leq \mu,\ \mathrm{I}(\mu_-) = \mathrm{I}(\mu_+)$$

$$I_a(t) = De^{-\alpha t^\beta}\left\{\frac{1}{2}\left(\mu^2 - t^2\right) + \frac{\alpha}{\beta+2}\left(\mu^{\beta+2} - t^{\beta+2}\right) + \mu\left(\left(t_1 - \mu\right) + \frac{\alpha}{\beta+1}\left(t_1^{\beta+1} - \mu^{\beta+1}\right)\right)\right\} \ldots(15)$$

and

$$I_a(t) = e^{-\alpha t^\beta} D\mu\left[\left(t_1 - t\right) + \frac{\alpha}{\beta+1}\left(t_1^{\beta+1} - t^{\beta+1}\right)\right], \quad \mu \leq t \leq t_1 \qquad \ldots(16)$$

From equation (2), we get

$$\frac{dI_b(t)}{dt} = -D\mu, \qquad t_1 \leq t \leq T,\ I_b(t_1) = 0 \qquad I_b(t) = -D\mu(t - t_1)$$

Total deterioration amount during $[0, t_1]$

$$D_c = D\int_0^\mu te^{\alpha t^\beta}\,dt + D\mu\int_\mu^{t_1} e^{\alpha t^\beta}\,dt - D\mu\int_\mu^{t_1} dt = D\left[S + \mu\frac{\alpha}{\beta+1}t_1^{\beta+1}\right] \qquad \ldots(18)$$

See Appendix-B

Total holding cost during $[0, t_1]$ is

$$HC = \int_0^{t_1} H_1(t) I_a(t)\,dt = \int_0^\mu (a + bt) I_a(t)\,dt + \int_\mu^{t_1} (a + bt) I_a(t)\,dt$$

$$HC = aD\left[\delta_1 + \gamma\left\{\left(t_1 - \mu\right) + \frac{\alpha}{\beta+1}\left(t_1^{\beta+1} - \mu^{\beta+1}\right)\right\} + \left\{\varepsilon + \frac{1}{2}\mu t_1^2 - t_1\gamma + t_1^{\beta+2}\psi + t_1^{\beta+1}\eta + t_1^{2\beta+2}\kappa\right\}\right]$$
$$+ bD\left[\pi + \rho\left(t_1 + \frac{\alpha}{\beta+1}t_1^{\beta+1}\right) + \left\{\frac{\mu}{6}t_1^3 - \rho t_1 + \xi + t_1^{\beta+3}\tau + t_1^{\beta+1}\vartheta + t_1^{2\beta+3}\Omega\right\}\right]$$

See Appendix-B

$$HC = aD\left[\left(\delta_1 - \gamma\mu - \frac{\alpha\mu^{\beta+1}\gamma}{\beta+1} + \varepsilon\right) + \frac{1}{2}\mu t_1^{\,2} + t_1^{\,\beta+1}\left(\frac{\alpha\gamma}{\beta+1} + \eta\right) + t_1^{\,\beta+2}\psi + t_1^{\,2\beta+2}\kappa\right] +$$
$$bD\left[\pi + \xi + t_1^{\,\beta+3}\tau + t_1^{\,\beta+1}\left(\frac{\alpha\rho}{\beta+1} + \vartheta\right) + t_1^{\,2\beta+3}\Omega + \frac{\mu}{6}t_1^{\,3}\right]$$

$$HC = aD\left[\frac{1}{2}\mu t_1^{\,2} + t_1^{\,2\beta+2}\kappa + t_1^{\,\beta+2}\psi + t_1^{\,\beta+1}\Pi + \Re\right] + bD\left[\frac{\mu}{6}t_1^{\,3} + t_1^{\,2\beta+3}\Omega + t_1^{\,\beta+3}\tau + t_1^{\,\beta+1}\Upsilon + \Psi\right] \ldots(19)$$

See Appendix-B

Lost sale amount during $[t_1, T]$ is

$$S_c = -\int_{t_1}^{T} I_b(t)\,dt = D\mu\int_{t_1}^{T}(t - t_1)dt = D\mu\left[\frac{1}{2}\left(T^2 - t_1^{\,2}\right) - t_1(T - t_1)\right] = D\mu\left(\frac{T^2}{2} + \frac{t_1^{\,2}}{2} - t_1 T\right) \ldots(20)$$

Order quantity is

$$OQ = \int_0^{\mu} De^{\alpha t^{\beta}}dt + D\mu\int_{\mu}^{t_1} dt + D\mu\int_{t_1}^{T}(t - t_1)\,dt = D\left(\frac{1}{2}\mu t_1^{\,2} + \mu t_1 - \mu t_1 T + Z_{12}\right) \qquad \ldots(21)$$

Total cost during $[0, T]$ = holding cost + lost sale amount + deterioration cost

$$TC_2(t_1) = HC + S_c + D_c$$

$$TC_2(t_1) = D\begin{bmatrix} a\left(\frac{1}{2}\mu t_1^{\,2} + t_1^{\,2\beta+2}\kappa + t_1^{\,\beta+2}\psi + t_1^{\,\beta+1}\Pi + \Re\right) + \mu\left(\frac{T^2}{2} + \frac{t_1^{\,2}}{2} - t_1 T\right) \\ +b\left(\frac{\mu}{6}t_1^{\,3} + t_1^{\,2\beta+3}\Omega + t_1^{\,\beta+3}\tau + t_1^{\,\beta+1}\Upsilon + \Psi\right) + S + \frac{\alpha\mu}{\beta+1}t_1^{\,\beta+1} \end{bmatrix} \qquad \ldots(22)$$

5.2.1 Solution:

$$\frac{\partial TC_2}{\partial t_1} = D\begin{bmatrix} a\mu t_1 + a(\beta+1)t_1^{\,\beta}\Pi + a(\beta+2)t_1^{\,\beta+1}\psi + a(2\beta+2)t_1^{\,2\beta+1}\kappa \\ +b(\beta+3)t_1^{\,\beta+2}\tau + b(\beta+1)t_1^{\,\beta}\Upsilon + b(2\beta+3)t_1^{\,2\beta+2}\Omega + \frac{b\mu}{2}t_1^{\,2} \\ +\alpha\mu t_1^{\,\beta} + \mu(t_1 - T) \end{bmatrix} \qquad \ldots(23)$$

$$\frac{\partial^2 TC_2}{\partial t_1^{\,2}} = D\begin{bmatrix} a\mu + a(\beta+1)\beta t_1^{\,\beta-1}\Pi + a(\beta+2)(\beta+2)t_1^{\,\beta}\psi + a(2\beta+2)(2\beta+1)t_1^{\,2\beta}\kappa \\ +b(\beta+3)(\beta+2)t_1^{\,\beta+1}\tau + b(\beta+1)\beta t_1^{\,\beta}\Upsilon + b(2\beta+3)(2\beta+2)t_1^{\,2\beta+1}\Omega + b\mu t_1 + \\ +\alpha\mu\beta t_1^{\,\beta-1} + \mu \end{bmatrix} \qquad \ldots(24)$$

Main objective to minimize the total relevant cost for the model starting without shortage, the necessary condition to minimize the total relevant cost is

$\frac{\partial TC_2}{\partial t_1} = 0$, we get

$$bD\Omega(2\beta+3)t_1^{\,2\beta+2} + aD\kappa(2\beta+2)t_1^{\,2\beta+1} + bD\tau(\beta+3)t_1^{\,\beta+2} + \frac{bD\mu}{2}t_1^{\,2}$$
$$+aD\psi(\beta+2)t_1^{\,\beta+1} + D\left((a\Pi + b\Upsilon)(\beta+1) + \alpha\mu\right)t_1^{\,\beta} + D\mu(a+1)t_1 - D\mu T = 0 \qquad \ldots(25)$$

Using the software Mathematica, we can calculate the optimal value of t_1 by equation (25) and the optimal value $TC_2(t_1)$ of the total relevant cost is determined by equation (22). The optimal value of t_1 satisfy the sufficient condition for minimizing total relevant cost $TC_2(t_1)$ is $\frac{\partial^2 TC_2}{\partial t_1^2} > 0$ ...(26)

The sufficient condition is satisfied.

5.2.2 Numerical Example:
Let us consider

$a = \$1/unit, b = \$0.5/unit, S_c = \$1/unit/year, d_c = \$5/unit, \alpha = 0.02, \beta = 2, T = 1 year,$

$D = 100, \mu = 0.8\, year.$

$\delta_1 = 0.17, \gamma = \phi = 0.637,\ \varepsilon = 0.25,\ \psi = 0.0013,\ \eta = -0.0042, \kappa = -0.000018,\ \pi = -0.153,$

$\xi = 0.136, \rho = 0.25, \tau = 0.0016,\ \vartheta = -0.0034,\ \Omega = -0.000011, \Re = -0.092,$

$\Pi = 0.000047, \Upsilon = -0.0017, \Psi = -0.017, S = 0.32,\ Z_{12} = 0.082, S = 0.32$

Thus, the optimal value of t_1 is $t_1^* \to 0.92 > \mu$. The optimal ordering quantity is $OQ = 42.06$. The minimum relevant cost is $TC_2 = 191.92$

VI. Models with Shortage

Now, inventory model starting with shortage during the period $[0, t_1]$ and is completely backlogged. Replenishment brings the inventory level up to Q after time t_1. The inventory level depletes and falls to zero at $t = T$ because of demand and deterioration during $[t_1, T]$. Two cases occur

(i) $t_1 < \mu$

(ii) $t_1 \geq \mu$

6.1 Case (i) $t_1 < \mu$

Therefore, the inventory $I(t)$ is described by the system of differential equations during $[0, T]$:

$$\frac{dq_1(t)}{dt} = -D(t),\ \ 0 \leq t \leq t_1,\ \ \ q(0) = 0 \qquad ...(27)$$

$$\frac{dq_2(t)}{dt} + \theta q_2(t) = -Dt\ \ t_1 \leq t \leq \mu,\ \ \ q(\mu_-) = q(\mu_+) \qquad ...(28)$$

$$\frac{dq_3(t)}{dt} + \theta q_3(t) = -D\mu\ \ \mu \leq t \leq T,\ \ \ q(T) = 0 \qquad ...(29)$$

From equation (27), we have

$$q_1(t) = -D\int_0^t t\,dt = -\frac{D}{2}t^2,\ \ 0 \leq t \leq t_1 \qquad ...(30)$$

From equation (28), we have

$$q_2(t) = e^{-\alpha t^\beta} D\left\{\int_t^\mu t e^{\alpha t^\beta} dt + \mu\int_\mu^T e^{\alpha t^\beta} dt\right\} = e^{-\alpha t^\beta} D\left(Z_{13} - \frac{1}{2}t^2 - \frac{\alpha}{\beta+2}t^{\beta+2}\right), t_1 \le t \le \mu \quad ...(31)$$

See Appendix-C

From equation (29), we have

$$q_3(t) = e^{-\alpha t^\beta} D\mu\int_t^T e^{\alpha t^\beta} dt = e^{-\alpha t^\beta} D\mu\left\{Z_{14} - t - \frac{\alpha}{\beta+1}t^{\beta+1}\right\}, \quad \mu \le t \le T \quad ...(32)$$

See Appendix-C

Total deterioration amount during $[t_1, T]$

$$D_c = e^{-\alpha t_1^\beta}\left\{\int_{t_1}^\mu D t e^{\alpha t^\beta} dt + D\mu\int_\mu^T e^{\alpha t^\beta} dt\right\} 3 = D\left[\left(1-\alpha t_1^\beta\right)\left\{\frac{t_1^2}{2} + \frac{\alpha}{\beta+2}t_1^{\beta+2} + C_2\right\}\right] \quad ...(33)$$

See Appendix-C

Total holding cost during $[t_1, T]$ is

$$HC = \int_{t_1}^T (a+bt)q(t)\,dt = D\left[\begin{array}{l} N_1 + \mu N_2 - aZ_{13}t_1 - \frac{1}{2}bZ_{13}t_1^2 + \frac{a}{6}t_1^3 + \frac{b}{8}t_1^4 + \left(\frac{\alpha b}{(\beta+2)(\beta+4)} - \frac{\alpha b}{2(\beta+4)}\right)t_1^{\beta+4} \\ + \left(\frac{a\alpha}{(\beta+2)(\beta+3)} - \frac{a\alpha}{2(\beta+3)}\right)t_1^{\beta+3} + \frac{\alpha b Z_{13}}{(\beta+2)}t_1^{\beta+2} + \frac{a\alpha Z_{13}}{\beta+1}t_1^{\beta+1} \\ - \frac{b\alpha^2}{(\beta+2)(2\beta+4)}t_1^{2\beta+4} - \frac{a\alpha^2}{(\beta+2)(2\beta+3)}t_1^{2\beta+3} \end{array}\right]$$

See Appendix-C

$$HC = D\left[\begin{array}{l} N_1 + \mu N_2 - aZ_{13}t_1 - \frac{1}{2}bZ_{13}t_1^2 + \frac{a}{6}t_1^3 + \frac{b}{8}t_1^4 + N_3 t_1^{\beta+4} + N_4 t_1^{\beta+3} \\ + \frac{\alpha b Z_{13}}{(\beta+2)}t_1^{\beta+2} + \frac{a\alpha Z_{13}}{\beta+1}t_1^{\beta+1} - N_5 t_1^{2\beta+4} - N_6 t_1^{2\beta+3} \end{array}\right] \quad ...(34)$$

Lost sale amount during $[0, t_1]$ is

$$S_c = D\int_0^{t_1}(t_1 - t)t\,dt = D\left\{\frac{1}{2}t_1^3 - \frac{1}{3}t_1^3\right\} = \frac{D}{6}t_1^3 \quad ...(35)$$

Order quantity is

$$OQ = D\int_0^{t_1} t\,dt + e^{-\alpha t^\beta}\left\{\int_{t_1}^\mu Dt\left(1+\alpha t^\beta\right)dt + D\mu\int_\mu^T\left(1+\alpha t^\beta\right)dt\right\}$$

$$OQ = D\left[\frac{1}{2}t_1^2 + \left(1-\alpha t^\beta\right)\left(\begin{array}{l} \frac{1}{2}\left(\mu^2 - t_1^2\right) + \frac{\alpha}{\beta+2}\left(\mu^{\beta+2} - t_1^{\beta+2}\right) \\ +\mu(T-\mu) + \frac{\alpha\mu}{\beta+1}\left(T^{\beta+1} - \mu^{\beta+1}\right) \end{array}\right)\right] \quad ...(36)$$

Total cost during $[0,T]$ =holding cost + lost sales amount + deterioration cost

$$TC_1'(t_1) = HC + S_c + D_c$$

$$TC_1' = D\begin{bmatrix} N_1 + \mu N_2 - aZ_{13}t_1 - \frac{1}{2}bZ_{13}t_1^2 + \frac{a}{6}t_1^3 + \frac{b}{8}t_1^4 + N_3 t_1^{\beta+4} + N_4 t_1^{\beta+3} + \frac{\alpha b Z_{13}}{(\beta+2)}t_1^{\beta+2} \\ + \frac{a\alpha Z_{13}}{\beta+1}t_1^{\beta+1} - N_5 t_1^{2\beta+4} - N_6 t_1^{2\beta+3} + \frac{1}{6}t_1^3 + e^{-\alpha t_1^\beta}\left(\frac{t_1^2}{2} + \frac{\alpha}{\beta+2}t_1^{\beta+2} + C_2\right) \end{bmatrix} \quad ...(37)$$

6.1.1 Solution:

$$\frac{\partial TC_1'}{\partial t_1} = D\begin{bmatrix} -aZ_{13} - bZ_{13}t_1 + \frac{a}{2}t_1^2 + \frac{b}{4}t_1^3 + N_3(\beta+4)t_1^{\beta+3} + N_4(\beta+3)t_1^{\beta+2} \\ +\alpha bZ_{13}t_1^{\beta+1} + a\alpha Z_{13}t_1^{\beta} - N_5(2\beta+4)t_1^{2\beta+3} - N_6(2\beta+3)t_1^{2\beta+2} \\ +\frac{1}{2}t_1^2 + (1-\alpha t_1^\beta)(t_1 + \alpha t_1^{\beta+1}) - \alpha\beta t_1^{\beta-1}\left(\frac{t_1^2}{2} + \frac{\alpha}{\beta+2}t_1^{\beta+2} + C_2\right) \end{bmatrix} \quad ...(38)$$

$$\frac{\partial^2 TC_1'}{\partial t_1^2} = D\begin{bmatrix} \frac{3b}{4}t_1^2 + N_3(\beta+4)(\beta+3)t_1^{\beta+2} + N_4(\beta+3)(\beta+2)t_1^{\beta+1} \\ +\alpha bZ_{13}(\beta+1)t_1^{\beta} + a\alpha Z_{13}\beta t_1^{\beta-1} - N_5(2\beta+4)(2\beta+3)t_1^{2\beta+2} \\ -N_6(2\beta+3)(2\beta+2)t_1^{2\beta+1} + (1-\alpha t_1^\beta)(1+\alpha(\beta+1)t_1^\beta) \\ -\alpha\beta t_1^{\beta-1}(t_1 + \alpha t_1^{\beta+1}) - \alpha\beta t_1^{\beta-1}(t_1 + \alpha t_1^{\beta+1}) - bZ_{13} + at_1 \\ -\alpha\beta(\beta-1)t_1^\beta\left(\frac{t_1^2}{2} + \frac{\alpha}{\beta+2}t_1^{\beta+2} + C_2\right) + t_1 \end{bmatrix} \quad ...(39)$$

Main objective to minimize the total relevant cost for the model starting with shortage, the necessary condition to minimize the total relevant cost is

$$\frac{\partial TC_1'}{\partial t_1} = 0$$

, we get

$$D\begin{bmatrix} -aZ_{13} - bZ_{13}t_1 + \frac{a}{2}t_1^2 + \frac{b}{4}t_1^3 + N_3(\beta+4)t_1^{\beta+3} + N_4(\beta+3)t_1^{\beta+2} \\ +\alpha bZ_{13}t_1^{\beta+1} + a\alpha Z_{13}t_1^{\beta} - N_5(2\beta+4)t_1^{2\beta+3} - N_6(2\beta+3)t_1^{2\beta+2} \\ +\frac{1}{2}t_1^2 + (1-\alpha t_1^\beta)(t_1 + \alpha t_1^{\beta+1}) - \alpha\beta t_1^{\beta-1}\left(\frac{t_1^2}{2} + \frac{\alpha}{\beta+2}t_1^{\beta+2} + C_2\right) \end{bmatrix} = 0 \quad ...(40)$$

Using the software Mathematica, we can calculate the optimal value of t_1 by equation (40) and the optimal value $TC_1'(t_1)$ of the total relevant cost is determined by equation (37). The optimal value of t_1 satisfy the sufficient condition for minimizing total relevant cost $TC_1'(t_1)$ is $\frac{\partial^2 TC_1'}{\partial t_1^2} > 0$ ...(41)

The sufficient condition is satisfied.

6.1.2 Numerical Example:
Let us consider

$a = \$1/unit, b = \$0.5/unit, S_c = \$1/unit, d_c = \$5/unit, \alpha = 0.02\, \beta = 2,$
$T = 1 year, D = 100, \delta = 1, \mu = 0.8\, year.$
$Z_{13} = 0.48, Z_{14} = 1.006, C_2 = 0.162, N_1 = 0.35, N_2 = 0.083, N_3 = -0.00042, N_4 = -0.001,$
$N_5 = 0.00000625, N_6 = 0.000014$

Thus, the optimal value of t_1 is $t_1^{'} \to 0.099 < \mu$. The optimal ordering quantity is $OQ = 48.5$. The minimum relevant cost is $TC_1' = 118.4$

6.2 Case (ii) $t_1 \ge \mu$

Therefore, the inventory $I(t)$ is described by the system of differential equations during $[0,T]$:

$$\frac{dq_1(t)}{dt} = -D(t), \quad 0 \le t \le \mu, \quad q(0) = 0 \qquad ...(42)$$

$$\frac{dq_2(t)}{dt} = -D(\mu), \quad \mu \le t \le t_1, \quad q(\mu_-) = q(\mu_+) \qquad ...(43)$$

$$\frac{dq_3(t)}{dt} + \theta q(t) = -D\mu, \quad t_1 \le t \le T, \quad q(T) = 0 \qquad ...(44)$$

From equation (42), we have

$$q_1(t) = -D\int_0^t t\,dt = -\frac{D}{2}t^2, \quad 0 \le t \le \mu \qquad ...(45)$$

From equation (43), we have

$$q_2(t) = -D\int_0^\mu t\,dt - D\mu\int_\mu^t dt = -D\left(\mu t - \frac{1}{2}\mu^2\right), \quad \mu \le t \le t_1 \qquad ...(46)$$

From equation (44), we have

$$q_3(t) = e^{-\alpha t^\beta} D\mu\int_t^T e^{\alpha t^\beta} dt = e^{-\alpha t^\beta} D\mu\left\{M_1 - t - \frac{\alpha}{\beta+1}t^{\beta+1}\right\}, \quad t_1 \le t \le T \qquad ...(47)$$

See Appendix-D

Total deterioration amount during $[t_1, T]$

$$D_c = e^{-\alpha t_1^\beta} D\mu\int_{t_1}^T e^{\alpha t^\beta} dt - (T - t_1)dt = D\mu\left\{\left(1 - \alpha t_1^\beta\right)\left(M_1 - t_1 - \frac{\alpha}{\beta+1}t_1^{\beta+1}\right) - (T - t_1)\right\} ...(48)$$

Total holding cost during $[t_1, T]$ is

$$HC = \int_{t_1}^T (a + bt) q_3(t)\,dt = a\left[\int_{t_1}^T q_3(t)\,dt\right] + b\left[\int_{t_1}^T t q_3(t)\,dt\right]$$

$$HC = D\mu\left[\begin{aligned} &D_1 + a\left(-M_1 t_1 + \frac{1}{2}t_1^2 + D_2 t_1^{\beta+2} + \frac{\alpha M_1}{(\beta+1)}t_1^{\beta+1} - D_3 t_1^{2\beta+2}\right) \\ &+b\left(-\frac{M_1}{2}t_1^2 + \frac{1}{3}t_1^3 + D_4 t_1^{\beta+3} + \frac{\alpha M_1}{(\beta+2)}t_1^{\beta+2} - D_5 t_1^{2\beta+3}\right)\end{aligned}\right] \qquad ...(49)$$

Lost sale amount during $\left[0,t_1\right]$ is

$$S_c=\int_0^{\mu} q_1(t)\,dt+\int_{\mu}^{t_1} q_2(t)\,dt=D\left[\frac{1}{6}\mu^3+\frac{1}{2}\mu t_1^{\,2}-\frac{1}{2}\mu^2 t_1\right] \qquad ...(50)$$

Ordered quantity is

$$OQ=D\int_0^{\mu} t\,dt+D\mu\int_{\mu}^{t_1} dt+e^{-\alpha t_1^{\beta}}D\mu\int_{t_1}^{T} e^{\alpha t^{\beta}}dt$$

$$OQ=D\left[-\frac{1}{2}\mu^2+\mu t_1+e^{-\alpha t_1^{\beta}}\mu\left(\left(T-t_1\right)+\frac{\alpha}{\beta+1}\left(T^{\beta+1}-t_1^{\beta+1}\right)\right)\right] \qquad ...(51)$$

Total cost during $\left[0,T\right]$ = holding cost + lost sale amount + deterioration cost

$$TC_2'(t_1)=HC+S_c+D_c$$

$$TC_2'(t_1)=D\begin{bmatrix}\frac{b\mu}{3}t_1^{\,3}-b\mu D_5 t_1^{\,2\beta+3}+b\mu D_4 t_1^{\,\beta+3}-a\mu D_3 t_1^{\,2\beta+2}+\frac{\alpha^2\mu}{\beta+1}t_1^{\,2\beta+1}\\ +\mu\left(aD_2+\frac{\alpha bM_1}{(\beta+2)}\right)t_1^{\,\beta+2}+\alpha\mu\left(\frac{aM_1}{(\beta+1)}-\frac{1}{\beta+1}+1\right)t_1^{\,\beta+1}\\ -\alpha\mu M_1 t_1^{\,\beta}+\frac{\mu}{2}(1+a-bM_1)t_1^{\,2}-\mu\left(aM_1+\frac{\mu}{2}\right)t_1\\ +\mu\left(\frac{1}{6}\mu^2-T+D_1+M_1\right)\end{bmatrix} \qquad ...(52)$$

6.2.1 Solution:

$$\frac{\partial TC_2'}{\partial t_1}=D\begin{bmatrix}b\mu t_1^{\,2}-b\mu D_5\left(2\beta+3\right)t_1^{\,2\beta+2}+b\mu D_4\left(\beta+3\right)t_1^{\,\beta+2}-a\mu D_3\left(2\beta+2\right)\\ t_1^{\,2\beta+1}+\frac{\alpha^2\mu\left(2\beta+1\right)}{\beta+1}t_1^{\,2\beta}+\mu\left(\beta+2\right)\left(aD_2+\frac{\alpha bM_1}{(\beta+2)}\right)t_1^{\,\beta+1}+\alpha\mu\\ \left(\beta+1\right)\left(\frac{aM_1}{(\beta+1)}-\frac{1}{\beta+1}+1\right)t_1^{\,\beta}-\alpha\mu\beta M_1 t_1^{\,\beta-1}+\mu\left(1+a-bM_1\right)t_1^{\,1}\\ -\mu\left(aM_1+\frac{\mu}{2}\right)\end{bmatrix} \qquad ...(53)$$

$$\frac{\partial^2 TC_2'}{\partial t_1^{\,2}}=D\begin{bmatrix}2b\mu t_1-b\mu D_5\left(2\beta+2\right)\left(2\beta+3\right)t_1^{\,2\beta+1}+b\mu D_4\left(\beta+2\right)\left(\beta+3\right)t_1^{\,\beta+1}\\ -a\mu D_3\left(2\beta+1\right)\left(2\beta+2\right)t_1^{\,2\beta}+\frac{2\alpha^2\mu\beta\left(2\beta+1\right)}{\beta+1}t_1^{\,2\beta-1}+\mu\left(\beta+1\right)\left(\beta+2\right)\\ \left(aD_2+\frac{\alpha bM_1}{(\beta+2)}\right)t_1^{\,\beta}+\alpha\mu\beta\left(\beta+1\right)\left(\frac{aM_1}{(\beta+1)}-\frac{1}{\beta+1}+1\right)t_1^{\,\beta-1}\\ -\alpha\mu\beta\beta-1M_1 t_1^{\,\beta-2}+\mu\left(1+a-bM_1\right)\end{bmatrix} \qquad ...(54)$$

Main objective to minimize the total relevant cost for the model starting with shortage, the necessary condition to minimize the total relevant cost is

$\frac{\partial TC_2'}{\partial t_1}=0$, we get

$$D\begin{bmatrix} b\mu t_1^2 - b\mu D_5(2\beta+3)t_1^{2\beta+2} + b\mu D_4(\beta+3)t_1^{\beta+2} - a\mu D_3(2\beta+2)t_1^{2\beta+1} \\ +\frac{\alpha^2\mu(2\beta+1)}{\beta+1}t_1^{2\beta} + \mu(\beta+2)\left(aD_2+\frac{\alpha bM_1}{(\beta+2)}\right)t_1^{\beta+1} \\ +\alpha\mu(\beta+1)\left(\frac{aM_1}{(\beta+1)}-\frac{1}{\beta+1}+1\right)t_1^{\beta} - \alpha\mu\beta M_1 t_1^{\beta-1} + \mu(1+a-bM_1)t_1^1 \\ -\mu\left(aM_1+\frac{\mu}{2}\right) \end{bmatrix} = 0 \quad ...(55)$$

Using the software Mathematica, we can calculate the optimal value of t_1 by equation (55) and the optimal value $TC_2'(t_1)$ of the total relevant cost is determined by equation (52). The optimal value of t_1 satisfy the sufficient condition for minimizing total relevant cost $TC_2'(t_1)$ is $\frac{\partial^2 TC_2'}{\partial t_1^2}>0$ ...(56)

The sufficient condition is satisfied.

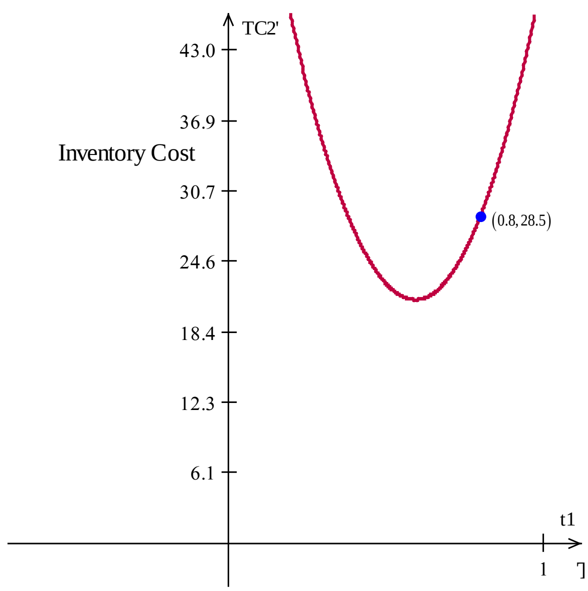

Figure 2: Graphical representation of total relevant cost with shortage.

6.2.2 Numerical Example:

Let us consider

$a=\$1/unit, b=\$0.5/unit, S_c=\$1/unit, d_c=\$5/unit, \alpha=0.02\ \beta=2$

$T=1year, D=100, \delta=1, \mu=0.8\,year.$

$M_1=1.007, D_1=0.587, D_2=-0.003, D_3=0.00002, D_4=-0.0026, D_5=0.000019$

We find $t_1 \to 0.62$, which does not satisfy $t_1 \geq \mu$, so the maximum point is max $\{t_1, \mu\} = \mu$. Thus, the optimal value of t_1 is $t_1^{'} \to 0.8 = \mu$. The optimal ordering quantity is $OQ' = 48.05$. The minimum relevant cost is $TC_2' = 28.5$

6.2.3 Sensitivity Analysis:

To know, how the optimal solution is affected by the values of parameters, we derive the sensitivity analysis for some parameters. The particular values of some parameters are increased or decreased by $+25\%$, -25% and $+50\%$, -50%. After that, we derive the value of t_1 and TC_1 with the help of increased or decreased values of S_c and d_c. The result of the minimum relevant cost is existing in the following table 2.

Table: 2

Parameters	Actual Values	+50% Increased	-50% Decreased	+25% Increased	-25% Decreased
S_c	1	1.5	0.5	1.25	0.75
d_c	5	7.5	2.5	6.25	3.75
t_1^*	0.8	0.8	0.8	0.8	0.8
TC_1^*	28.5	41.5	15.5	35.0	22.0
OQ^*	48.05	48.05	48.05	48.05	48.05

From the result of above table, we observe that total relevant cost and ordering quantity is much affected by deterioration cost and shortage cost, other parameters are less sensitive.

VII. Conclusion

We developed order level inventory models with time varying holding cost for weibull deteriorating items. We considered ramp-type demand rate and holding cost is not constant varying over time. The deteriorating item is deteriorated with two parameters weibull distribution deterioration. The models are developed under two different policies (i) without shortage and (ii) with shortage, which is completely backlogged. The total relevant cost with constant holding cost is less than the total relevant cost with time varying holding cost. The total inventory cost with constant holding cost is more realistic than the total inventory cost with time varying holding cost. Advance research in this way can be carried out such as stochastic demand and finite replenishment rate, quantity discounts and permissible delay in payments.

Appendix-A:

Where,

$$P = -\frac{\alpha^2 b}{2(\beta+2)^2}, R = \frac{\alpha b}{2(\beta+2)} - \frac{b\alpha}{(\beta+2)(\beta+4)} - \frac{\alpha b}{2(\beta+2)} + \frac{\alpha b}{2(\beta+4)}$$

$$Q = \frac{a\alpha^2}{(\beta+2)(2\beta+3)} - \frac{a\alpha^2}{(\beta+1)(\beta+2)}, S = \frac{a\alpha}{\beta+2} - \frac{a\alpha}{(\beta+2)(\beta+3)} - \frac{a\alpha}{2(\beta+1)} + \frac{a\alpha}{2(\beta+3)}$$

$$A_1 = \mu\left(3T^2 - 3T\mu + \mu^2\right), \quad A_2 = \frac{D}{2}\mu^2 + D\mu(T-\mu)$$

Appendix-B:

Where,

$$S = \frac{\mu^2}{2} + \frac{\alpha}{\beta+2}\mu^{\beta+2} - \frac{\alpha\mu^{\beta+2}}{\beta+1}$$

$$\delta_1 = \frac{\mu^3}{3} + \frac{\alpha(\beta+2)\mu^{\beta+3}}{(\beta+1)(\beta+3)} - \frac{(\beta+2)\alpha^2\mu^{2\beta+3}}{(\beta+1)(2\beta+3)}, \gamma = \mu^2 - \frac{\alpha\mu^{\beta+2}}{(\beta+1)}, \varepsilon = \frac{\mu^3}{2} - \frac{\alpha\mu^2}{\beta+1}$$

$$+\frac{\alpha\mu^{\beta+3}}{(\beta+1)(\beta+2)}, \psi = \frac{2\alpha\mu}{\beta+2} - \frac{\alpha\mu}{(\beta+1)(\beta+2)} - \frac{\alpha\mu}{\beta+1}, \eta = \frac{\alpha^2\mu^{\beta+2}}{(\beta+1)^2} - \frac{\alpha\mu^2}{\beta+1},$$

$$\kappa = \frac{\alpha^2\mu}{(\beta+1)(2\beta+2)} - \frac{\alpha^2\mu}{(\beta+1)^2}\pi = \frac{\mu^4}{8} + \frac{\alpha\beta\mu^{\beta+4}}{2(\beta+2)(\beta+4)} - \frac{\alpha^2\mu^{2\beta+4}}{2(\beta+2)^2} - \left(\frac{\mu^3}{2} - \frac{\alpha\mu^{\beta+3}}{(\beta+2)}\right)$$

$$\left(\mu + \frac{\alpha\mu^{\beta+1}}{\beta+1}\right), \vartheta = \frac{\alpha^2\mu^{\beta+3}}{(\beta+1)(\beta+2)} - \frac{\alpha\mu^3}{(\beta+1)}\rho = \frac{\mu^3}{2} - \frac{\alpha\mu^{\beta+3}}{(\beta+2)}, \xi = \frac{\mu^4}{3} + \frac{\alpha\mu^{\beta+4}}{(\beta+1)(\beta+3)}$$

$$-\frac{\alpha\mu^{\beta+4}}{\beta+3} - \frac{\alpha^2\mu^{2\beta+4}}{(\beta+1)(2\beta+3)}\tau = \frac{\alpha\mu}{(\beta+1)}\left(\frac{1}{2} - \frac{\alpha}{\beta+2} + \frac{\alpha-1}{\beta+3}\right),$$

$$\Omega = \frac{\alpha^2\mu}{(\beta+1)(2\beta+3)} - \frac{\alpha^2\mu}{(\beta+1)(\beta+2)}$$

$$\Re = \delta_1 - \gamma\mu - \frac{\alpha\mu^{\beta+1}\gamma}{\beta+1} + \varepsilon, \Pi = \frac{\alpha\gamma}{\beta+1} + \eta, \ \Upsilon = \frac{\alpha\rho}{\beta+1} + \vartheta, \Psi = \pi + \xi$$

$$Z_{12} = \frac{\mu T^2}{2} - \frac{\mu^2}{2} + \frac{\alpha\mu^{\beta+2}}{\beta+2}$$

Appendix-C:

Where

$$Z_{13} = \frac{1}{2}\mu^2 + \frac{\alpha}{\beta+2}\mu^{\beta+2} + \mu\left\{(T-\mu) + \frac{\alpha}{\beta+1}\left(T^{\beta+1} - \mu^{\beta+1}\right)\right\}$$

$$Z_{14} = T + \frac{\alpha}{\beta+1}T^{\beta+1}, \quad C_2 = \mu\left(T - \mu + \frac{\alpha}{\beta+1}\left(T^{\beta+1} - \mu^{\beta+1}\right)\right)$$

$$N_1 = \begin{bmatrix} aZ_{13}\mu - \frac{a\mu^3}{6} - \frac{a\alpha\mu^{\beta+3}}{(\beta+2)(\beta+3)} - \frac{a\alpha Z_{13}\mu^{\beta+1}}{\beta+1} + \frac{a\alpha\mu^{\beta+3}}{2(\beta+3)} + \frac{a\alpha^2\mu^{2\beta+3}}{(\beta+2)(2\beta+3)} \\ + \frac{bZ_{13}\mu^2}{2} - \frac{b\mu^4}{8} - \frac{\alpha b\mu^{\beta+4}}{(\beta+2)(\beta+4)} - \frac{\alpha bZ_{13}\mu^{\beta+2}}{(\beta+2)} + \frac{\alpha b\mu^{\beta+4}}{2(\beta+4)} + \frac{b\alpha^2\mu^{2\beta+4}}{(\beta+2)(2\beta+4)} \end{bmatrix}$$

$$N_2 = \begin{bmatrix} aZ_{14}(T-\mu) - \frac{a\alpha\left(T^{\beta+2} - \mu^{\beta+2}\right)}{(\beta+1)(\beta+2)} - \frac{a\alpha Z_{14}\left(T^{\beta+1} - \mu^{\beta+1}\right)}{(\beta+1)} + \frac{a\alpha\left(T^{\beta+2} - \mu^{\beta+2}\right)}{(\beta+2)} \\ + \frac{a\alpha^2\left(T^{2\beta+2} - \mu^{2\beta+2}\right)}{(\beta+1)(2\beta+2)} + \frac{bZ_{14}\left(T^2 - \mu^2\right)}{2} - \frac{b\left(T^3 - \mu^3\right)}{3} - \frac{b\alpha\left(T^{\beta+3} - \mu^{\beta+3}\right)}{(\beta+1)(\beta+3)} \\ - \frac{\alpha bZ_{14}\left(T^{\beta+2} - \mu^{\beta+2}\right)}{\beta+2} + \frac{\alpha b\left(T^{\beta+3} - \mu^{\beta+3}\right)}{(\beta+3)} + \frac{b\alpha^2\left(T^{2\beta+3} - \mu^{2\beta+3}\right)}{(\beta+1)(2\beta+3)} - \frac{a\left(T^2 - \mu^2\right)}{2} \end{bmatrix}$$

$$N_3 = \frac{\alpha b}{(\beta+2)(\beta+4)} - \frac{\alpha b}{2(\beta+4)}, N_4 = \frac{a\alpha}{(\beta+2)(\beta+3)} - \frac{a\alpha}{2(\beta+3)}, N_5 = \frac{b\alpha^2}{(\beta+2)(2\beta+4)}, N_6 = \frac{a\alpha^2}{(\beta+2)(2\beta+3)}$$

See Appendix-D:

Where, $M_1 = T + \frac{\alpha}{\beta+1} T^{\beta+1}$

$$D_1 = a\left\{ M_1 T - \frac{1}{2}T^2 - \frac{\alpha}{(\beta+1)(\beta+2)}T^{\beta+2} - \frac{\alpha M_1}{(\beta+1)}T^{\beta+1} + \frac{\alpha}{(\beta+2)}T^{\beta+2} + \frac{\alpha^2}{(\beta+1)(2\beta+2)}T^{2\beta+2} \right\} +$$
$$b\left\{ \frac{M_1}{2}T^2 - \frac{1}{3}T^3 - \frac{\alpha}{(\beta+1)(\beta+3)}T^{\beta+3} - \frac{\alpha M_1}{(\beta+2)}T^{\beta+2} + \frac{\alpha}{(\beta+3)}T^{\beta+3} + \frac{\alpha^2}{(\beta+1)(2\beta+3)}T^{2\beta+3} \right\}$$

$$D_2 = \frac{\alpha}{(\beta+1)(\beta+2)} - \frac{\alpha}{(\beta+2)}, D_3 = \frac{\alpha^2}{(\beta+1)(2\beta+2)}, D_4 = \frac{\alpha}{(\beta+1)(\beta+3)} - \frac{\alpha}{(\beta+3)},$$

$$D_5 = \frac{\alpha^2}{(\beta+1)(2\beta+3)}$$

ACKNOWLEDGMENT

The authors would like to very thankful to the referees for their valuable guidance.

[1] R.P. Covert, and G.C. Philip, "An EOQ model for items with Weibull distribution deterioration," AIEE Transactions, 5, 1973, 323-326.

[2] K.S.Wu, and L.Y. Ouyang, "A replenishment policy for deteriorating items with ramp type demand rate," Proc, Natt. Sci Counc.ROC (A), 24, 2000, 279-286.

[3] K.S.Wu, "An EOQ inventory model for items with Weibull distribution deterioration, ramp type demand rate and partial backlogging," Production Planning and Control, 12, 2001, 787-793.

[4] P.M. Ghare, and G.H.Schrader, "An inventory model for exponentially deteriorating items," Journal of industrial Engg. 14, 1963, 238-243.

[5] S.K. Goyal, and, B.C. Giri, "Recent trends in modeling of deteriorating inventory," European Journal of Operational Research, 134, 2001, 1-16.

[6] E.A. Silver, "A simple inventory replenishment decision rule for a linear trend in demand," J. Oper. Res. Q, 30, 1979, 71-75.

[7] S.K.Ghosh, and K.S. Chaudhuri, "An order-level inventory model for a deteriorating item with Weibull distribution deterioration, time quadratic demand and shortages," Adv. Model. Optim. 6(1), 1979, 21-35.

[8] S.S, Sanni, "An economic order quantity inventory model with time dependent weibull deterioration and trended demand," M.Sc. thesis, University of Nigeria, 2012.

[9] B.Mandal, and A.K. Pal, "Order level inventory system with ramp type demand rate for deteriorating items," Journal of Interdisciplinary Mathematics, 1, 1998, 49-66.

[10] E. Naddor, "Inventory System," John Wiley and Sons, New York, 1966.

[11] A. Roy, "An inventory model for deteriorating items with price dependent demand and time varying holding cost," Advanced Modeling and Optimization, 10, 2008, 25-37.

[12] R. P. Tripati, "Inventory model with cash flow oriented and time-dependent holding cost under permissible delay in payments," Yugoslav Journal of Operations Research, 23(1), 2013.

[13] B. Van deer Veen, "Introduction to the Theory of Operational Research," Philip Technical Library, Springer-Verlag, New York, 1967.

[14] H.J. Weiss, "EOQ models with non-linear holding cost," European Journal of Operational Research, 9, 1982, 56-60.

[15] S. Jain, and M. Kumar, "An EOQ inventory model for items with ramp-type demand three parameter Weibull distribution deterioration and starting with shortage," Yugoslav Journal of Operations Research, 20(2), 2010, 249-259.

[16] A.K. Jalan, B.C. Giri, and K.S.Chadhuri, "EOQ model for items with weibull distribution deterioration, shortages and ramp type demand," Recent Development in Operations Research, Narosa Publishing House, New Delhi, India, 2001, 207-223.

[17] S. Pal, A. Goswami, and K. S. Chaudhuri, "A deterministic inventory model for deteriorating items with stock-dependent demand rate," International Journal of Production Economics, 32(3), 1993, 291-299.

[18] G.P.Samanta, and A. Roy, "A Production inventory model with deteriorating items and shortages," Yugoslav Journal of Operation Research, 14(2), 2004, 219-230.

[19] K.Skouri, and I.Konstantaras, "Order level inventory models for deteriorating seasonable products with time dependent demand and shortages," Hindawi Publishing Corporation Mathematical Problems in Engineering article ID 679736, 2009.

[20] K.Skouri, and I.Konstantaras, S.Papachristos, and I.Ganas, "Inventory models with ramp type demand rate, partial backlogging deteriorating and weibull deterioration rate," European Journal of Operational Research, 192, 2009, 79-92.

[21] A.K. Jalan, R.R. Giri, and K.S. Chaudhuri, "EOQ model for items with weibull distribution deterioration, shortages and trended demand," International Journal of System Science, 27, 1996, 851-856.

[22] S.Jain, and M. Kumar, "An EOQ inventory model with ramp-type demand, Weibull distribution deterioration and starting with shortage," Opsearch 44, 2007, 240-250.

[23] B. Karmakar, and K.D. Choudhury, "Inventory models with ramp-type demand rate for deteriorating items with partial backlogging and time-varying holding cost," Yugoslav Journal of Operations Research, 24, Number 2, 2014, 249-266.

AUTHORS

First Author – Ritu Raj, Research Scholar, Centre for Mathematical Sciences, Banasthali University, P.O. Banasthali Vidyapith, Banasthali – 304022, Rajasthan, India, e-mail: rrituiit@gmail.com, Mob:09711717006.
Second Author – Naresh Kumar Kaliraman, Research Scholar, Centre for Mathematical Sciences, Banasthali University, P.O. Banasthali Vidyapith, Banasthali – 304022, Rajasthan, India,e-mail:dr.nareshkumar123@gmail.com, Mob:09891008069.
Third Author – Dr Shalini Chandra, Associate Professor, Centre for Mathematical Sciences, Banasthali University, P.O. Banasthali Vidyapith, Banasthali – 304022, Rajasthan, India, e-mail: chandrshalini@gmail.com, Mob: 07891434849.
Fourth Author – Dr Harish Chaudhary, Assistant Professor, Department of Management Studies, Indian Institute of Technology, Delhi, New Delhi-110016, India, e-mail: hciitd@gmail.com, Mob: 09911555875.