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State Preference Theorem: Describing the Traditional Model

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Abstract- This paper reviews the State preference theory from a traditional perspective. It explains the model under which the state preference theory and gives rigorous explanations of the assumptions and their normative explanations. The paper goes further to reflect its focus on the state preference framework and also explains the aspects of state prices and their relation to aggregate wealth. The paper gives a technical view of risk neutral valuation in relation to state preference theorem, insights that allows for transforming of the state prices into a common discount factor known as risk neutral probabilities. The paper looks touches on a few criticisms and finally concludes that the model provides an elegant and general framework for the analysis of financial markets and yields a pricing rule for securities. The state preference theory is remarkably acknowledged for providing useful addition knowledge to a financial economist's toolkit and with a basic understanding of financial markets and prices that forms the bread and butter of financial management candidature.

Index Terms- State Preference, State Prices, risk neutral, valuation

I. Introduction

1.1 Background introduction

The state-preference approach to uncertainty was introduced by Arrow (1953) and further detailed by Debreu (1959). It was made famous in the late 1960s, with the work of Hirshleifer (1965, 1966) in the theory of investment after which the model found more and more emerging candidates.

According to Copeland, Weston and Shastri (2005) the candidates of asset pricing theory all have one central question: how do individuals allocate scarce resources through a price system based on the valuation of risky assets?

Hirshleifer (1966) asserts that the state preference approach resolves the assets or securities into distributions of dated contingent claims to income defined over the set of all possible states of the world.

The basic proposition of the state-preference approach to uncertainty is that commodities can be differentiated not only by their physical properties and location in space and time but also by their location in "state". By this, it for instance, means that "ice cream when it is raining" is a *different* commodity than "ice cream when it is sunny" and thus are treated differently by agents and can command different prices. Choice is an act, whereas preferences are a state of mind.

In real world, people do not necessarily rank or order their preferences in a consistent way. In preference theory, some idealized conditions are regularly imposed on the preferences of economic actors. One of the most important of these idealized conditions is the *axiom of transitivity*

For preference theory to be useful mathematically, there needs to be an assumption of continuity. Continuity simply means that there are no 'jumps' in people's preferences: if one prefer very large oranges to apples, he/she will prefer large oranges to apples as well. The continuity assumption is "too strong" in the sense that it indeed guarantees the existence of a *continuous utility function* representation.

1.2 Assumptions and their Normative descriptions

1.2.1 Assumptions

The basic theoretical assumptions of state preference theory are enlisted below;

i) The consumer's preferences are independent of prices or other changes
ii) Order-theoretic: acyclicity, transitivity, the semi order property, completeness
iii) Topological: continuity, openness or closedness of the preference sets
iv) Linear-space: convexity, homogeneity, translation-invariance

1.2.2 Normative interpretations of assumptions (axioms)

The axioms are an attempt to model the decision maker's preferences, not over the actual choice, but over the type of desirable procedure (a procedure that any human being would like to follow).

One crucial assumption in the Arrow-Debreu world is the completeness of the market. The market is said to be complete if every payoff structure is achievable, i.e. if the asset's returns span the s states. Formally completeness is achieved if every ADS $\underline{e}_s$ can be constructed through a portfolio $\underline{x}_s$. That means $D\underline{x}_s = \underline{e}_s$, s = 1, · · · , S.

Another key assumption which has to be made is the No-Arbitrage profit condition. Interestingly, this condition was stated as a "by-product" in the original works e.g. a necessary condition in relation to the single-price law of markets, as in Hirshleifer [1966], or excluded through assumptions about the prices of ADS, as in Arrow [1953]. At that time nobody thought of Arbitrage itself as a powerful tool for the valuation of assets and a basis for sophisticated asset pricing theories on its own. In modern textbooks which start with the introduction of an Arrow-Debreu-like financial market the Arbitrage argument is followed more elaborately and conclusions are made that go far beyond the original works of SPT. Consumers whose preference structures violate transitivity would get exposed to being milked by some unscrupulous person.

The axiom of completeness implies that some choice will be made, an assertion that is more philosophically questionable. In most applications, the set of consumption alternatives is infinite and the consumer is not conscious of all preferences. For example, one does not have to choose over going on holiday by plane or by train: if one does not have enough money to go on holiday anyway then it is not necessary to attach a preference order to those alternatives. However, preference can be interpreted as a hypothetical choice that could be made rather than a conscious state of mind. In this case, completeness amounts to an assumption that the consumers can always make up their mind whether they are indifferent or prefer one option when presented with any pair of options.

In extremes, there is no "rational" choice available. For instance, if asked to choose which one of one's children will be killed, as in, there is no rational way out of it. In that case preferences would be incomplete, since "not being able to choose" is not the same as "being indifferent".

2. The state preference model

2.1 State Preference Framework (SPT)

The SPT framework features two points in time: t_o as today and t_1 as tomorrow. Trading and portfolio optimization only occur in t_o. The uncertainty in this framework is characterized through various mutually exclusive and exhaustive future states that can occur at time t_1 from the finite set $\Omega = \{w_1, ... w_s\}$ with cardinality S. The investor might know the different probabilities of the states, but he does not know which one is going to occur. Securities can therefore be seen as a set of possible payoffs each occurring in a mutually exclusive state of nature. Mathematically speaking, they can be represented as a vector 1 of state contingent claims or as a random variable. Represented as vectors, securities assign a payoff to every possible state w_s:

$$\underline{a}_j = \begin{pmatrix} a_j(w_1) \\ \bullet \\ \bullet \\ \bullet \\ a_j(w_s \end{pmatrix}$$

At time t_0 it is not known which state will occur, but the individuals know each possible payoff. The set A = $\{a_1 ... a_j\}$ represents the securities and has cardinality J. At time t_0 the prices of the existing securities are given by the vector

$$\underline{p} = \begin{pmatrix} p_1 \\ \bullet \\ \bullet \\ \bullet \\ p_j \end{pmatrix}$$

Where: each p_j is the price of a security $\underline{a}_j$. An important concept to be introduced are the

Arrow-Debreu-Securities, ADS henceforth, $(\underline{e}_1, \underline{e}_2, ..., \underline{e}_s)$. Those securities yield a payoff of one monetary unit in a certain state s and zero otherwise .

$$\underline{e}_s = \begin{pmatrix} 0 \\ \bullet \\ \bullet \\ \bullet \\ 1 \\ \bullet \\ \bullet \\ \bullet \\ 0 \end{pmatrix}$$

This concept allows for the intuitive decomposition of every payoff into a linear combination

of ADS. One can now further examine the array of possible payoff structures. To do so one can condense the elements introduced so far in a S × J payoff matrix D, that can be seen as one of the simplest representations of a financial market.

Each row represents a state and each column represents a security:

$$D = \begin{bmatrix} a_1(w_1) & a_2(w_1) \ldots & a_J(w_1) \\ a_1(w_2) & a_2(w_2) \ldots & a_J(w_2) \\ . & . & . \\ . & . & . \\ . & . & . \\ a_1(w_S) & a_2(w_s) \ldots & a_J(w_s) \end{bmatrix}$$

Furthermore, a portfolio $\underline{x}$ is defined as a linear combination of securities of the following form, where each x_j denotes the number of each security held:

$$\underline{x} = \begin{pmatrix} x_1 \\ \bullet \\ \bullet \\ \bullet \\ x_j \end{pmatrix}$$

2.2 State prices and their relation to aggregate wealth

Dividing the state prices by their respective probabilities of the state occurring one obtains the probability-adjusted willingness to pay. As Dybvig and Ross (2003) argue that the marginal utility of consumption is proportional to the relative scarcity. Defining the state price density as

$$\rho_s = \frac{v_s}{\pi_s}$$

The asset pricing equation can be written as an expected value:

$$p_j = \sum_{s=1}^{S} \pi_S \rho_s a_{sj} = E\left[\overline{\rho}\,\overline{a}_j\right]$$

Dividing the expression above by p_j will yield a more convenient representation for further calculations. By common economic reasoning the quotient $\overline{a}_j / p_j$ denotes one plus the (uncertain) expected return on security j written as $(1+\overline{R}_j)$ [Zimmermann, 1998, p. 39]. Furthermore, $(1+\overline{R}_j)$ will be denoted as $\overline{X}_j$. Thus, one can write:

$$1 = E\left[\overline{\rho}\frac{\overline{a}_j}{p_j}\right] \equiv E\left[\overline{\rho}(1+\overline{R}_j\right] = E\left[\overline{\rho}\,\overline{X}_j\right]$$

Denoting the return on a riskless asset by X = (1 + R_0) it can be proven that the following relationship holds for the expected return on security j:

$$E\left[\overline{X}_j\right] = X - X\,\mathrm{cov}\left[\overline{\rho}, \overline{X}_j\right]$$

Expected return of a security depends on its covariance with the state price density. The more negative the covariance the higher the return. The interpretation is very valuable for our understanding of asset prices: a negative covariance means that asset payoffs are high when the state price density is low (hence, the willingness to pay is low) and vice versa having assumed risk aversion investors to be those are the states of the world where aggregate wealth is high.

Bearing this risk is rewarded with more return. Securities with a high proportion of non-diversifiable risk will have higher expected rates of return. The securities that do not share that economy risk will have lower rates of expected return since they do not involve a lot of risk bearing in terms of aggregate wealth levels.

3. Risk Neutral Valuation

One can construct a security that yields a payoff of one monetary unit in each state, thus, making it risk free. Such a security would be a pure discount bond trading at a risk free interest rate discount. Thus, the sum of the state prices should equal the price of the riskless investment:

$$\sum_{k=1}^{S} V_k = \frac{1}{1+R_0} \quad \text{...} \ (7)$$

This insight allows transforming the state prices into a common discount factor known as risk neutral probabilities. Following Debreu [1959] define ψ as

$$\underline{\psi} = \begin{pmatrix} \psi_1 \\ . \\ . \\ . \\ \Psi_s \end{pmatrix} \equiv (1+R_0)\begin{pmatrix} V_1 \\ . \\ . \\ . \\ V_s \end{pmatrix}$$

Where $\psi_s >> 0$ and $\sum_{s=1}^{S} \psi_s = 1$

The ψ vector can be interpreted as a vector of probabilities since they are all between zero and one and sum to one. Those probabilities are called risk neutral probabilities – of course they are not the "real probabilities", but using them simplifies mathematical finance since one can use the rich mathematical toolkit known from statistics. The value of a cash flow under risk neutral valuation is its expected value under risk neutral probabilities discounted at the risk free rate:

$$p_j = \frac{1}{1+R_0}\sum_{s=1}^{S} a_{sj}\Psi_s = \frac{1}{1+R_o}E^{\Psi}\left[\overline{a}_j\right]$$

Valuing with risk neutral probabilities is different from the "traditional" approach. In the traditional approach the asset j is valued by taking the expected value of the cash flows under statistical probabilities denoted $E^p\left[\overline{a}_j\right]$ and discounting it with a risk adjusted rate of return denoted R_j. Thus, the risk adjustment takes place in the denominator. Under risk neutral probabilities the risk adjustment takes place in the numerator when taking the expected value $E^{\Psi}\left[\overline{a}_j\right]$. To illustrate this mathematically:

$$p_j = \frac{E^p\left[a_j\right]}{1+R_j} \quad \text{or} \quad \frac{E^{\mu}\left[a_j\right]}{1+R_0}$$

4. Criticisms of SPT

Some critics say that rational theories of choice and preference theories rely too heavily on the assumption of invariance, which states that the relation of preference should not depend on the description of the options or on the method of elicitation. Modigliani (1974) concedes that only an "infinite" liquidity preference (an unlimited demand for money) will block return to full-employment equilibrium in a free market.

But, as it be seen, heavy speculative demand for money speeds the adjustment process. Moreover, the demand for money could never be *infinite* because people must always continue consuming, on some level, regardless of their expectations. Since people must continue consuming, they must also continue producing, so that there can be adjustment and full employment regardless of the degree of hoarding.

It is assumed that the consumer's preferences are independent of prices or other changes. This assumption is not realistic. The consumer's preferences are bound to be affected by changes in prices, or say, changes in fashion.

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6. **Conclusion**

The State Preference Theory provides an elegant and general framework for the analysis of financial markets and yields a pricing rule for securities. This so-called state price vector can be inferred from existing security prices in a complete capital market and can value any new security introduced into the market.

The SPT provides us with useful addition knowledge to a financial economist's toolkit and with a basic understanding of financial markets and prices.

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