ISSN 2250-3153

Bounds for the Zeros of Polynomials

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Abstract: In this paper we find bounds for the zeros of a class of polynomials whose coefficients or their real and imaginary parts are restricted to certain conditions. Our results improve and generalize many known results in this direction.

Mathematics Subject Classification: 30C10, 30C15
Keywords and phrases: Coefficient, Polynomial, Zero.

1. Introduction and Statement of Results

The following result, known as the Enestrom-Kakeya Theorem [5] , is well-known in the theory of distribution of zeros of polynomials:

Theorem A: Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n such that

$$a_n \geq a_{n-1} \geq ...... \geq a_1 \geq a_0 > 0 .$$

Then P(z) has all its zeros in the closed unit disk $|z| \leq 1$.

In the literature, there exist several generalizations and extensions of this result. Joyal, Labelle and Rahman [4] extended it to polynomials with general monotonic coefficients by proving the following result:

Theorem B: Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n such that

$$a_n \geq a_{n-1} \geq ...... \geq a_1 \geq a_0 .$$

Then P(z) has all its zeros in the disk

$$|z| \leq \frac{a_n - a_0 + |a_0|}{|a_n|} .$$

Aziz and Zargar [1] relaxed the hypothesis of Theorem A and proved the following result:

Theorem C: Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n such that

$$ka_n \geq a_{n-1} \geq ...... \geq a_1 \geq a_0 .$$

Then P(z) has all its zeros in the disk

$$|z+k-1| \leq \frac{ka_n - a_0 + |a_0|}{|a_n|} .$$

Recently M. H. Gulzar [3] proved the following results:

Theorem D: Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n such that for some $\lambda \neq 1,\ 1 \le k \le n$, $a_{n-k} \neq 0$, $\rho \ge 0, 0<\tau \le 1$,

$$\rho + a_n \ge a_{n-1} \ge ...... \ge a_{n-k+1} \ge \lambda a_{n-k} \ge a_{n-k-1} \ge ...... \ge a_1 \ge \tau a_0 .$$

If $a_{n-k-1} > a_{n-k}$, then all the zeros of P(z) lie in the disk

$$\left| z+\frac{\rho}{a_n} \right| \le \frac{\rho + a_n + (\lambda -1)a_{n-k} + |\lambda -1||a_{n-k}| + 2|a_0| - \tau(a_0 + |a_0|)}{|a_n|},$$

and if $a_{n-k} > a_{n-k+1}$, then all the zeros of P(z) lie in the disk

$$\left| z+\frac{\rho}{a_n} \right| \le \frac{\rho + a_n + (1-\lambda)a_{n-k} + |1-\lambda||a_{n-k}| + 2|a_0| - \tau(a_0 + |a_0|)}{|a_n|}.$$

Remark 1: Under the conditions of the above theorem, it follows that all the zeros of P(z) lie in

$$|z| \le \frac{|\rho| + \rho + a_n + (\lambda -1)a_{n-k} + |\lambda -1||a_{n-k}| + 2|a_0| - \tau(a_0 + |a_0|)}{|a_n|},$$

if $a_{n-k-1} > a_{n-k}$ and in

$$|z| \le \frac{|\rho| + \rho + a_n + (1-\lambda)a_{n-k} + |1-\lambda||a_{n-k}| + 2|a_0| - \tau(a_0 + |a_0|)}{|a_n|},$$

if $a_{n-k} > a_{n-k+1}$.

Theorem E: Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n such that for some $\lambda \neq 1,\ 1 \le k \le n$, $a_{n-k} \neq 0$, $\rho, 0<\tau \le 1$,

$$|\rho + a_n| \ge |a_{n-1}| \ge ...... \ge |a_{n-k+1}| \ge \lambda |a_{n-k}| \ge |a_{n-k-1}| \ge ...... \ge |a_1| \ge \tau |a_0|$$

and for some real α and β,

$$|\arg a_j - \beta| \le \alpha \le \frac{\pi}{2},\ j=0,1,......,n .$$

If $|a_{n-k-1}| > |a_{n-k}|$ (i.e. $\lambda > 1$), then all the zeros of P(z) lie in the disk

$$\left| z+\frac{\rho}{a_n} \right| \le \frac{1}{|a_n|}[(|\rho| + |a_n|)(\cos\alpha + \sin\alpha) - |a_{n-k}|(\cos\alpha - \sin\alpha - \lambda\cos\alpha - \lambda\sin\alpha - \lambda + 1)$$

$$+\tau|a_0|(\sin\alpha - \cos\alpha - 1) + 2|a_0| + 2\sin\alpha \sum_{j=1, j\neq n-k}^{n-1} |a_j|],$$

and if $|a_{n-k}| > |a_{n-k+1}|$ (i.e. $\lambda < 1$), then all the zeros of P(z) lie in the disk

$$\left| z+\frac{\rho}{a_n} \right| \le \frac{1}{|a_n|}[(|\rho| + |a_n|)(\cos\alpha + \sin\alpha) + |a_{n-k}|(\cos\alpha + \sin\alpha - \lambda\cos\alpha + \lambda\sin\alpha - \lambda + 1)$$

$$+\tau|a_0|(\sin\alpha - \cos\alpha - 1) + 2|a_0| + 2\sin\alpha \sum_{j=1, j\neq n-k}^{n-1} |a_j|].$$

Remark 2: Under the conditions of the above theorem, it follows that all the zeros of P(z) lie in

$$|z| \le \frac{1}{|a_n|}[|\rho| + (|\rho| + |a_n|)(\cos\alpha + \sin\alpha) - |a_{n-k}|(\cos\alpha - \sin\alpha - \lambda\cos\alpha - \lambda\sin\alpha - \lambda + 1)$$

$$+\tau|a_0|(\sin\alpha - \cos\alpha - 1) + 2|a_0| + 2\sin\alpha \sum_{j=1, j\ne n-k}^{n-1}|a_j|],$$

if $|a_{n-k-1}| > |a_{n-k}|$ and in

$$|z| \le \frac{1}{|a_n|}[|\rho| + (|\rho| + |a_n|)(\cos\alpha + \sin\alpha) + |a_{n-k}|(\cos\alpha + \sin\alpha - \lambda\cos\alpha + \lambda\sin\alpha - \lambda + 1)$$

$$+\tau|a_0|(\sin\alpha - \cos\alpha - 1) + 2|a_0| + 2\sin\alpha \sum_{j=1, j\ne n-k}^{n-1}|a_j|],$$

if $|a_{n-k}| > |a_{n-k+1}|$.

The aim of this paper is to generalize the above results. More precisely, we are going to prove the following results:

Theorem 1: Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with $\operatorname{Re}(a_j) = \alpha_j, \operatorname{Im}(z) = \beta_j, j = 0,1,......,n$ such that for some ρ, σ $, 0 < \tau \le 1$, $1 \le k \le n$, $\alpha_{n-k} \ne 0$,

$$\rho + \alpha_n \ge \alpha_{n-1} \ge ...... \ge \alpha_{n-k+1} \ge \sigma + \alpha_{n-k} \ge \alpha_{n-k-1} \ge ...... \ge \alpha_1 \ge \tau\alpha_0 .$$

If $\alpha_{n-k-1} > \alpha_{n-k}$, then all the zeros of P(z) lie in $r_1 \le |z| \le R_1$, where

$$r_1 = \frac{|a_0|}{|a_n| + |\rho| + \rho + \alpha_n + |\sigma| + \sigma - \tau(|\alpha_0| + \alpha_0) + |\alpha_0| + |\beta_0| + |\beta_n| + 2\sum_{j=1}^{n-1}|\beta_j|},$$

and

$$R_1 = \frac{|\rho| + \rho + \alpha_n + \sigma + |\sigma| - \tau(|\alpha_0| + \alpha_0) + 2|\alpha_0| + |\beta_0| + |\beta_n| + 2\sum_{j=1}^{n-1}|\beta_j|}{|a_n|},$$

and if $\alpha_{n-k} > \alpha_{n-k+1}$, then all the zeros of P(z) lie in $r_2 \le |z| \le R_2$
where

$$r_2 = \frac{|a_0|}{|a_n| + |\rho| + \rho + \alpha_n + |\sigma| - \sigma - \tau(|\alpha_0| + \alpha_0) + |\alpha_0| + |\beta_0| + |\beta_n| + 2\sum_{j=1}^{n-1}|\beta_j|},$$

and

$$R_2 = \frac{|\rho| + \rho + \alpha_n + |\sigma| - \sigma - \tau(|\alpha_0| + \alpha_0) + 2|\alpha_0| + |\beta_0| + |\beta_n| + 2\sum_{j=1}^{n-1}|\beta_j|}{|a_n|}.$$

If a_j are real i.e. $\beta_j = 0, j = 0,1,......,n,$ Theorem 1 immediately gives the following result, which reduces to Theorem D by taking $\sigma = (\lambda - 1)a_{n-k}$:

Corollary 1: Let $P(z)=\sum_{j=0}^{n}a_jz^j$ be a polynomial of degree n such that for some ρ,σ $,0<\tau\leq 1, 1\leq k\leq n$, $a_{n-k}\neq 0$,

$$\rho+a_n\geq a_{n-1}\geq ......\geq a_{n-k+1}\geq \sigma+a_{n-k}\geq a_{n-k-1}\geq ......\geq a_1\geq \tau a_0 .$$

If $a_{n-k-1}>a_{n-k}$, then all the zeros of P(z) lie in $r_1\leq |z|\leq R_1$, where

$$r_1=\frac{|a_0|}{|a_n|+|\rho|+\rho+a_n+|\sigma|+\sigma-\tau(|a_0|+a_0)+|a_0|},$$

and

$$R_1=\frac{|\rho|+\rho+a_n+\sigma+|\sigma|-\tau(|a_0|+a_0)+2|a_0|}{|a_n|},$$

and if $a_{n-k}>a_{n-k+1}$, then all the zeros of P(z) lie in $r_2\leq |z|\leq R_2$,where

$$r_2=\frac{|a_0|}{|a_n|+|\rho|+\rho+a_n+|\sigma|-\sigma-\tau(|a_0|+a_0)+|a_0|},$$

and

$$R_2=\frac{|\rho|+\rho+a_n+|\sigma|-\sigma-\tau(|a_0|+a_0)+2|a_0|}{|a_n|}.$$

Remark 3: Taking $\rho=0,\sigma=0$ $,\tau=1$ and $a_0>0$, Cor.1 immediately gives Theorem A.

For different values of the parameters ρ,σ $,\tau$ and k, we get many other interesting results. For example, if we take $\rho=0$, we get the following result:

Corollary 2: Let $P(z)=\sum_{j=0}^{n}a_jz^j$ be a polynomial of degree n with $\mathrm{Re}(a_j)=\alpha_j, \mathrm{Im}(z)=\beta_j, j=0,1,......,n$ such that for some, σ $,0<\tau\leq 1$, $1\leq k\leq n$, $\alpha_{n-k}\neq 0$,

$$\alpha_n\geq \alpha_{n-1}\geq ......\geq \alpha_{n-k+1}\geq \sigma+\alpha_{n-k}\geq \alpha_{n-k-1}\geq ......\geq \alpha_1\geq \tau\alpha_0 .$$

If $\alpha_{n-k-1}>\alpha_{n-k}$, then all the zeros of P(z) lie in $r_1\leq |z|\leq R_1$, where

$$r_1=\frac{|a_0|}{|a_n|+\alpha_n+|\sigma|+\sigma-\tau(|\alpha_0|+\alpha_0)+|\alpha_0|+|\beta_0|+|\beta_n|+2\sum_{j=1}^{n-1}|\beta_j|},$$

and

$$R_1=\frac{\alpha_n+\sigma+|\sigma|-\tau(|\alpha_0|+\alpha_0)+2|a_0|+|\beta_0|+|\beta_n|+2\sum_{j=1}^{n-1}|\beta_j|}{|a_n|},$$

and if $\alpha_{n-k}>\alpha_{n-k+1}$, then all the zeros of P(z) lie in $r_2\leq |z|\leq R_2$
where

$$r_2=\frac{|a_0|}{|a_n|+\alpha_n+|\sigma|-\sigma-\tau(|\alpha_0|+\alpha_0)+|\alpha_0|+|\beta_0|+|\beta_n|+2\sum_{j=1}^{n-1}|\beta_j|},$$

and

$$R_2=\frac{\alpha_n+|\sigma|-\sigma-\tau(|\alpha_0|+\alpha_0)+2|\alpha_0|+|\beta_0|+|\beta_n|+2\sum_{j=1}^{n-1}|\beta_j|}{|a_n|}.$$

If a_j are real i.e. $\beta_j=0, j=0,1,......,n,$ Cor.2 gives the following result:

Corollary 3: Let $P(z)=\sum_{j=0}^{n}a_j z^j$ be a polynomial of degree n such that for some σ $,0<\tau\le 1,1\le k\le n$, $a_{n-k}\ne 0$,

$$a_n\ge a_{n-1}\ge......\ge a_{n-k+1}\ge\sigma+a_{n-k}\ge a_{n-k-1}\ge......\ge a_1\ge\tau a_0.$$

If $a_{n-k-1}>a_{n-k}$, then all the zeros of P(z) lie in $r_1\le|z|\le R_1$, where

$$r_1=\frac{|a_0|}{|a_n|+a_n+|\sigma|+\sigma-\tau(|a_0|+a_0)+|a_0|},$$

and

$$R_1=\frac{a_n+\sigma+|\sigma|-\tau(|a_0|+a_0)+2|a_0|}{|a_n|},$$

and if $a_{n-k}>a_{n-k+1}$, then all the zeros of P(z) lie in $r_2\le|z|\le R_2$
where

$$r_2=\frac{|a_0|}{|a_n|+a_n+|\sigma|-\sigma-\tau(|a_0|+a_0)+|a_0|},$$

and

$$R_2=\frac{a_n+|\sigma|-\sigma-\tau(|a_0|+a_0)+2|a_0|}{|a_n|}.$$

Theorem 2: Let $P(z)=\sum_{j=0}^{n}a_j z^j$ be a polynomial of degree n such that for some $\lambda\ne 1,\ 1\le k\le n$, $a_{n-k}\ne 0$, $\sigma,\rho,0<\tau\le 1$,

$$|\rho+a_n|\ge|a_{n-1}|\ge......\ge|a_{n-k+1}|\ge\sigma+|a_{n-k}|\ge|a_{n-k-1}|\ge......\ge|a_1|\ge\tau|a_0|$$

and for some real α and β,

$$|\arg a_j-\beta|\le\alpha\le\frac{\pi}{2}, j=0,1,......,n.$$

If $|a_{n-k-1}|>|a_{n-k}|$, then all the zeros of P(z) lie in $r_3\le|z|\le R_3$,

where $r_3=\dfrac{|a_0|}{(|a_n|+|\rho|)(\cos\alpha+\sin\alpha+1)+|\sigma|(\cos\alpha+\sin\alpha+1)-\tau|a_0|(\cos\alpha-\sin\alpha+1)+|a_0|+2\sin\alpha\sum_{j=1}^{n-1}|a_j|}$ and

$$R_3=\frac{1}{|a_n|}[(|\rho|+|a_n|)(\cos\alpha+\sin\alpha)+|\sigma|(\cos\alpha+\sin\alpha+1)-\tau|a_0|(\cos\alpha-\sin\alpha+|a_0| +2\sin\alpha\sum_{j=1}^{n-1}|a_j|+|\rho|]$$

and if $|a_{n-k}| > |a_{n-k+1}|$, then all the zeros of P(z) lie in $r_4 \le |z| \le R_4$,
where

$$r_4 = \frac{|a_0|}{[(|a_n|+|\rho|)(\cos\alpha+\sin\alpha+1)+|\sigma|-(|\sigma+a_{n-k}|-|a_{n-k}|)(\cos\alpha-\sin\alpha)-\tau|a_0|(\cos\alpha-\sin\alpha+1) + |a_0| + 2\sin\alpha\sum_{j=1}^{n-1}|a_j|]}$$

and

$$R_4 = \frac{1}{|a_n|}[|\rho|+(|\rho|+|a_n|)(\cos\alpha+\sin\alpha)+|\sigma|(1+\sin\alpha)+(|a_{n-k}|-|\sigma+a_{n-k}|)\cos\alpha - \tau|a_0|(\cos\alpha-\sin\alpha+1)+|a_0|+2\sin\alpha\sum_{j=1}^{n-1}|a_j|].$$

Remark 4: Taking $\sigma=(\lambda-1)|a_{n-k}|$, Theorem E immediately follows from Theorem 2.
For different values of the parameters ρ, σ , τ and k, we get many other interesting results.
For example , if we take $\rho = 0$, we get the following result:

Corollary 3: Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n such that for some $\lambda \ne 1,\ 1 \le k \le n$,
$a_{n-k} \ne 0$, $\sigma, \rho, 0 < \tau \le 1$,

$$|a_n| \ge |a_{n-1}| \ge ...... \ge |a_{n-k+1}| \ge \sigma + |a_{n-k}| \ge |a_{n-k-1}| \ge ...... \ge |a_1| \ge \tau|a_0|$$

and for some real α and β,

$$|\arg a_j - \beta| \le \alpha \le \frac{\pi}{2},\ j = 0,1,......,n.$$

If $|a_{n-k-1}| > |a_{n-k}|$, then all the zeros of P(z) lie in $r_3 \le |z| \le R_3$,
where $r_3 = \dfrac{|a_0|}{|a_n|(\cos\alpha+\sin\alpha+1)+|\sigma|(\cos\alpha+\sin\alpha+1)-\tau|a_0|(\cos\alpha-\sin\alpha+1)+|a_0|+2\sin\alpha\sum_{j=1}^{n-1}|a_j|}$ and

$$R_3 = \frac{1}{|a_n|}[|a_n|)(\cos\alpha+\sin\alpha)+|\sigma|(\cos\alpha+\sin\alpha+1)-\tau|a_0|(\cos\alpha-\sin\alpha+|a_0| + 2\sin\alpha\sum_{j=1}^{n-1}|a_j|+|\rho|]$$

and if $|a_{n-k}| > |a_{n-k+1}|$, then all the zeros of P(z) lie in $r_4 \le |z| \le R_4$,
where

$$r_4 = \frac{|a_0|}{[|a_n|(\cos\alpha+\sin\alpha+1)+|\sigma|-(|\sigma+a_{n-k}|-|a_{n-k}|)(\cos\alpha-\sin\alpha)-\tau|a_0|(\cos\alpha-\sin\alpha+1) + |a_0| + 2\sin\alpha\sum_{j=1}^{n-1}|a_j|]}$$

and

$$R_4 = \frac{1}{|a_n|}[|a_n|)(\cos\alpha+\sin\alpha)+|\sigma|+(|a_{n-k}|-|\sigma+a_{n-k}|)\cos\alpha$$

ISSN 2250-3153

$$-\tau|a_0|(\cos\alpha-\sin\alpha+1)+|a_0|+2\sin\alpha\sum_{j=1}^{n-1}|a_j|]$$

2.Lemma

For the proofs of the above results, we need the following lemma due to Govil and Rahman [2] :

Lemma : For any two complex numbers b_0 and b_1 such that $|b_0|\ge|b_1|$ and and for some real numbers α and β satisfying

$$\left|\arg b_j-\beta\right|\le\alpha\le\frac{\pi}{2},\ j=0,1,$$

$$|b_0-b_1|\le(|b_0|-|b_1|)\cos\alpha+(|b_0|+|b_1|)\sin\alpha\,.$$

3. Proofs of Theorems

Proof of Theorem 1: Consider the polynomial

$$\begin{aligned}F(z)&=(1-z)P(z)\\&=(1-z)(a_nz^n+a_{n-1}z^{n-1}+......+a_1z+a_0)\\&=-a_nz^{n+1}+(a_n-a_{n-1})z^n+(a_{n-1}-a_{n-2})z^{n-1}+......\\&\quad+(a_2-a_1)z^2+(a_1-a_0)z+a_0\\&=-a_nz^{n+1}+(\alpha_n-\alpha_{n-1})z^n+(\alpha_{n-1}-\alpha_{n-2})z^{n-1}+.....(\alpha_2-\alpha_1)z^2\\&\quad+(\alpha_1-\alpha_0)z+a_0+i\sum_{j=1}^{n}(\beta_j-\beta_{j-1})z^j\\&=a_0+q(z),\end{aligned}$$

where

$$\begin{aligned}q(z)&=-a_nz^{n+1}+(\alpha_n-\alpha_{n-1})z^n+(\alpha_{n-1}-\alpha_{n-2})z^{n-1}+.....+(\alpha_2-\alpha_1)z^2.\\&\quad+(\alpha_1-\alpha_0)z+i\sum_{j=1}^{n}(\beta_j-\beta_{j-1})z^j\end{aligned}$$

If $\alpha_{n-k-1}>\alpha_{n-k}$, then $\alpha_{n-k+1}>\alpha_{n-k}$ and we have

$$\begin{aligned}q(z)&=-a_nz^{n+1}-\rho z^n+(\rho+\alpha_n-\alpha_{n-1})z^n+(\alpha_{n-1}-\alpha_{n-2})z^{n-1}+......\\&\quad+(\alpha_{n-k+1}-\alpha_{n-k})z^{n-k+1}+\sigma z^{n-k}+(\sigma+\alpha_{n-k}-\alpha_{n-k-1})z^{n-k}\\&\quad+(\alpha_{n-k-1}-\alpha_{n-k-2})z^{n-k-1}+......+(\tau-1)\alpha_0z\\&\quad+(\alpha_1-\tau\alpha_0)z+i\sum_{j=1}^{n}(\beta_j-\beta_{j-1})z^j\,.\end{aligned}$$

For $|z|<1$,

$$\begin{aligned}|q(z)|&\le|a_n|+|\rho|+\rho+\alpha_n-\alpha_{n-1}+\alpha_{n-1}-\alpha_{n-2}+......+\alpha_{n-k+1}-\alpha_{n-k}\\&\quad+|\sigma|+\sigma+\alpha_{n-k}-\alpha_{n-k-1}+\alpha_{n-k-1}-\alpha_{n-k-2}+......\\&\quad+\alpha_1-\tau\alpha_0+(1-\tau)|\alpha_0|+\sum_{j=1}^{n}(|\beta_j|+|\beta_{j-1}|)\end{aligned}$$

$$= |a_n| + |\rho| + \rho + \alpha_n + |\sigma| + \sigma - \tau(|\alpha_0| + \alpha_0) + |\alpha_0| + |\beta_0| + |\beta_n| + 2\sum_{j=1}^{n-1}|\beta_j|$$

$$= M_1$$

Since q(z) is analytic for $|z| < 1$ and q(0)=0, it follows , by Rouche's theorem, that

$$|q(z)| \le M_1|z| \text{ for } |z| < 1.$$

Hence, it follows that, for $|z| < 1$,

$$\begin{aligned} |F(z)| &= |a_0 + q(z)| \\ &\ge |a_0| - |q(z)| \\ &\ge |a_0| - M_1|z| \\ &> 0 \end{aligned}$$

if

$$|z| < \frac{|a_0|}{M_1}.$$

This shows that P(z) has all its zeros in $|z| \ge \frac{|a_0|}{M_1} = r_1$ or in $r_1 \le |z|$.

If $\alpha_{n-k} > \alpha_{n-k+1}$, then $\alpha_{n-k} > \alpha_{n-k-1}$ and we have

$$\begin{aligned} q(z) = &-a_n z^{n+1} - \rho z^n + (\rho + \alpha_n - \alpha_{n-1})z^n + (\alpha_{n-1} - \alpha_{n-2})z^{n-1} + \ldots\ldots \\ &+ (\alpha_{n-k+1} - \sigma - \alpha_{n-k})z^{n-k+1} + \sigma z^{n-k+1} + (\alpha_{n-k} - \alpha_{n-k-1})z^{n-k} \\ &+ (\alpha_{n-k-1} - \alpha_{n-k-2})z^{n-k-1} + \ldots\ldots + (\tau - 1)\alpha_0 z \\ &+ (\alpha_1 - \tau\alpha_0)z + i\sum_{j=1}^{n}(\beta_j - \beta_{j-1})z^j . \end{aligned}$$

For $|z| < 1$,

$$\begin{aligned} |q(z)| \le &|a_n| + |\rho| + \rho + \alpha_n - \alpha_{n-1} + \alpha_{n-1} - \alpha_{n-2} + \ldots\ldots + \alpha_{n-k+1} - \sigma - \alpha_{n-k} \\ &+ |\sigma| + \alpha_{n-k} - \alpha_{n-k-1} + \alpha_{n-k-1} - \alpha_{n-k-2} + \ldots\ldots \\ &+ \alpha_1 - \tau\alpha_0 + (1-\tau)|\alpha_0| + \sum_{j=1}^{n}(|\beta_j| + |\beta_{j-1}|) \\ = &|a_n| + |\rho| + \rho + \alpha_n + |\sigma| - \sigma - \tau(|\alpha_0| + \alpha_0) + |\alpha_0| + |\beta_0| + |\beta_n| + 2\sum_{j=1}^{n-1}|\beta_j| \\ = &M_2 \end{aligned}$$

Since q(z) is analytic for $|z| < 1$ and q(0)=0, it follows , by Rouche's theorem, that

$$|q(z)| \le M_2|z| \text{ for } |z| < 1.$$

Hence, it follows that, for $|z| < 1$,

$$\begin{aligned} |F(z)| &= |a_0 + q(z)| \\ &\ge |a_0| - |q(z)| \\ &\ge |a_0| - M_2|z| \\ &> 0 \end{aligned}$$

if

$$|z| < \frac{|a_0|}{M_2}.$$

This shows that P(z) has all its zeros in $|z| \geq \frac{|a_0|}{M_2} = r_2$ or in $r_2 \leq |z|$.

On the other hand, if $\alpha_{n-k-1} > \alpha_{n-k}$, then $\alpha_{n-k+1} > \alpha_{n-k}$ and we have for $|z| > 1$

$$|F(z)| \geq |z|^{n+1}|a_n z + \rho| - |z|^n[\rho + \alpha_n - \alpha_{n-1} + \frac{\alpha_{n-1} - \alpha_{n-2}}{|z|} + ... + \frac{\alpha_{n-k+1} - \alpha_{n-k}}{|z|^{k-1}}$$

$$+ \frac{\sigma + \alpha_{n-k} - \alpha_{n-k-1}}{|z|^k} + \frac{|\sigma|}{|z|^k} + \frac{\alpha_{n-k-1} - \alpha_{n-k-2}}{|z|^{k+1}} + ...$$

$$+ \frac{\alpha_1 - \tau\alpha_0}{|z|^{n-1}} + \frac{(1-\tau)|\alpha_0|}{|z|^{n-1}} + \sum_{j=1}^{n}(\frac{|\beta_j| + |\beta_{j-1}|}{|z|^{n-j}})]$$

$$> |z|^n[|a_n z + \rho| - \{\rho + \alpha_n - \alpha_{n-1} + \alpha_{n-1} - \alpha_{n-2} + ... + \alpha_{n-k+1} - \alpha_{n-k} + ...$$

$$+ \sigma + \alpha_{n-k} - \alpha_{n-k-1} + |\sigma| + \alpha_{n-k-1} - \alpha_{n-k-2} + ... + \alpha_2 - \alpha_1$$

$$+ \alpha_1 - \tau\alpha_0 + (1-\tau)|\alpha_0| + |\beta_0| + |\beta_n| + 2\sum_{j=1}^{n-1}|\beta_j|\}]$$

$$= |z|^n[|a_n z + \rho| - \{\rho + \alpha_n + \sigma + |\sigma| - \tau(|\alpha_0| + \alpha_0) + |\alpha_0| + |\beta_0| + |\beta_n| + 2\sum_{j=1}^{n-1}|\beta_j|\}]$$

$$> 0$$

if

$$|a_n z + \rho| > \rho + \alpha_n + \sigma + |\sigma| - \tau(|\alpha_0| + \alpha_0) + |\alpha_0| + |\beta_0| + |\beta_n| + 2\sum_{j=1}^{n-1}|\beta_j| = M_3 .$$

This shows that those zeros of F(z) whose modulus is greater than 1 lie in $\left|z + \frac{\rho}{a_n}\right| \leq \frac{M_3}{|a_n|}$. But the zeros of F(z) whose modulus is less than or equal to 1 already lie in $\left|z + \frac{\rho}{a_n}\right| \leq \frac{M_3}{|a_n|}$. Therefore, it follows that all the zeros of F(z) and hence P(z) lie in $\left|z + \frac{\rho}{a_n}\right| \leq \frac{M_3}{|a_n|}$, In other words, all the zeros of P(z) lie in

$$|z| \leq \frac{M_3 + |\rho|}{|a_n|} = R_1.$$

If $\alpha_{n-k} > \alpha_{n-k+1}$, then $\alpha_{n-k} > \alpha_{n-k-1}$ and we have for $|z| > 1$

$$|F(z)| \geq |z|^{n+1}|a_n z + \rho| - |z|^n[\rho + \alpha_n - \alpha_{n-1} + \frac{\alpha_{n-1} - \alpha_{n-2}}{|z|} + ... + \frac{\alpha_{n-k+1} - \sigma - \alpha_{n-k}}{|z|^{k-1}}$$

$$+ \frac{|\sigma|}{|z|^{k-1}} + \frac{\alpha_{n-k} - \alpha_{n-k-1}}{|z|^k} + \frac{\alpha_{n-k-1} - \alpha_{n-k-2}}{|z|^{k+1}} + ...$$

$$+\frac{\alpha_1-\tau\alpha_0}{|z|^{n-1}}+\frac{(1-\tau)|\alpha_0|}{|z|^{n-1}}+\sum_{j=1}^{n}(\frac{|\beta_j|+|\beta_{j-1}|}{|z|^{n-j}})]$$

$$>|z|^n[|a_nz+\rho|-\{\rho+\alpha_n-\alpha_{n-1}+\alpha_{n-1}-\alpha_{n-2}+......+\alpha_{n-k+1}-\sigma-\alpha_{n-k}+......$$
$$+|\sigma|+\alpha_{n-k}-\alpha_{n-k-1}+\alpha_{n-k-1}-\alpha_{n-k-2}+......+\alpha_2-\alpha_1$$
$$+\alpha_1-\tau\alpha_0+(1-\tau)|\alpha_0|+|\beta_0|+|\beta_n|+2\sum_{j=1}^{n-1}|\beta_j|\}]$$
$$=|z|^n[|a_nz+\rho|-\{\rho+\alpha_n-\sigma+|\sigma|-\tau(|\alpha_0|+\alpha_0)+|\alpha_0|+|\beta_0|+|\beta_n|+2\sum_{j=1}^{n-1}|\beta_j|\}]$$
$$>0$$

if

$$|a_nz+\rho|>\{\rho+\alpha_n+|\sigma|-\sigma-\tau(|\alpha_0|+\alpha_0)+|\alpha_0|+|\beta_0|+|\beta_n|+2\sum_{j=1}^{n-1}|\beta_j|=M_4\ .$$

This shows that those zeros of F(z) whose modulus is greater than 1 lie in $\left|z+\frac{\rho}{a_n}\right|\le\frac{M_4}{|a_n|}$. But the zeros of F(z) whose modulus is less than or equal to 1 already lie in $\left|z+\frac{\rho}{a_n}\right|\le\frac{M_4}{|a_n|}$. Therefore, it follows that all the zeros of F(z) and hence P(z) lie in $\left|z+\frac{\rho}{a_n}\right|\le\frac{M_4}{|a_n|}$, In other words, all the zeros of P(z) lie in

$|z|\le\frac{M_4+|\rho|}{|a_n|}=R_2$ in this case.

That completes the proof of Theorem 1.

Proof of Theorem 2: Consider the polynomial

$$F(z)=(1-z)P(z)$$
$$=(1-z)(a_nz^n+a_{n-1}z^{n-1}+......+a_1z+a_0)$$
$$=-a_nz^{n+1}+(a_n-a_{n-1})z^n+(a_{n-1}-a_{n-2})z^{n-1}+......+(a_{n-k+1}-a_{n-k})$$
$$+(a_{n-k}-a_{n-k-1})+......+(a_2-a_1)z^2+(a_1-a_0)z+a_0$$
$$=a_0+q(z),$$

where

$$q(z)=-a_nz^{n+1}+(a_n-a_{n-1})z^n+(a_{n-1}-a_{n-2})z^{n-1}+......+(a_{n-k+1}-a_{n-k})z^{n-k+1}$$
$$+(a_{n-k}-a_{n-k-1})z^{n-k}+......+(a_2-a_1)z^2+(a_1-a_0)z\ .$$
$$=-a_nz^{n+1}-\rho z^n+(\rho+a_n-a_{n-1})z^n+(a_{n-1}-a_{n-2})z^{n-1}+......+(a_{n-k+1}-a_{n-k})z^{n-k+1}$$
$$+(a_{n-k}-a_{n-k-1})z^{n-k}+......+(a_2-a_1)z^2+(a_1-\tau a_0)z+(\tau-1)a_0z$$

If $|a_{n-k-1}| > |a_{n-k}|$, then $|a_{n-k+1}| > |a_{n-k}|$ and we have, by using the lemma, for $|z|<1$,

$$|q(z)| = \left| \begin{array}{l} -a_n z^{n+1} - \rho z^n + (\rho + a_n - a_{n-1})z^n + (a_{n-1} - a_{n-2})z^{n-1} + ... + (a_{n-k+1} - a_{n-k})z^{n-k+1} \\ + (\sigma + a_{n-k} - a_{n-k-1})z^{n-k} - \sigma z^{n-k} + (a_{n-k-1} - a_{n-k-2})z^{n-k-1} + ... + (a_2 - a_1)z^2 \\ + (a_1 - \tau a_0)z + (\tau - 1)a_0 \end{array} \right|$$

$$\le |a_n| + |\rho| + |\rho + a_n - a_{n-1}| + |a_{n-1} - a_{n-2}| + ... + |a_{n-k+1} - a_{n-k}| + |\sigma + a_{n-k} - a_{n-k-1}|$$
$$+ |\sigma| + |a_{n-k-1} - a_{n-k-2}| + ... + |a_2 - a_1| + |a_1 - \tau a_0| + (1-\tau)|a_0|$$

$$\le |a_n| + |\rho| + (|\rho + a_n| - |a_{n-1}|)\cos\alpha + (|\rho + a_n| + |a_{n-1}|)\sin\alpha + (|a_{n-1}| - |a_{n-2}|)\cos\alpha$$
$$+ (|a_{n-1}| + |a_{n-2}|)\sin\alpha + ... + (|a_{n-k+1}| - |a_{n-k}|)\cos\alpha + (|a_{n-k+1}| + |a_{n-k}|)\sin\alpha + |\sigma|$$
$$+ (|\sigma + a_{n-k}| - |a_{n-k-1}|)\cos\alpha + (|\sigma + a_{n-k}| + |a_{n-k-1}|)\sin\alpha + |\sigma|. + ...$$
$$+ (|a_{n-k-1}| - |a_{n-k-2}|)\cos\alpha + (|a_{n-k-1}| + |a_{n-k-2}|)\sin\alpha + ... + (|a_2| - |a_1|)\cos\alpha$$
$$+ (|a_2| + |a_1|)\sin\alpha + (|a_1| - \tau|a_0|)\cos\alpha + (|a_1| + \tau|a_0|)\sin\alpha + (1-\tau)|a_0|$$
$$\le (|a_n| + |\rho|)(\cos\alpha + \sin\alpha + 1) + |\sigma|(\cos\alpha + \sin\alpha + 1)$$
$$- \tau|a_0|(\cos\alpha - \sin\alpha + 1) + |a_0| + 2\sin\alpha \sum_{j=1}^{n-1} |a_j|$$
$$= M_5 .$$

Since q(z) is analytic and q(0) =0, by Rouche's Theorem, it follows that for $|z|<1$,

$$|q(z)| \le M_5 |z|$$

Hence, for $|z|<1$,

$$|F(z)| = |a_0 + q(z)|$$
$$\ge |a_0| - |q(z)|$$
$$\ge |a_0| - M_5|z|$$
$$> 0$$

if

$$|z| < \frac{|a_0|}{M_5} .$$

This shows that all the zeros of P(z) lie in $|z| \ge \frac{|a_0|}{M_5} = r_3$.

If $\alpha_{n-k} > \alpha_{n-k+1}$, then $\alpha_{n-k} > \alpha_{n-k-1}$ and we have

$$q(z) = -a_n z^{n+1} - \rho z^n + (\rho + a_n - a_{n-1})z^n + (a_{n-1} - a_{n-2})z^{n-1} + ...$$

$$+ (a_{n-k+1} - \sigma - a_{n-k})z^{n-k+1} + \sigma z^{n-k+1} + (a_{n-k} - a_{n-k-1})z^{n-k}$$
$$+ (a_{n-k-1} - a_{n-k-2})z^{n-k-1} + ... + (\tau - 1)a_0 z$$
$$+ (a_1 - \tau a_0)z .$$

For $|z|<1$, we have, by using the lemma,

$$
\begin{aligned}
|q(z)| &\le |a_n| + |\rho| + |\rho + a_n - a_{n-1}| + |a_{n-1} - a_{n-2}| + \dots\dots + |a_{n-k+1} - \sigma - a_{n-k}| \\
&\quad + |\sigma| + |a_{n-k} - a_{n-k-1}| + |a_{n-k-1} - a_{n-k-2}| + \dots\dots \\
&\quad + |a_1 - \tau a_0| + (1-\tau)|a_0|) \\
&= |a_n| + |\rho| + (|\rho + a_n| - |a_{n-1}|)\cos\alpha + (|\rho + a_n| + |a_{n-1}|)\sin\alpha + (|a_{n-1}| - |a_{n-2}|)\cos\alpha \\
&\quad + (|a_{n-1}| + |a_{n-2}|)\sin\alpha + \dots\dots + (|a_{n-k+1}| - |\sigma + a_{n-k}|)\cos\alpha \\
&\quad + (|a_{n-k+1}| + |\sigma + a_{n-k}|)\sin\alpha + |\sigma| + (|a_{n-k}| - |a_{n-k-1}|)\cos\alpha \\
&\quad + (|a_{n-k}| + |a_{n-k-1}|)\sin\alpha + \dots\dots + (|a_1| - \tau|a_0|)\cos\alpha + (|a_1| + \tau|a_0|)\sin\alpha \\
&\quad + (1-\tau)|a_0|) \\
&\le (|a_n| + |\rho|)(\cos\alpha + \sin\alpha + 1) + |\sigma| - |\sigma + a_{n-k}|(\cos\alpha - \sin\alpha) \\
&\quad + |a_{n-k}|(\cos\alpha + \sin\alpha) - \tau|a_0|(\cos\alpha - \sin\alpha + 1) + |a_0| + 2\sin\alpha\sum_{j=1}^{n-1}|a_j| \\
&= M_6
\end{aligned}
$$

Since q(z) is analytic for $|z| < 1$ and q(0)=0, it follows , by Rouche's theorem, that

$|q(z)| \le M_6|z|$ for $|z| < 1$.

Hence, it follows that, for $|z| < 1$,

$$
\begin{aligned}
|F(z)| &= |a_0 + q(z)| \\
&\ge |a_0| - |q(z)| \\
&\ge |a_0| - M_6|z| \\
&> 0
\end{aligned}
$$

if

$$|z| < \frac{|a_0|}{M_6}.$$

This shows that P(z) has all its zeros in $|z| \ge \frac{|a_0|}{M_6} = r_4$ or in $r_4 \le |z|$.

On the other hand, if $|a_{n-k-1}| > |a_{n-k}|$, then $|a_{n-k+1}| > |a_{n-k}|$ and we have, by using the lemma, fo$|z| > 1$,

$$
\begin{aligned}
|F(z)| &\ge |z|^n[|a_n z + \rho| - \{|\rho + \alpha_n - \alpha_{n-1}| + \frac{|\alpha_{n-1} - \alpha_{n-2}|}{|z|} + \dots\dots \\
&\quad + \frac{|a_{n-k+1} - a_{n-k}|}{|z|^{k-1}} + \frac{|\sigma|}{|z|^{k-1}} + \frac{|\sigma + a_{n-k} - a_{n-k-1}|}{|z|^k} \\
&\quad + \frac{|a_{n-k-1} - a_{n-k-2}|}{|z|^{k+1}} + \dots\dots + \frac{|a_2 - a_1|}{|z|^{n-2}} + \frac{|a_1 - \tau a_0|}{|z|^{n-1}} + \frac{(1-\tau)|a_0|}{|z|^{n-1}}\}]. \\
&> |z|^n[|a_n z + \rho| - \{|\rho + \alpha_n - \alpha_{n-1}| + |\alpha_{n-1} - \alpha_{n-2}| + \dots. \\
&\quad + |a_{n-k+1} - a_{n-k}| + |\sigma| + |\sigma + a_{n-k} - a_{n-k-1}| \\
&\quad + |a_{n-k-1} - a_{n-k-2}| + \dots\dots + |a_2 - a_1| + |a_1 - \tau a_0| + (1-\tau)|a_0|\}] \\
&\ge |z|^n[|a_n z + \rho| - \{|\rho + a_n| - |a_{n-1}|)\cos\alpha + |\rho + a_n| + |a_{n-1}|)\sin\alpha
\end{aligned}
$$

$$+(|a_{n-1}|-|a_{n-2}|)\cos\alpha+(|a_{n-1}|+|a_{n-2}|)\sin\alpha+......$$
$$+(|a_{n-k+1}|-|a_{n-k}|)\cos\alpha+(|a_{n-k+1}|+|a_{n-k}|)\sin\alpha$$
$$+|\sigma|+(|\sigma+a_{n-k}|-|a_{n-k-1}|)\cos\alpha+(|\sigma+a_{n-k}|+|a_{n-k-1}|)\sin\alpha$$
$$+......+(|a_2|-|a_1|)\cos\alpha+(|a_2|+|a_1|)\sin\alpha+(|a_1|-\tau|a_0|)\cos\alpha$$
$$+(|a_1|+\tau|a_0|)\sin\alpha+(1-\tau)|a_0|\}]$$
$$\geq|z|^n[|a_nz+\rho|-\{(|\rho|+|a_n|)(\cos\alpha+\sin\alpha)+|\sigma|(\cos\alpha+\sin\alpha+1)$$
$$-\tau|a_0|(\cos\alpha-\sin\alpha+|a_0|+2\sin\alpha\sum_{j=1}^{n-1}|a_j|\}]$$
$$>0$$

if $$|a_nz+\rho|>(|\rho|+|a_n|)(\cos\alpha+\sin\alpha)+|\sigma|(\cos\alpha+\sin\alpha+1)-\tau|a_0|(\cos\alpha-\sin\alpha)+|a_0|$$
$$+2\sin\alpha\sum_{j=1}^{n-1}|a_j|$$

i.e.,
$$\left|z+\frac{\rho}{a_n}\right|>\frac{1}{|a_n|}[(|\rho|+|a_n|)(\cos\alpha+\sin\alpha)+|\sigma|(\cos\alpha+\sin\alpha+1)-\tau|a_0|(\cos\alpha-\sin\alpha+|a_0|$$
$$+2\sin\alpha\sum_{j=1}^{n-1}|a_j|]$$
$$=\frac{M_7}{|a_n|}\ .$$

This shows that those zeros of F(z) whose modulus is greater than 1 lie in $\left|z+\frac{\rho}{a_n}\right|\leq\frac{M_7}{|a_n|}$. But the zeros of F(z) whose modulus is less than or equal to 1 already lie in $\left|z+\frac{\rho}{a_n}\right|\leq\frac{M_7}{|a_n|}$. Therefore, it follows that all the zeros of F(z) and hence P(z) lie in $\left|z+\frac{\rho}{a_n}\right|\leq\frac{M_7}{|a_n|}$, In other words, all the zeros of P(z) lie in

$|z|\leq\frac{M_7+|\rho|}{|a_n|}=R_3$ in this case.

If $|a_{n-k}|>|a_{n-k+1}|$, then $|a_{n-k}|>|a_{n-k-1}|$ and we have, by using the lemma, fo $|z|>1$,

$$|F(z)|\geq|z|^n[|a_nz+\rho|-\{|\rho+a_n-a_{n-1}|+\frac{|a_{n-1}-a_{n-2}|}{|z|}+......$$
$$+\frac{|a_{n-k+1}-\sigma-a_{n-k}|}{|z|^{k-1}}+\frac{|\sigma|}{|z|^{k-1}}+\frac{|a_{n-k}-a_{n-k-1}|}{|z|^k}$$
$$+\frac{|a_{n-k-1}-a_{n-k-2}|}{|z|^{k+1}}+......+\frac{|a_2-a_1|}{|z|^{n-2}}+\frac{|a_1-\tau a_0|}{|z|^{n-1}}+\frac{(1-\tau)|a_0|}{|z|^{n-1}}\}].$$
$$>|z|^n[|a_nz+\rho|-\{|\rho+a_n|-|a_{n-1}|)\cos\alpha+|\rho+a_n|+|a_{n-1}|)\sin\alpha$$

$$+(|a_{n-1}|-|a_{n-2}|)\cos\alpha+(|a_{n-1}|+|a_{n-2}|)\sin\alpha+......$$

$$+(|a_{n-k+1}|-|\sigma+a_{n-k}|)\cos\alpha+(|a_{n-k+1}|+|\sigma+a_{n-k}|)\sin\alpha$$
$$+|\sigma|+(|a_{n-k}|-|a_{n-k-1}|)\cos\alpha+(|a_{n-k}|+|a_{n-k-1}|)\sin\alpha$$
$$+......+(|a_2|-|a_1|)\cos\alpha+(|a_2|+|a_1|)\sin\alpha+(|a_1|-\tau|a_0|)\cos\alpha$$
$$+(|a_1|+\tau|a_0|)\sin\alpha+(1-\tau)|a_0|\}]$$
$$\geq|z|^n[|a_n z+\rho|-\{(|\rho|+|a_n|)(\cos\alpha+\sin\alpha)+|\sigma|(1+\sin\alpha)$$
$$+(|a_{n-k}|-|\sigma+a_{n-k}|)\cos\alpha\ -\tau|a_0|(\cos\alpha-\sin\alpha+1)+|a_0|+2\sin\alpha\sum_{j=1}^{n-1}|a_j|\}]$$
$$>0$$

if $|a_n z+\rho|>(|\rho|+|a_n|)(\cos\alpha+\sin\alpha)+|\sigma|(1+\sin\alpha)$

$$+(|a_{n-k}|-|\sigma+a_{n-k}|)\cos\alpha\ -\tau|a_0|(\cos\alpha-\sin\alpha+1)+|a_0|+2\sin\alpha\sum_{j=1}^{n-1}|a_j|$$

i.e.,

$$\left|z+\frac{\rho}{a_n}\right|>\frac{1}{|a_n|}[|\rho|+|a_n|)(\cos\alpha+\sin\alpha)+|\sigma|(1+\sin\alpha)$$
$$+(|a_{n-k}|-|\sigma+a_{n-k}|)\cos\alpha\ -\tau|a_0|(\cos\alpha-\sin\alpha+1)+|a_0|+2\sin\alpha\sum_{j=1}^{n-1}|a_j|]$$

$$=\frac{M_8}{|a_n|}.$$

This shows that those zeros of F(z) whose modulus is greater than 1 lie in $\left|z+\frac{\rho}{a_n}\right|\leq\frac{M_8}{|a_n|}$. But the zeros of F(z) whose modulus is less than or equal to 1 already lie in $\left|z+\frac{\rho}{a_n}\right|\leq\frac{M_8}{|a_n|}$. Therefore, it follows that all the zeros of F(z) and hence P(z) lie in $\left|z+\frac{\rho}{a_n}\right|\leq\frac{M_8}{|a_n|}$, In other words, all the zeros of P(z) lie in

$|z|\leq\frac{M_8+|\rho|}{|a_n|}=R_4$ in this case.

That completes the proof of Theorem 2.

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