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Improved Technique for Constructing Doubly-even Magic Squares using Basic Latin Squares

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Abstract: The technique developed by Tomba (July 2012) for finding (*n* x *n*) magic squares (when *n* is divisible by 4) using basic Latin squares is re-discussed. The steps include fixing the column associated with the elements, adjacent to the pivot element and arranging in an orderly manner, making symmetric transformations that generates a magic parametric constant (T) and finally derived with minor adjustment on the pair-numbers of satisfying T [12]. Nordgren's method can be applied so as to avoid symmetric transformation, adjustment on the pair numbers satisfying T and thereby shortening in 3 steps only.

The improved technique for construction of doubly-even magic squares needs three steps with no symmetric transformation, minor adjustments etc. on the pair numbers satisfying T. As such, lot of labour and time is saved with the use of this improved technique.

Key-words: Latin square (basic), doubly even magic square (normal), magic parametric constant (Tomba's constant),

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1. INTRODUCTION

In a Latin square, Latin letters are seen once in each row and column whereas the sums of rows and columns are equal but not the sums of diagonals. The idea of Latin squares and basic Latin squares, normal magic squares have studied in [11] and [12]. A (*n* x *n*) basic Latin square is symmetric and non-duplicated (if *n* is odd) and symmetric but duplicated (if *n* is even).

Normal magic square has the following properties;

(a) Elements or numbers ($n \geq 0$) are consecutive and not repeated
(b) Sums of the rows, columns and diagonals (using diagonal notation as d_{ij}) are equal to the magic sum, S

$$\Rightarrow \sum_i b_{ij} = \sum_j b_{ij} = \sum_i d_{ij} = \sum_j d_{ij} \;\; i, j = 1, 2, \ldots n$$

(c) Equality property of the sum of rows, columns and diagonals remain unaltered for rotations and reflections.

An (n x n) array $\{b_{ij}\}$ (with diagonal notation d_{ij}) satisfying the following properties are weak magic squares;

(a) Elements or numbers ($n \geq 0$) are consecutive and not repeated
(b) Sums of diagonals are equal to the magic sum, S

$$\Rightarrow \sum_i d_{ij} = \sum_j d_{ij} = S; \; i, j = 1, 2, ..n$$

(c) Sums of the rows, columns are equal to the magic sum (S), except for some *i* and *j*

$$\Rightarrow \sum_i b_{ij} = \sum_j b_{ij} = S; \;\; i, j = 1, 2, ..n \qquad \ldots \quad [1]$$

(d) Equality property of the sum of rows, columns and diagonals remain unaltered for rotations and reflections.

Then the matrix $\{b_{ij}\}$ is a weak magic square.

Let $\{a_{ij}\}$ be a magic square satisfying the properties (a) to (c). The alternate structures of a magic square can be expressed (clockwise or anticlockwise rotation) as

$\{a_{ij}(k)\}$ for $(k\frac{\pi}{2};\; k = \pm 1, \; \pm 2, \ldots \pm m)$

where, $\{a_{ij}\} = \{a_{ij}(k)\}$ for all $i = 0, 4, 8, \ldots$

2. TECHNIQUE (TOMBA, 2012)
(For constructing doubly-even magic squares)

The construction process is expressed in the following five steps:

Step-1: First the consecutive numbers (1 to n^2) in *n* rows and *n* columns be arranged in basic Latin square format. The pivot element lies between two numbers, $(\frac{n^2}{2})$ and $\left(\frac{n^2}{2}+1\right)$. Find the magic parametric constant, T= $\{n^2+1\}$

Step-2: Select the column associated with these two numbers, assign this column as main diagonal elements and arrange other (row) elements in an orderly manner to give diagonals sums equal

Step-3: Make symmetric transformations of other elements (retaining the diagonal elements unchanged) to construct the extreme corner and central blocks of (2 x 2) each.

Step-4: Reverting $\frac{1}{2}\{n-4\}$ rows and columns in a systematic manner, a magic parametric constant (T) and a set of sub-magic parametric constants are generated. The reversion process will satisfy the following conditions;
(i) n (singly even),
$2 \le \left\{\frac{n}{2} \pm 2k\right\} < (n-2)$ for k = 0, 1, 2 ... n
(ii) n (doubly even),
$2 \le \left\{\frac{n-2}{2} \pm 2k\right\} < (n-2)$ for k = 0, 1, 2 ...n

Step-5: Main adjustments should be made on the pair-numbers satisfying T, whereas minor adjustments should be made on other elements of sub-magic parametric constants to get the magic square for any doubly-even n.

Note: The technique generates weak magic squares for any singly-even n. Different weak magic squares can be derived taking the central block arbitrarily any two of the pair numbers satisfying T.

3. IMPROVED TECHNIQUE

(For constructing doubly-even magic squares)

The technique of constructing doubly-even magic square using basic Latin square can be improved with the application of Nordgren's method after step-2. The technique can be expressed as follows:

Let the n^2 matrix $\{a_{ij}\}$; $i, j=1,2,...n$ with the consecutive numbers be arranged in basic Latin square format as;

$$\begin{bmatrix} a_{11} & a_{12} & ... & a_{1n-1} & a_{1n} \\ a_{22} & a_{23} & .... & a_{2n} & a_{21} \\ ... & ... & ... & ... & ... \\ a_{nn} & a_{n1} & ... & a_{nn-2} & a_{nn-1} \end{bmatrix}$$

where, $\sum_i a_{ij} = S$ for all i and S = $\frac{1}{2}\{n(n^2+1)\}$ ... [2]

This condition is true for all n (odd or even) due to basic Latin square property. The pivot element is unique (for n is odd) but it lies between two numbers (for n is even).

Since the pivot element is not fixed, we select the column (row in case of LSF-2), associated with the numbers adjacent to the pivot element, assign it as the diagonal elements and arrange the other row (column) elements in an orderly manner to get a new matrix satisfying the property

$$\sum_i d_{ij} = \sum_j d_{ij} = S \text{ for all } i \text{ and } j \qquad [3]$$

Reflecting odd columns of 1st, 3rd, ... (< $\left[\frac{n}{2}\right]$th) and even columns of nth, (n-2)th, ... (> $\left[\frac{n}{2}+1\right]$th) followed by reflection of odd rows of 1st, 3rd, ... (< $\left[\frac{n}{2}\right]$th) and even rows of nth, (n-2)th, ... (> $\left[\frac{n}{2}+1\right]$th), the desired magic square can be achieved.

The process of symmetric transformation and adjustment on the pair satisfying T is not required It gives $\{b_{ij}\}$satisfying , $\sum_i b_{ij} = \sum_j b_{ij} = \sum_i d_{ij} = \sum_j d_{ij}$ [4]

$\Rightarrow \{b_{ij}\}$; $i, j=1,2,..n$ is the doubly even magic square.

4. STEPS FOR CONSTRUCTION FOR THE IMPROVED TECHNIQUE

(For doubly-even magic squares)

Improved method for construction of magic square using basic Latin square is expressed in the following steps:

Step-1: First arrange the consecutive numbers 1 to n^2 or (a_{11} *to* a_{nn}) in basic Latin square format. Find T=$[n^2+1]$.

Step-2: The pivot element lies between $a_{\frac{n}{2},\frac{n}{2}+1}$ and $a_{\frac{n}{2}+1,\frac{n}{2}+1}$. It is associated with $\left[\frac{n}{2}+1\right]$ th column (or row in case of BLSF-2).

Select this column, assign it as diagonal elements and arrange other row (column) elements in an orderly manner to give the diagonals sums equal. The process gives the central block of (2x2).

Step-3: Reflecting the first odd columns, less than $\left[\frac{n}{2}\right]$ and last even columns greater than $\left[\frac{n}{2}+1\right]$ and again reflecting the first odd rows, less than $\left[\frac{n}{2}\right]$ and last even rows, greater than $\left[\frac{n}{2}+1\right]$, the required magic square is generated.

The process will not affect the central (2x2) block.

5. ADVANTAGES

The improved technique needs no adjustments on the pair numbers satisfying T (Tomba's constants). The process diverts from the earlier method after step-2. In lieu of Step-3 to step-5, Nordgren's method is applied. The construction of doubly even magic squares can be completed in three steps only in lieu of 5 steps needed for the earlier method, developed by Tomba.

It generates doubly even magic squares of any *n* (divisible by 4) with different values in the central (2x2) block. The technique does not generate extreme corner (2x2) blocks. Therefore, lot of labour and time is saved with the use of this improved technique.

6. NUMERICAL EXAMPLES

6.1: To construct a (4 x 4) magic square

Step-1: The numbers (1 to 16) in 4 rows and 4 columns, arranged in basic Latin square format be:

1	2	3	4
6	7	8	5
11	12	9	10
16	13	14	15

The arrangement gives the column totals equal, $\sum_i a_{ij} = 34$ for all i. Here, S = 34 and T = 17.

Step-2: Select the column associated with the numbers 8 and 9 (say 3, 8, 9, 14) and assign it as diagonal elements. Rearranging the other elements in an orderly manner gives a new matrix satisfying $\sum_i d_{ij} = \sum_j d_{ij} = 34$

15	11	7	3
16	12	8	4
13	9	5	1
14	10	6	2

Step-3: Reflecting the odd columns, less than 2 and even columns greater than 3 and again reflecting the same odd rows less than 2 and even rows greater than 3

$\Rightarrow$ Reflecting the first column, 4th column and again reflecting the first row and fourth row

3	7	11	15
16	12	8	4
13	9	5	1
2	6	10	14

2	7	11	14
13	12	8	1
16	9	5	4
3	6	10	15

It represents the (4x4) magic square with T = 17

6.2 To construct (8 x 8) magic square

Step-1: Let the consecutive numbers be (1, 2, 3, ... 64) in 8 rows and 8 columns be arranged in basic Latin Square format as:

1	2	3	4	5	6	7	8
10	11	12	13	14	15	16	9
19	20	21	22	23	24	17	18
28	29	30	**31**	**32**	25	26	27
37	38	39	**40**	**33**	34	35	36
46	47	48	41	42	43	44	45
55	56	49	50	51	52	53	54
64	57	58	59	60	61	62	63

Step-2: Select the 5th column, associated with the element 32 and 33, assign it as diagonal elements.

Rearranging the other row elements in an orderly manner gives a new matrix satisfying the diagonal sums equal.

61	53	45	37	29	21	13	5
62	54	46	38	30	22	14	6
63	55	47	39	31	23	15	7
64	56	48	**40**	**32**	24	16	8
57	49	41	**33**	**25**	17	9	1
58	50	42	34	26	18	10	2
59	51	43	35	27	19	11	3
60	52	44	36	28	20	12	4

Step-3: Reflecting 1st, 3rd columns and 6th and 8th columns; and again reflecting 1st, 3rd rows and 6th and 8th rows, the 8x8 magic square is generated.

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60	53	44	37	29	20	13	4
59	54	43	38	30	19	14	3
58	55	42	39	31	18	15	2
57	56	41	**40**	**32**	17	16	1
64	49	48	**33**	**25**	24	9	8
63	50	47	34	26	23	10	7
62	51	46	35	27	22	11	6
61	52	45	36	28	21	12	5

4	13	20	29	37	44	53	60	**260**
59	54	43	38	30	19	14	3	**260**
2	15	18	31	39	42	55	58	**260**
57	56	41	**40**	**32**	17	16	1	**260**
64	49	48	**33**	**25**	24	9	8	**260**
7	10	23	26	34	47	50	63	**260**
62	51	46	35	27	22	11	6	**260**
5	12	21	28	36	45	52	61	**260**
260	**260**	**260**	**260**	**260**	**260**	**260**	**260**	**260**

It represents the 8x8 magic square with T = 65

6.3 To construct (12 x 12) magic square

Step 1: Let the numbers 1 to 144 be arranged in Latin Square format be;

1	2	3	4	5	6	7	8	9	10	11	12
14	15	16	17	18	19	20	21	22	23	24	13
27	28	29	30	31	32	33	34	35	36	25	26
40	41	42	43	44	45	46	47	48	37	38	39
53	54	55	56	57	58	59	60	49	50	51	52
66	67	68	69	70	71	**72**	61	62	63	64	65
79	80	81	82	83	84	**73**	74	75	76	77	78
92	93	94	95	96	85	86	87	88	89	90	91
105	106	107	108	97	98	99	100	101	102	103	104
118	119	120	109	110	111	112	113	114	115	116	117
131	132	121	122	123	124	125	126	127	128	129	130
144	133	134	135	136	137	138	139	140	141	142	143

Step 2: The pivot lies between 72 and 73.

Select the 7th i.e the $\left[\frac{n}{2}+1\right]$ 'th column, associated with these elements and assign it as diagonal elements.

Rearranging the other elements in an orderly manner to get a new (12 x 12) array, satisfying column sums equal

139	127	115	103	91	79	67	55	43	31	19	7
140	128	116	104	92	80	68	56	44	32	20	8
141	129	117	105	93	81	69	57	45	33	21	9
142	130	118	106	94	82	70	58	46	34	22	10
143	131	119	107	95	83	71	59	47	35	23	11
144	132	120	108	96	84	72	60	48	36	24	12
133	121	109	97	85	73	61	49	37	25	13	1
134	122	110	98	86	74	62	50	38	26	14	2
135	123	111	99	87	75	63	51	39	27	15	3
136	124	112	100	88	76	64	52	40	28	16	4
137	125	113	101	89	77	65	53	41	29	17	5
138	126	114	102	90	78	66	54	42	30	18	6

It gives the central block of (2x2) as

84	72
73	61

Step 3: Reflecting the 1st, 3rd, 5th columns (less than 6th) 8th, 10th and 12th columns (greater than 7th) and again reflecting the same odd and even rows (1st, 3rd, 5th rows and 8th, 10th, 12th rows), the 12 x 12 magic square is generated.

138	127	114	103	90	79	67	54	43	30	19	6
137	128	113	104	89	80	68	53	44	29	20	5
136	129	112	105	88	81	69	52	45	28	21	4
135	130	111	106	87	82	70	51	46	27	22	3
134	131	110	107	86	83	71	50	47	26	23	2
133	132	109	108	85	84	72	49	48	25	24	1
144	121	120	97	96	73	61	60	37	36	13	12
143	122	119	98	95	74	62	59	38	35	14	11
142	123	118	99	94	75	63	58	39	34	15	10
141	124	117	100	93	76	64	57	40	33	16	9
140	125	116	101	92	77	65	56	41	32	17	8
139	126	115	102	91	78	66	55	42	31	18	7

6	19	30	43	54	67	79	90	103	114	127	138
137	128	113	104	89	80	68	53	44	29	20	5
4	21	28	45	52	69	81	88	105	112	129	136
135	130	111	106	87	82	70	51	46	27	22	3
2	23	26	47	50	71	83	86	107	110	131	134
133	132	109	108	85	84	72	49	48	25	24	1
144	121	120	97	96	73	61	60	37	36	13	12
11	14	35	38	59	62	74	95	98	119	122	143
142	123	118	99	94	75	63	58	39	34	15	10
9	16	33	40	57	64	76	93	100	117	124	141
140	125	116	101	92	77	65	56	41	32	17	8
7	18	31	42	55	66	78	91	102	115	126	139

The central block remains unchanged.

7. CONCLUSION

The technique can be used for finding magic squares from basic Latin squares of any order ($n \geq 1$, for n is doubly-even). The construction is done by fixing the column (row) elements associated with the pivot element, assigning it as diagonal element and arranging other elements in an orderly manner, reflecting first odd columns less than $\left[\frac{n}{2}\right]$ and last even columns, greater than $\left[\frac{n}{2}+1\right]$, and reflecting again the same first odd rows and last even rows. The magic square generated by this process gives different central block as compared to that of Nordgren [9].

The pair-numbers satisfying the magic parametric constant, T can be used for checking the result in a very comprehensive manner.

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