Dynamical Behavior of Logistic Equation through Symbiosis Model

M.Sambasiavm

Assistant Professor in Mathematics, J.N.R.M, Port Blair

Abstract-The stability of the fixed points of two-dimensional dynamical system is analyzed through Jacobian matrix and using its Eigen values. The species disappears for $\lambda \in (0, 0.75)$ due to non stable co-existence. The species is synchronized to a stable non-vanishing fixed quantity when $\lambda \in (0.75, 0.866)$.Each one of the species oscillates out of phase between the same two fixed values when $\lambda \in (0.866, 0.957)$ and the species oscillate among infinitely many different states when $\lambda \in (0.957, 1.03)$ but symbiosis model moves towards the chaotic region when the value of the parameter λ crosses one.

Index Terms: Discrete dynamics, chaos, eigen values, symbiosis, attractor.

I.INTRODUCTION

In Recent Mathematics, chaotic maps are maps which exhibit some sort of dynamical behaviour. These maps are parameterized by a discrete time or continuous time parameter. These maps are often seen in dynamical system. Logistic map is one of the examples for the chaotic map in discrete dynamical system [1]. It also describes how the dynamical system becomes chaos. The first mathematical model for population dynamic was studied by Malthus in 1798 [7] in the form of equation $X_{n+1} = rX_n$ … (1). He tells that the human population was growing geometrically while the food supply was growing arithmetically.

The complication with the equation (1) is that the population continues to grow unlimited over time. In 1845, Verhulst showed that the population growth not only depends on the population size but also on how far the size is from its upper limit [9]. The equation (1) is redefined as $\frac{dp}{dt}$ = kp – np^2 . Verhulst called this equation as Logistic equation. This equation is the basis of modern chaos theory, namely the example for the period doubling. The discrete version of Verhulst model for the evolution of the population of a species is given by $X_{n+1} = rX_n\,(1 - X_n)$ . . . . . . (2) where X_n represents the population of an isolated species after n generations and assume that this variable is bounded in the range $0 < X_n < 1$. The parameter r lies between 0 and 4 in order to assure that $0 < X_n < 1$. The expanding plane is controlled by the term rX_n proportional to the current population X_n and to the constant growth rate r. The population dynamics of one species can be investigated through one dimensional logistic map and the study reveals that, the dynamics of one species exhibits chaos[8]. The logistic equation of one species is extended for the purpose of understanding the dynamic behaviour of 2-species.

II.COUPLED LOGISTIC EQUATION

Suppose, if two species (X_n, Y_n) are living together, and then each species evolve in the following logistic type dynamics $X_{n+1} = \mu_x(Y_n)X_n(1 - X_n)$; $Y_{n+1} = \mu_y(X_n)Y_n(1 - Y_n)$

The interaction between the species causes the growth rate μ_x. The term μ_x is a parameter because it varies with time. The parameter μ_x largely depends on other population size and on a positive constant λ . The positive constant λ measures the strength of the mutual interaction.

When the two species interact, there could be a chance for linear increasing and decreasing. This linear increasing and decreasing is based on the growth rate μ and therefore the parameter μ is given by the functions $\mu_1 = \lambda(3X + 1)$ and $\mu_2 = \lambda(-3X + 4)$ respectively and it lies in the interval [1, 4].

III.SYMBIOSIS MODEL

The symbiosis model is used to study the dynamics of two isolated symbiotic species. The discrete version for the symbiosis model is,

$$\left.\begin{aligned} X_{n+1} &= \lambda(3Y_n + 1)X_n(1 - X_n) \\ Y_{n+1} &= \lambda(3X_n + 1)Y_n(1 - Y_n) \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots(3)$$

here λ is the parameter, which measures the mutual benefit between the species. Therefore the parameter λ is also called as mutual benefit. The study reveals that the λ is valid or meaningful only at the range of 0< λ<1.08, which is discussed in Results and discussion.

This application can be represented by

$$T_{\lambda}: [0,1] \times [0,1] \rightarrow R^2$$

$$T_{\lambda}(X_n, Y_n) = (X_{n+1}, Y_{n+1})$$

Where, λ is a real and adjustable parameter. Suppose X_n=0 or Y_n=0 then, the logistic dynamics of one species is recovered [6].

The symbiosis model is derived from the logistic type components, thus, the effect of parameter $'\lambda'$ of logistics map is lost and a new scenario emerges. The symbiotic relationship between the species makes the system to reach different stable states.

The behaviour of the symbiosis model is based on $'\lambda'$ (mutual benefit). By varying the value of this $'\lambda'$, the system is taken to the different states such as an extinction state, a fixed synchronized state [2], a bi-stable lag synchronized state, and an oscillating dynamics among infinitely many possible states [4]. These effects are caused by the symbiotic coupling of the species and it is not predictable from the properties of the individual logistic evolution of each one of them. Moreover this interaction implies a mutual profit for both the species.

One remarkable property is observed in the symbiosis model, i.e. the property which is occurred between the parameter of isolated and coupled species. In the isolated species, at the parameter value r<1 there is no existence of species whereas in the coupled species existence of species is possible at the same parameter value.

IV. EIGEN VALUES FOR THE FIXED POINTS OF SYMBIOSIS MODEL

In dynamics, bifurcation plays an important role and these bifurcations happen in the interval 0< λ <1.0843. In this range there exists stable attractor P_i for each value of λ.

Finding the fixed point is an important task to analyze the behaviour of the dynamical system. The existence of non-trivial fixed point at each 'n' ensures the non-trivial evolution of the system [5]. Here the fixed point of the equation (3) and their restricted maps fixed points are calculated.

- The restriction of $T_{\lambda}(X_n, Y_n) = (X_{n+1}, Y_{n+1})$ to the diagonal, gives a one dimensional cubic map. Therefore the equation (3) is modified as

$$\left.\begin{aligned} X_{n+1} &= \lambda(3X_n + 1)X_n(1 - X_n) \\ Y_{n+1} &= \lambda(3Y_n + 1)Y_n(1 - Y_n) \end{aligned}\right\} \quad \ldots\ldots(4)$$

The solution for $X_{n+1} = X_n$ and $Y_{n+1}=Y_n$ are the fixed points of the equation (4). Therefore

- $X_{n+1} = \lambda(3X_n + 1)X_n(1 - X_n)$ becomes $X_n = \lambda(3X_n + 1)X_n(1 - X_n)$
  $\Rightarrow X_n = \frac{1 \pm \sqrt{4 - \frac{3}{\lambda}}}{3}$

Similarly for $Y_{n+1}=Y_n$

- $Y_{n+1} = \lambda(3Y_n + 1)Y_n(1 - Y_n)$ becomes $Y_n = \lambda(3Y_n + 1)Y_n(1 - Y_n)$
  $\Rightarrow Y_n = \frac{1 \pm \sqrt{4 - \frac{3}{\lambda}}}{3}$

Since the map T is restricted to the diagonal, so the points are equal i.e. $X_n = Y_n$.Therefore the fixed points are

$$P_3 = \left[\frac{1 - \sqrt{4 - \frac{3}{\lambda}}}{3}, \frac{1 - \sqrt{4 - \frac{3}{\lambda}}}{3}\right]$$

$$P_4 = \left[\frac{1 + \sqrt{4 - \frac{3}{\lambda}}}{3}, \frac{1 + \sqrt{4 - \frac{3}{\lambda}}}{3}\right]$$

The restriction of map $T_{\lambda}(X_n, Y_n) = (X_{n+1}, Y_{n+1})$ to the axes is reduced to the logistic map $X_{n+1} = f(X_n)$ with $f(X_n) = \lambda X_n(1 - X_n)$. Thus the equation (3) is modified as

$$\left.\begin{aligned} X_{n+1} &= \lambda X_n(1 - X_n) \\ Y_{n+1} &= \lambda Y_n(1 - Y_n) \end{aligned}\right\} \quad \ldots\ldots\ldots(5)$$

The solution for $X_{n+1} = X_n$ and $Y_{n+1}= Y_n$ are the fixed points of the equation (5). Therefore

- $X_{n+1} = \lambda X_n(1 - X_n)$ became $X_n = \lambda X_n(1 - X_n)$

$\Rightarrow X_n = 1 - \frac{1}{\lambda}$

- $Y_{n+1} = \lambda Y_n(1 - Y_n)$
  $Y_n = \lambda Y_n(1 - Y_n)$
  $\Rightarrow Y_n = 1 - \frac{1}{\lambda}$

Therefore equation (5) has the following fixed points

$P_1 = \left(1 - \frac{1}{\lambda}, 0\right)$(since Y_n=0)

$P_2 = \left(0, 1 - \frac{1}{\lambda}\right)$(since X_n=0)

The fixed points P_0, P_3, P_4 are on the diagonal and P_1, P_2 are on the axes, where P_0 is a trivial fixed point.

The stability of the fixed points of two-dimensional dynamical system is analyzed through Jacobian matrix J and using Eigen values r. If the fixed point is stable then $|r|$<1, otherwise the point is unstable. The Eigen values of the fixed points are calculated by finding the Jacobian matrix for the dynamical system and compute the characteristic equation for it.

The Jacobian matrix for 2D dynamical system

$$J = \begin{pmatrix} \partial X_{n+1}/\partial X_n & \partial X_{n+1}/\partial Y_n \\ \partial Y_{n+1}/\partial X_n & \partial Y_{n+1}/\partial Y_n \end{pmatrix}$$

Jacobian matrix for equation (3) is

$$J = \begin{pmatrix} \lambda(3Y_n + 1)(1 - 2X_n) & 3\lambda(X_n - X_n^2) \\ 3\lambda(Y_n - Y_n^2) & \lambda(3X_n + 1)(1 - 2Y_n) \end{pmatrix}$$

Now explore the Jacobian matrix for all the fixed points and identifythe corresponding Eigen values.

Let$J^0 = J \; at \; P_0(0, 0)$

$$J\big|_{P_0} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$$

The characteristic equation for J^0 is $|J^0 - r_o I| = 0$ (where r_o is Eigen value and I is unit matrix)

$$\begin{vmatrix} \lambda - r_o & 0 \\ 0 & \lambda - r_o \end{vmatrix} = 0$$

$(\lambda - r_o)^2 = 0 \Rightarrow \lambda = r_o$

Therefore Eigen value of the fixed point P_0 is $r_o = \lambda$

Let$J^1 = J \; at \; P_1 = \left(1 - \frac{1}{\lambda}, 0\right)$

$$J\big|_{P_1} = \begin{pmatrix} 2 - \lambda & 3 - \frac{3}{\lambda} \\ 0 & -3 + 4\lambda \end{pmatrix}$$

The characteristic equation for J^1 is $|J^1 - r_1 I|$=0

$$\begin{vmatrix} 2 - \lambda - r_1 & 3 - \frac{3}{\lambda} \\ 0 & -3 + 4\lambda - r_1 \end{vmatrix} = 0$$

$$\Rightarrow r_1^2 + r_1(1 - 3\lambda) + 11\lambda - 4\lambda^2 - 6 = 0$$

The Eigen values of the fixed point P_1 are

$$r_1 = 2 - \lambda r_1' = -3 + 5\lambda$$

Let $J^2 = J \; at \; P_2 = \left(0, 1 - \frac{1}{\lambda}\right)$

$$J\big|_{P_2} = \begin{pmatrix} -3 + 6\lambda & 0 \\ 3 - \frac{3}{\lambda} & 2 - \lambda \end{pmatrix}$$

The characteristic equation for J^2 is$|J^2 - r_2 I|$=0

$$\begin{vmatrix} -3 + 6\lambda - r_2 & 0 \\ 3 - \frac{3}{\lambda} & 2 - \lambda - r_2 \end{vmatrix}$$

$$\Rightarrow r_2^2 + r_2(1 - 5\lambda) + 15\lambda - 6\lambda^2 - 6 = 0$$

$$r_2 = 2 - \lambda$$

$$\Rightarrow r_2' = 3(1+2\lambda)$$

The Eigen values of the fixed point P_2 are

$$r_2 = 2 - \lambda, \; r_2' = 3(1+2\lambda)$$

Let $J^3 = J \; at \; P_3 = \frac{1}{3}\left(1 - \sqrt{4 - 3/\lambda}, 1 - \sqrt{4 - 3/\lambda}\right)$

$$J\big|_{P_3} = \begin{pmatrix} 2 + (-2 + \sqrt{4 - 3/\lambda})\lambda & \frac{1}{3}(3 - (2 + \sqrt{4 - 3/\lambda})\lambda) \\ \frac{1}{3}(3 - (2 + \sqrt{4 - 3/\lambda})\lambda) & 2 + (-2 + \sqrt{4 - 3/\lambda})\lambda \end{pmatrix}$$

The characteristic equation for J^3 is $|J^3 - r_3 I|$=0

$$\begin{vmatrix} 2 + (-2 + \sqrt{4 - 3/\lambda})\lambda - r_3 & \frac{1}{3}(3 - (2 + \sqrt{4 - 3/\lambda})\lambda) \\ \frac{1}{3}(3 - (2 + \sqrt{4 - 3/\lambda})\lambda) & 2 + (-2 + \sqrt{4 - 3/\lambda})\lambda - r_3 \end{vmatrix} = 0$$

$$\Rightarrow r_3^2 - 2r_3\left(2+\left(-2+\sqrt{4-\frac{3}{\lambda}}\right)\lambda\right)+\frac{1}{9}\left(27+42\left(-2+\sqrt{4-\frac{3}{\lambda}}\right)\lambda\right) - 8\left(-8+5\sqrt{4-3/\lambda}\right)\lambda^2)=0$$

$$\Rightarrow r_3=\frac{1}{3}\left(6-6\lambda+3\lambda\sqrt{4-\frac{3}{\lambda}}\right)\pm\frac{1}{3}\left(9+12\lambda-6\lambda\sqrt{4-3/\lambda}-28\lambda^2+4\lambda^2\sqrt{4-3/\lambda}+9(4-\frac{3}{\lambda})\lambda^2\right)^{1/2}$$

The Eigen values of the fixed point P_3 are

$$r_3=\frac{1}{3}\left(6-6\lambda+3\lambda\sqrt{4-\frac{3}{\lambda}}\right)\pm\frac{1}{3}\left(9+12\lambda-6\lambda\sqrt{4-3/\lambda}-28\lambda^2+4\lambda^2\sqrt{4-3/\lambda}+9(4-\frac{3}{\lambda})\lambda^2\right)^{1/2}$$

Let $J^4 = J \; at \; P_4 = \frac{1}{3}\left(1+\sqrt{4-{}^{3}/_{\lambda}}, 1+\sqrt{4-{}^{3}/_{\lambda}}\right)$

$$J\Big|P_0 = \begin{pmatrix} 2-(2+\sqrt{4-3/\lambda})\lambda & \frac{1}{3}(3-(2+\sqrt{4-3/\lambda})\lambda) \\ \frac{1}{3}(3-(2+\sqrt{4-3/\lambda})\lambda) & 2-(2+\sqrt{4-3/\lambda})\lambda \end{pmatrix}$$

The characteristic equation for J^4 is $|J^4 - r_4 I| = 0$

$$\begin{vmatrix} 2-(2+\sqrt{4-3/\lambda})\lambda & \frac{1}{3}(3-(2+\sqrt{4-3/\lambda})\lambda) \\ \frac{1}{3}(3-(2+\sqrt{4-3/\lambda})\lambda) & 2-(2+\sqrt{4-3/\lambda})\lambda \end{vmatrix} = 0$$

$$\Rightarrow r_4^2 - 2r_4\left(-2+\left(2+\sqrt{4-3/\lambda}\right)\right)+\frac{1}{9}\left(27-6\left(14+5\sqrt{4-3/\lambda}\right)\lambda+32\left(2+\sqrt{4-3/\lambda}\right)\lambda^2\right)=0$$

The Eigen values of the fixed point P_4 are

$$r_4 = \frac{1}{3}\left(6-3\left(2+\sqrt{4-3/\lambda}\right)\lambda\right)\pm\left(\frac{1}{3}\left(9-3\left(5+2\sqrt{4-3/\lambda}\right)+4\left(2+\sqrt{4-3/\lambda}\right)\lambda^2\right)^{1/2}\right)$$

Using these Eigen values, the range at which the existence of attractor, repellor and bifurcation of fixed points are explained in the following lines.

The point P_0 (0, 0) is an attractor node in the interval 0< λ <1, because the modulus of the Eigen value of P_0 is less than one in this interval or beyond this range the point P_0 becomes a repelling point.

The points P_1 and P_2 values exist but they are unstable for every value of λ. That is, the modulus of the Eigen values of P_1 and P_2 is always greater than one.

The points P_3, P_4 are complex points in the range 0< λ <0.75, so the bifurcation of the system begins beyond this range. When λ=0.75, there is a saddle node bifurcation on the diagonal. This bifurcation point generates two points namely P_3 *and* P_4. Among these two points P_3 is a saddle point and P_4 is an attractive node in the range 0.75< λ <0.866. The value of λ crosses 0.866, the point P_4 suffers a flip bifurcation. This bifurcation again generates stable orbits of period two P_5, P_6 outside the diagonal. These points are obtained by solving the quadratic equation $\lambda(4\lambda+3)X^2 - 4\lambda(\lambda+1)X_1 + \lambda = 0$.

The solutions are

$$P_5 = \left(\frac{2\lambda(\lambda+1)+\sqrt{\lambda(\lambda+1)(4\lambda^2-3)}}{\lambda(4\lambda+3)}, \frac{2\lambda(\lambda+1)-\sqrt{\lambda(\lambda+1)(4\lambda^2-3)}}{\lambda(4\lambda+3)}\right)$$

$$P_6 = \left(\frac{2\lambda(\lambda+1)-\sqrt{\lambda(\lambda+1)(4\lambda^2-3)}}{\lambda(4\lambda+3)}, \frac{2\lambda(\lambda+1)+\sqrt{\lambda(\lambda+1)(4\lambda^2-3)}}{\lambda(4\lambda+3)}\right)$$

The two fixed points P_5, P_6 serve as an attractor in the range 0.866< λ <0.957. When λ takes 0.957, the period-2 symmetric points lose its stability via Neimark Hopf-bifurcation. The stable closed invariant curves correspond to P_5, P_6 grows when λ lies in the range 0.957< λ <1. In this range, frequency locking windows are also obtained for some values of λ. The period two cycles on the axes appear by a period doubling bifurcation and are found solving the cubic equation.

$$\lambda^3X^3 - 2\lambda^3X + (\lambda^2+\lambda^3)X + 1 - \lambda^2 = 0$$

. They have existence for $\lambda < 3$. The solutions are

$$P_7 = \left(\frac{(\lambda+1)-(\lambda+1)(\lambda-3)^{1/2}}{2\lambda}, 0\right)$$

$$P_8 = \left(\frac{(\lambda+1)+(\lambda+1)(\lambda-3)^{1/2}}{2\lambda}, 0\right)$$

$$P_9 = \left(0, \frac{(\lambda+1)-(\lambda+1)(\lambda-3)^{1/2}}{2\lambda}\right)$$

$$P_{10} = \left(0, \frac{(\lambda+1)+(\lambda+1)(\lambda-3)^{1/2}}{2\lambda}\right)$$

V. RESULTS AND DISCUSSION

The symbiosis model moves towards the chaotic region when the value of the parameter λ crosses one[3]. Here we summarize the dynamical behavior of the symbiosis model when λ is modified.

Case (i) $\lambda \in (0,0.75)$

The extinction of species occurred at 2346th iteration when $\lambda = 0.738$ with initial values x(0) =0.9 and y(0)=0.49 . The corresponding output is

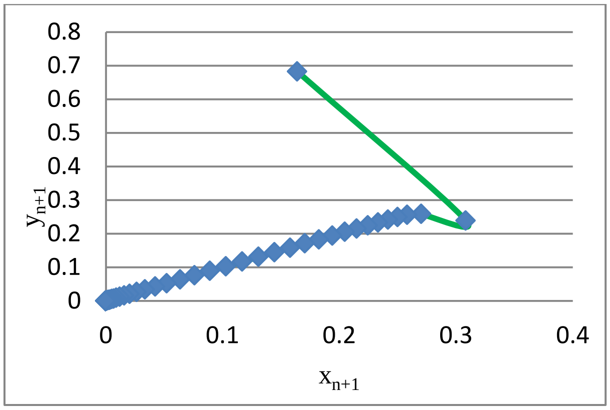

Fig 01

The mutual benefit λ is not big enough to allow a stable co-existence of both species, so the species will disappear.

Case (ii) $\lambda \in (0.75,0.866)$

The two species converge to the fixed value 0.551551 at 45th iteration when $\lambda = 0.839$ with initial values x(0) =0.98 and y(0)=0.439 . The corresponding output is

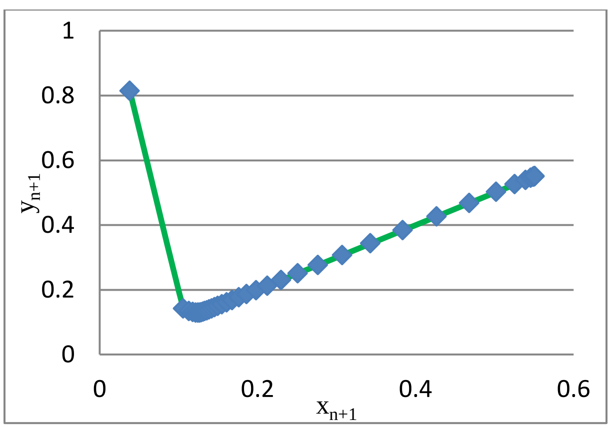

Fig 02

A sudden change is obtained when λ is greater than 0.75. Both the populations are synchronized to a stable non-vanishing fixed quantity.

Case (iii) $\lambda \in (0.866,0.957)$

The two species oscillate between the two points (0.483189, 0.669157) and (0.669157, 0.483189) beyond 127th iteration when $\lambda = 0.891$ with initial values x(0) =0.151 and y(0)=0.22 . The corresponding output is

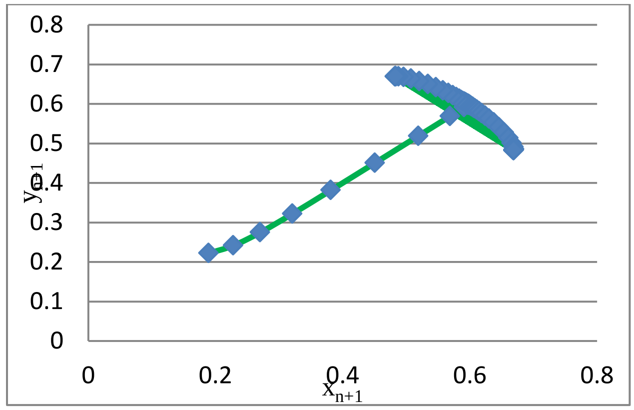

Fig 03

The system is in the state of bi-stable when the value of λ between this range. Each one of the species oscillates out of phase between the same two fixed values. This is said to be a stable 2-period orbit.

Case (iv) $\lambda \in (0.957,1.03)$

From the iteration values, the species lost its periodic oscillation and moved towards the quasi periodic oscillation when $\lambda = 0.957$ with initial values x(0) =0.91 and y(0)=0.5. The ocrresponding output is

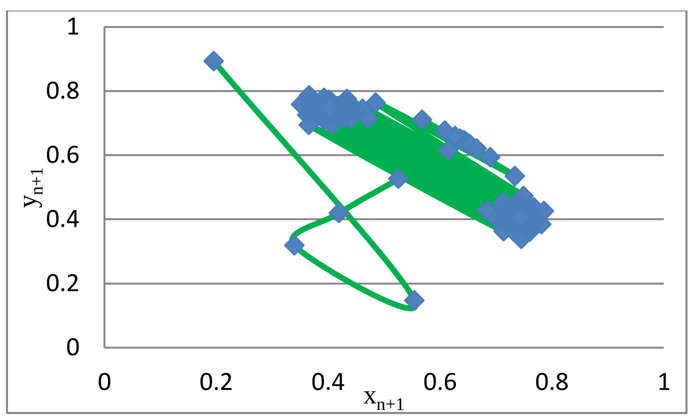

Fig 04

In this range, the system is not in the state of periodic orbit. It acquires a new frequency called quasi-periodic. In this range both populations oscillate among infinitely many different states and the synchronization is lost.

Case (v) $\lambda \in (1.03, 1.0843)$

The iteration values are in random order and therefore the species are in chaotic stage when λ =1.032 with initial values x(0) =0.8 and y(0)=0.9. The corresponding output is

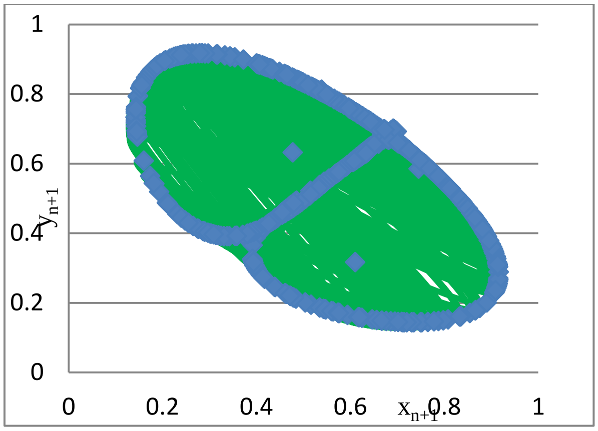

Fig 05

In this range, the system is in a chaotic regime, with non-periodic unpredictable twists.

VI. CONCLUSION

In this model the interaction of two species results in mutual benefit. That is the growth of one species does not affect the growth of other species and therefore the parameter λ in this model is called as "mutual benefit". The complex behaviour of the symbiosis is described when the value of λ is modified. The mutual benefit λ takes the system to various stages such as, for smaller value of λ, the species become extinct, and for the increasing value of λ the system oscillates in a 2-periodic orbit. The chaotic behaviour of the system evolves for larger vaule of λ. The above mentioned effects are caused by the symbiotic coupling of the species and it is not predictable from the properties of the individual logistic evolution of each one of them.

ACKNOWLEDGEMENT

I would like to thank Dr. Nandikeswari R for her kind Cooperation and continuous encouragements. Also I thank Dr.Vivek Kumar Sahu for his valuable suggestions.

REFERENCES

[1] Atlee Jackson. E., [1990] "Perspective of Non-Linear Dynamics", Vol. 2, Cambridge.

[2] Boccaletti, S., Kurths, J., Osipov, G., Valladares, D.L. and Zhou, C.S. [2002] "The synchronization of chaotic systems", Phys. Rep. 366, pp. 1-101.

[3] Frouzakis, C., Gardini, L., Kevrekidis, I., Millerioux, G. and Mira, C. [1997] "On some properties of invariant sets of two-dimensional noninvertible maps", International Journal of Bifurcation and Chaos 7 (6), pp. 1167-1194.

[4] Heagy, J.F., Carroll, T.L. and Pecora, L.M. [1995] "Desynchronization by periodic orbits", Phys. Rev. E 52, R1253–R1256.

[5] López-Ruiz, R. and Pérez-Garcia, C. [1991] "Dynamics of maps with a global multiplicative coupling," Chaos, Solitons & Fractals 1 (6),pp. 511- 528.

[6] Lopez-Ruiz. R and Fournier- Prunaret, D.,[2004] " Complex behavior in a discrete coupled logistic model for the symbiotic interaction of two pecies", Mathematical Bioscience and Engineering, 1, pp. 307-324.

[7] May, R.M. [1976], "Simple mathematical models with very complicated dynamics", Nature 261, pp. 459-467.

[8] Sambasivam, M. and Grurajan, N. [2010]," Chaos-Logistic map – Tent map –Corresponding Cellular automata", Mapana journal of Sciences,Vol.9 (2), pp. 28-34.

[9] Verhulst, P.F. [1845] "Recherches mathématiques sur la loi d'accroissement de la population.", Nouv. Mém. de l'Academie Royale des Sci. et Belles-Lettres de Bruxelles 18, pp. 1- 41.

AUTHORS

Corresponding Author – M.SAMBASIVAM, Assistant Professor, J.N.R.M (Govt. P. G. College), Port Blair, Affiliated to Pondicherry University.
Email: **navinpraem@gmail.com**