Analysis of GI/GI/1 Queue by Push-Out Simulation Technique

M. S. Rawat*, Harish Rawat and S.R. Ansari*****

* Deptt. of Mathematics, H.N.B. Garhwal University Srinagar Garhwal, 246174
** Deptt. of Mathematics, H.N.B. Garhwal University Srinagar Garhwal, 246174
*** Deptt. of Mathematics, H.N.B. Garhwal University Srinagar Garhwal, 246174

Abstract- This paper is concerned with the sensitivity analysis of GI/GI/1 queueing model by "push-out" simulation technique, which is based on discrete event dynamic system using structural parameters. The study of "push-out" technique is extension of score function simulation method for sensitivity analysis and stochastic optimization.

Index Terms- Simulation, score function, structural parameter, joint distribution.
Mathematical Subject Classification**:** 60K25, 60K30.

I. Introduction

In this paper we discuss the sensitivity analysis by simulation transformation technique called "push-out". We consider GI/GI/1 queueing model which is based on discrete events dynamic system using structural parameters. The customers arrive and take services as general distribution using single server. The study of the GI/GI/1 queue is extension of score function simulation method for sensitivity analysis and stochastic optimization.

Several researchers have discussed sensitivity analysis and stochastic optimization for many queueing models in different frame-works, Asmussen and Melamed [1] considered regenerative simulation of TES progress. Dussault et al. [2] discussed combining stochastic counterpart and stochastic approximation methods. Glasserman [3] discussed gradient estimation via perturbation analysis. Sensitivity analysis of GI/GI/m/B queue with respect to buffer size by the score function method was discussed by Kriman [5]. Convergent rates for steady- state a derivative estimator was considered by L' Ecuyer [6]. Marti [7] discussed stochastic optimization method of structural design and considered efficiency of score function method for sensitivity analysis and optimization of queueing networks. Rubinstein [10] discussed sensitivity analysis of discrete events dynamic system by the "push- out" method.

This term is derived from the fact that we push-out the parameter v_2 from the original sample performance $L(Y, v_2)$ into an auxiliary probability function (pdf) via a suitable transformation, and then apply the score function simulation method to perform sensitivity analysis and optimization. Then we differentiate the resulting (auxiliary) sample performance with respect to $v = (v_1, v_2)$.

II. Preliminaries

It is desired to estimate the steady- state expected waiting time $\ell(v)$ in stable G/G/1 queue as

$$\ell(v) = E_{v_1}\{L(Y_t, v_2)\} \tag{1}$$

where $L(Y_t, v_2)$ is sample performance depending on the parameter vector v_2 and input sequence $Y_t = \{y_1, y_2 \ldots\ldots y_t\}$ of independent identically distributed random vectors with common pdf (probability density function) $f(y, v_1)$. Subscript v_1 in $E_{v_1}\{L\}$ indicates that the expectation is taken with respect to the pdf $f(y,v_1)$ and combined vector of parameter given here by $v = \{v_1, v_2\}$. We assume f depends on the parameter vector v_1 but not on v_2 and L depend on v_2 but not on v_1. Pflug [8] defined the parameter v_1 and v_2 as the probability measure and random process respectively.

III. "PUSH-OUT" TECHNIQUE

We consider "push- out" technique which shows that typically smoothes out the sample performance function $L(y, v_2)$ with respect to v_2 by rendering it independent of v_2. To determine the idea of the "push-out" technique, let there exists a vector valued function $x = x(y, v_2)$ and the real valued function $\bar{L}(x)$ independent of v_2, such that

$$L(y, v_2) = \bar{L}\{x(y, v_2)\} \tag{2}$$

Suppose that $Y \sim f(y, v_2)$ and the corresponding random vector $X = x(Y, v_2)$ for which the pdf is $\bar{f}(x, v_1, v_2)$, then

$$L(v) = \int_{x \in X} \bar{L}(x)\bar{f}(x, v)\,dx = E_{\bar{f}}\{\bar{L}(x)\} \tag{3}$$

where expectation is now taken with respect to the pdf $\bar{f}(x, v_1, v_2)$. As mentioned in (1) the term "push- out" derived from the fact that the parameter v_2 is pushed out from $L\{y, v_2\}$ to an auxiliary pdf $\bar{f}(x, v_1, v_2)$.

The derivative of $\nabla \ell$(v) is similar, but one needs to make use of the identity

$$\frac{d}{dv}\int_{a(v)}^{b(v)} g(v,x)dx = \int_{a(v)}^{b(v)} \frac{\partial}{\partial v} g(v,x)dx + \frac{db(v)}{dv} g(v,b(v)) - \frac{da(v)}{dv} g(v,a(v)) \quad (4)$$

A representation of L(y, v_2) of the form (1) and the subsequent transformation (2) are not always available, and if available, it may be hard to calculate $\bar{f}(x, v_1, v_2)$. Let for every v_2, x = x(y, v_2) be one-to-one and have an inverse y = y(x, v_2) and assumed to be continuously differentiable in each component of x, then

$$\bar{f}(x,v_1,v_2) = f\{y(x, v_2), v_1\}.\left|\frac{\partial y(x,v_1)}{\partial x}\right| \quad (5)$$

where $\frac{\partial y}{\partial x}$ denotes absolute value of the determinant of the Jacobean of y(x, v_2) with respect to x.

Definition 1. [11]. Let G be the probability measure with pdf (probability density function) g(z) so that
dG(z) = g(z)dz.

sup $\{f(z, v_1)\} \subset$ sup {g(z)}, then ℓ (v) = E_g\{L(Z, v_2)W(Z, v_1)\}
(6)

where W(Z, v_1) = f(z, v_1)/g(z) is likelihood ratio discussed in Glynn [4], and subscript g indicate that the expectation is taken with respect to dominating pdf (probability density function) g(z). Under the standard regulatory condition admitting the interchangeability of the expectation and differentiation operator, we have

$$\nabla_{v_1}^{k}\ell(v) = E_g\{L(Z, v_2)\nabla_{v_1}^{k}W(Z,v_1)\} \text{ for } k = 1, 2, \ldots.. \quad (7)$$

$$\nabla_{v_2}^{k}\ell(v) = E_g\{\nabla_{v_2}^{k} L(Z, v_2) W(Z,v_1)\} \text{ for } k = 1, 2, \ldots. \quad (8)$$

Definition 2. [9]. Let g(z) be a pdf (probability density function) that dominates the pdf $\tilde{f}$ (z, v), then from (6), (7), (8) we have

$$\nabla^{k}\ell(v) = E_g\{\tilde{L}(Z)\nabla^{k}\hat{W}(Z,u)\}$$

where $\hat{W}(Z,u) = \tilde{f}$ (z, u)/g(z) and Z ~ g(z). (9)

Now for a given sample $\{Z_1, Z_2 \ldots Z_n\}$ from g(z) $\nabla^{k}\ell(v)$ may be estimated as

$$\nabla^{k}\tilde{\ell}_N(v) = \frac{1}{N}\sum_{i=1}^{N}\tilde{L}(Z_i)\nabla^{k}\hat{W}(Z_i,v) \quad \text{for } k = 0, 1, 2\ldots$$

where $\nabla^{0}\ell_N(v) = \ell_N(v)$ and estimate $\nabla^{k}\tilde{\ell}_N(v)$ referred to as push-out score function (POSF).

IV. MODEL DESCRIPTIONS

Let L_n and $£_n$ be sojourn time and waiting time distribution of n^{th} customer respectively, and

y_{1n} = service time of n^{th} customer,
y_{2n} = time between arrival of the n^{th} and $(n-1)^{th}$ customer,
where $y_{10} = y_{20} = 0$.

Let X be steady-state random variable with steady-state cumulative density function (cdf)

$F_X(x, v) = p_v(X \le x) = \lim_{n\to\infty} p_v(X_n \le x)$, and $f_x(x_1)$ is pdf of random variable such that $X \in \{L, £\}$ and $f_x(x, v) = \frac{\partial F_x(x,v)}{\partial x}$.

Estimation of $F_x(x, v)$ and $f_x(x, v)$ for related random variables such as virtual waiting time and queue length are discussed by Rubinstein [11]. Let τ be the number of customers served during a busy period in a steady-state GI/GI/1 queue with FCFS (first come first served) discipline. Let v be the service time distribution variable of f(y, v).

V. MAIN RESULTS

Theorem1. Let $X \in \{L, £\}$ the steady-state expected waiting time cdf (cumulative density function)

$F_X(x, v) = p_v(X \le x)$ of the sample performance is estimated as

$$\tilde{F}_{XN}(x,v) = \frac{1}{N}\sum_{i=1}^{N}\tilde{I}(Z_i)\tilde{W}(Z_i,u).$$

Proof: Since $X \in \{L, £\}$ be random variable of the waiting time distribution such that

$X = \sum_{t=1}^{\tau} L_t$ where τ is number of customers served during busy period in the steady-state. Using likelihood ratio which is discussed in Glynn [4], we have

$$F_X(x, v) = E_g\{I_{(-\infty, 0)}(X - x)\tilde{W}_t(Z_\tau, v)\}, \quad (10)$$

where $\tilde{W}_t = \prod_{j=1}^{t} W_j$, $W_j = \frac{f(Z_j,v)}{g(Z_j)}$, $Z_j \sim g(z)$,
and $Z_\tau = (Z_1, Z_2, \ldots\ldots\ldots\ldots Z_\tau)$. The higher order derivatives of F_X may be similarly represented.

Standard likelihood ratio Glynn [4], estimators of $F_X(x, v)$ based on (1) required calculation of the estimator function $I_{(-\infty, 0)}(X - x)$ for each value of x separately. Estimation of $F_X(x, v)$ for multiple values of x and v from single simulation run is as:
$F_X(x, v) = E_v\{I_{(-\infty, 0)}(X - x)\}$.

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Since L_n and $£_n$ are sojourn and waiting time of n^{th} customer, then
$L_n = £_n + Y_{1n}$, for n= 1, 2, 3……..

Now $F_X(x, v)$ may be written as

$F_X(x, v) = E_u\{ I_{(-\infty, 0)}(\tilde{X})\}$ and u = (x, v)

where $\tilde{X} = \sum_{n=1}^{\tau-1} L_n + £_\tau + \tilde{Y}_\tau$,

$\tilde{Y}_\tau = Y_\tau - x$ and $\tilde{Y}_\tau \sim \tilde{f}(y, u)$ for $\tilde{f}(y, 0, u) = f(y, u)$.

So the cdf $F_X(x, v)$ may be represented as

$F_X(x, v) = E_g\{ \tilde{I}_\tau \tilde{W}_\tau(u)\}$ where $\tilde{W}_\tau(u) = \hat{W}_\tau(u) \prod_{j=1}^{\tau-1} W_j(u)$,

$W_j(v) = \dfrac{f(Z_j, v)}{g(Z_j)}$, j = 1, 2 ….. (τ – 1),

$\hat{W}_\tau(u) = \dfrac{\bar{f}(Z_\tau, u)}{g(Z_\tau)}$ for $Z_k \sim g(z)$, k = 1, 2 ….. τ, and indicator function

$\tilde{I}_\tau = I_{(-\infty, 0)}(\sum_{n=1}^{\tau-1} L_n + £_\tau + Z_\tau)$.

Now we may estimate $F_X(x, v)$ as

$$\tilde{F}_{XN}(x, v) = \frac{1}{N}\sum_{i=1}^{N} \tilde{I}(Z_{\tau i})\tilde{W}(Z_{\tau i}, u). \quad (11)$$

where $Z_{\tau i} = (Z_{1i}, Z_{2i}, \ldots\ldots Z_{\tau i})$ for i = 1, 2, 3……..N is a sample from g(z). At steady-state when τ = 1 in equation (11), we get the required result.

Theorem 2. In the sample performance $\tilde{I}_\tau = I_{(-\infty, 0)}(\sum_{n=1}^{\tau-1} L_n + £_\tau + Z_\tau)$, the sojourn time and waiting time distribution $\sum_{n=1}^{\tau-1} L_n$ and $£_n$ can be obtained by variate Z_τ from the dominating pdf (probability density function) g(z).

Proof: Let L be the random variable of sojourn time distribution, then estimation of steady-state cdf $F_L(x, v) = p_v(L \le x)$
Arguing similar to theorem 1, we have
$F_L(x, v) = E_v\{ I_{(-\infty, 0)}(L - x)\}$.

or we can represent $F_L(x, v) = E_u\{ I_{(-\infty, 0)}(\tilde{L}_n)\}$ and u = (x, v)

where $\tilde{L}_n = Z_n + \tilde{Y}_{1n}$, $Y_{1n} \sim f(y, v)$, $\tilde{Y}_{1n} = Y_{1n} - x$, $\tilde{Y}_{1n} \sim \bar{f}(y, v)$ and

$I_n = I_{(-\infty, 0)}(\tilde{L}_n)$ represent regenerative process across busy cycle.

But $$F_L(x, v) = \frac{E_u\{\sum_{n=1}^{\tau} I_n\}}{E_u(\tau)}. \quad (12)$$

Now using (12) and above arguments, we obtain

$$F_L(x, v) = \frac{E_g\{\sum_{n=1}^{\tau} \tilde{I}_n \tilde{W}_n(u)\}}{E_g\{\sum_{n=1}^{\tau} \tilde{W}_n(u)\}} \quad (13)$$

where $\tilde{I}_n = I_{(-\infty, 0)}(£_n + Z_n)$, $\tilde{W}_n(u) = \hat{W}_n(u)\prod_{j=1}^{n-1} W_j(v)$,

$W_j(v) = \dfrac{f(Z_j, v)}{g(Z_j)}$ and

$\hat{W}_n(u) = \dfrac{\tilde{f}(Z_n, v)}{g(Z_n)}$, $Z_n \sim g(z)$ for n = 1, 2, 3, ….τ.

Since $\tilde{Y}_1 = Y_1 + x$, then we choose the dominating probability density function (pdf) g(y) as $\tilde{f}(y, x_0, v_0)$ provided $x_0 \ge x$ and $y \ge 0$. Now $F_L(x, v)$ may be estimated as

$$\tilde{F}_L(x, v) = \frac{\sum_{i=1}^{N} \sum_{n=1}^{\tau_i} \tilde{I}_{ni}(Z_{ni}) \tilde{W}_{ni}(Z_{ni}, u)\}}{\sum_{i=1}^{N} \sum_{n=1}^{\tau_i} \tilde{W}_{ni}(Z_{ni}, u)\}}.$$

where $\{Z_{ni}\} = \{(Z_{1i}, Z_{2i}, \ldots\ldots Z_{ni}): n = 1, 2, \ldots\ldots \tau_i, i = 1, 2, \ldots\ldots N)\}$ is a random sample from dominating pdf g(z).

Remark: When both variate $£_n$ and L_n are associated with the sample performance
$\tilde{I}_n = I_{(-\infty, 0)}(£_n + Z_n)$, can be obtained by simulation and $\tilde{Y}$ can be similarly generated from the dominating pdf, g(y) = $\tilde{f}(y, x_0, v_0)$.
To estimate the steady-state pdf f of the sojourn time process $\{L_n\}$ we have to differentiate (7) with respect to x and similarly for the estimation of higher order derivatives $F_L(x, v)$.

The condition is applicable to estimate $F_L(x, v)$ when the underlying process $\{£_n\}$ is not known analytically.

Example: Let the probability density function of waiting time distribution $f(y, v) = v \exp(-vy)$ be known, then we can estimate the waiting time auxiliary pdf (probability density function).

Solution: Since $f(y, v) = v \exp(-vy)$ then

$\tilde{f}(y, x, v) = v \exp(-v(y + x)), y \geq -x.$

Now dominating pdf may be selected as

$g(y) = v_0 \exp(-v_0(y + x_0))$ where $x < x_0$, $y > -x_0$ and

$\tilde{f}(y, x, v) = f(y, v)$ for $n = 1, 2 \ldots\ldots \tau - 1.$

So, $\tilde{f}(y, x, v) = v \exp(-v(y + x))$ for $n = \tau$.

Similarly we can estimate the dominating pdf g(y).

It is important to note that, the procedure above can be adapted to general queueing networks, particularly to those in which the distribution of neither L_n nor $£_n$ is not analytically available.

VI. Discussion

In this paper we discuss the sensitivity analysis of GI/GI/1 queueing model by a transformation technique called "push-out" simulation technique. The study of "push-out" technique is extension of score function method for sensitivity analysis and stochastic optimization. We discuss the conditions under which such types of transformation are useful.

When sojourn and waiting time of any customer in the queueing system are not available analytically, then using suitable transformation and score function method, we can estimate the waiting and sojourn time distribution of the customer for the given model. "push-out" technique is a transformation technique, which is totally a model dependent.

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Authors

First Author – M. S. Rawat, Professor, M.Sc., Ph.D., Deptt. of Mathematics, H.N.B. Garhwal University Srinagar Garhwal, 246174

Second Author – Harish Rawat, Research Scholar, M.Sc., , Deptt. of Mathematics, H.N.B. Garhwal University Srinagar Garhwal, 246174

Third Author – S.R. Ansari, Professor, M.Sc., Ph.D., , Deptt. of Mathematics, H.N.B. Garhwal University Srinagar Garhwal, 246174

Correspondence Author – Harish Rawat, E–mail: rawatharishrch@gmail.com, Contact number. 09720560402