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# Development of Mathematical Model and Stability Analysis for UAH

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***Abstract-*** Unmanned Aerial Vehicle become popular and very useful today. Unmanned Aerial Vehicle types have fixed wing UAV and rotary wing UAV. Among rotary wing UAV, helicopter type is chosen to implement the mathematical model for UAH. Since these helicopter mathematical model is very complex and inflexible, simple model "minimum-complexity helicopter simulation math model" (MCHSMM) is used to develop a non-linear mathematical model R-50 helicopter for this work. This modeling consists of three blocks: (1) Rigid body dynamics, (2) Force and torque dynamics, and (3) Flapping and thrust dynamics. Each of these are implemented in MATALB SIMULINK. There are four control inputs: $u_{lat}$ (lateral control input), $u_{long}$ (longitudinal control input), $u_{col}$ (collective control input) and $u_{ped}$ (pedal control input). By alternating positive, negative, and zero to four control inputs, the resultant responses of positions, Euler angles, translatory velocities, angular velocities and flapping angles are simulated. The stability analysis for UAH are also analyzed and simulation results provides the real world approaches for UAH system.

***Index Terms***- Unmanned Aerial Helicopter (UAH), R-50 helicopter, Mathematical Model, MATLAB SIMULINK, MCHSMM

## I. Introduction

In order to successfully control an unmanned aerial helicopter (UAH), a mathematical model is generally developed based on either Newton's laws of motion or the Euler-Lagrange equation for motion. Helicopters have become an interesting area of study due to their flight capabilities of hovering, vertical takeoff and landing (VTOL), flying forward, backwards, and laterally Such VTOL vehicles will provide reconnaissance, surveillance and target acquisition assistance to the ground troops, and offering major advantages over fixed-wing unmanned aircrafts, like flying in very low altitudes, taking off and landing almost everywhere [4].

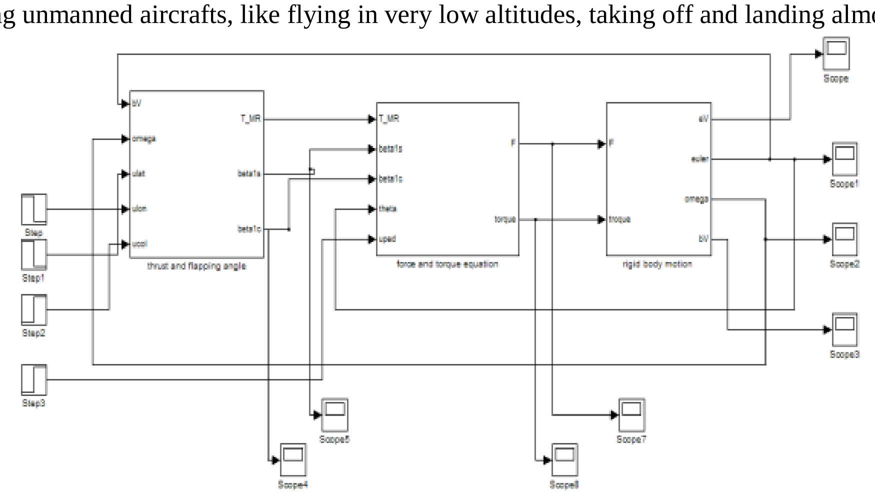


Figure1. Overall nonlinear SIMULINK model for R-50 helicopter

Rigid body dynamic can be used not only for fixed wing aircraft but also rotary wing aircrafts. Force and torque are inputs for rigid body block and Euler angles, translational velocities, angular velocity and positions are outputs. For force and torque equations block, main rotor thrust, tail rotor thrust and flapping angles (Control rotor flapping and Main rotor flapping) are inputs. For flapping and thrust equations, there are four control inputs: three inputs on main rotor and one input on tail rotor. While modeling, some parameters are assumed constant or zero such as wind velocity, air density and center of gravity. The equations derived by using top-down modeling are implemented with the help of SIMULINK to obtain the non-linear mathematical model for R-50 helicopter [1]. The main purpose of this project is to understand the basics of helicopter modeling, develop a mathematical simulation model of an autonomous helicopter based on information of a Yamaha R-50 model helicopter**.**

## II. MATHEMATICAL MODEL OF R-50 HELICOPTER

Among the categories of unmanned helicopter system, Ursa Magna Series-Yamaha R-50 is chosen for the development of mathematical model. Yamaha R-50 was originally developed in Japan for pesticide spraying in rice fields. The Yamaha R-50 is powered by a water-cooled, single cylinder, 12-hp, and 98 cc two-stoke gasoline engine. A special external engine starter is required for the engine. The engine is very reliable and powerful because it can carry 20 kg of payload. Its length, height and width are 3.58 meters, 0.7 meters and 1.08 meters respectively. Its rotor diameter and dry weight are 3.070 meters and 44 kg. Figure 1 shows the Yamaha R-50 helicopter [2].

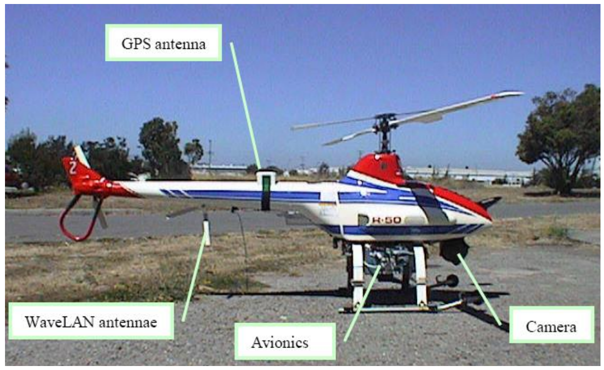


Figure 2. Yamaha R-50 Helicopter

The modelling of the Yahama-R-50 helicopter will be performed using a top-down principle. The modeling with top-down principle consists of three parts. It is shown in Figure 2 [1].

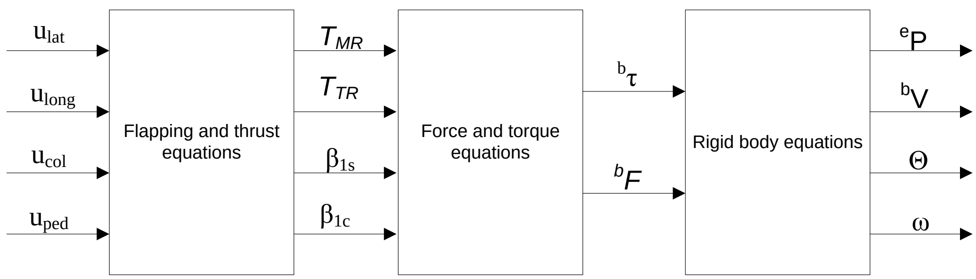


Figure 3. The three parts of the top down modeling with appertaining inputs and outputs

There are four basic control channels on a helicopter: main rotor collective pitch ($u_{col}$), longitudinal cyclic ($u_{long}$), lateral cyclic ($u_{lat}$) and tail rotor collective pitch ($u_{ped}$). Four basic control channels ($u_{col}$, $u_{long}$, $u_{lat}$, and $u_{ped}$) make the helicopter to perform a roll, to perform a pitch, to perform vertical movement and to perform a yaw respectively. The thrust generated by the main rotor is perpendicular to the tip path plane (TPP) and essentially controls the altitude of the helicopter. The tail rotor thrust direction is opposite to the main rotor thrust direction. The tail rotor thrust is for upwards and main rotor is for heading at the moment of helicopter hovering.

The Euler rotation to the roll, pitch and yawl axes are as follow;

$$C_x(\phi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & cos\phi & sin\phi \\ 0 & -sin\phi & cos\phi \end{bmatrix}$$
$$C_y(\theta) = \begin{bmatrix} cos\theta & 0 & -sin\theta \\ 0 & 1 & 0 \\ sin\theta & 0 & cos\theta \end{bmatrix} \qquad (1)$$
$$C_z(\psi) = \begin{bmatrix} cos\psi & sin\psi & 0 \\ -sin\psi & cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

where $\phi$, $\theta$ $and$ $\psi$ are Euler angles between the SF to BF.

The relationship between the Euler angle rates $(\dot{\phi}, \dot{\theta}, \dot{\psi})$ and the angular velocities in the body frame $(p, q, r)$ can be derived to give the matrix equation below:

$$\begin{bmatrix}\dot{\phi}\\ \dot{\theta}\\ \dot{\psi}\end{bmatrix} = \begin{bmatrix}1 & s\phi t\theta & c\phi t\theta\\ 0 & c\phi & -s\phi\\ 0 & \frac{s\phi}{c\theta} & \frac{c\phi}{c\theta}\end{bmatrix}\begin{bmatrix}p\\ q\\ r\end{bmatrix} \quad (2)$$

Helicopter consists of Translational dynamics and Rotational dynamics equations.

$$F = m\frac{d}{dt}V|_b + \omega \times mV \quad (3)$$

$$M = m\frac{d}{dt}H|_b + \omega \times H \quad (4)$$

where F is the forces in body frame, V is the aircraft velocity, M is torques, H is the angular momentum, m is the mass of the helicopter and $\omega$ is the total angular velocity.

Among the categories of unmanned helicopter system, Ursa Magna Series-Yamaha R-50 is chosen for the development of mathematical model. Yamaha R-50 was originally developed in Japan for the crop-dusting. Lately, it is also used for both civil and military applications. The Yamaha R-50 is the type of using two blades teetering main rotor and it is powered by a water-cooled, single cylinder, 12-hp, and 98 cc two-stroke gasoline engines. A special external engine starter is required for the engine. The engine is very reliable and powerful because it can carry 20 kg of payload. Its length, height and width are 3.58 meters, 0.7 meters and 1.08 meters respectively. Its rotor diameter and dry weight are 3.070 meters and 44 kg [2].

## III. DEVELOPMENT OF FORCE AND THRUST EQUATIONS

For the part of force and torque equations, main rotor thrust $T_{MR}$, tail rotor thrust $T_{TR}$ and flapping angles ( $\beta_{1s}$ and $\beta_{1c}$ ) are used as inputs to produce force and torque outputs which are inputs for rigid body equations part [1].

For forces generated from the main rotor thrust ${}^bF_{MR}$, the forces component along ${}^bx$ and ${}^bz$ depend on only the main rotor thrust while the force component along ${}^by$ depend on the both the main and tail rotor thrust. For the forces generated from the tail rotor thrust ${}^bF_{TR}$, the force is only in the ${}^by$. But the force due to gravitational acceleration ${}^bF_g$ is along ${}^sz$ direction and located in the spatial frame (SF). Therefore, the rotation matrix from SF to BF ($R_{bs}$) is needed [1]. So Final force equations are as follows

$${}^bF = \begin{bmatrix}{}^bf_x\\ {}^bf_y\\ {}^bf_z\end{bmatrix} = {}^bF_{MR} + {}^bF_{TR} + {}^bF_g = \begin{bmatrix}-T_{MR}.\sin(\beta_{1c}) - \sin(\theta).m.g\\ T_{MR}.\sin(\beta_{1s}) + T_{TR} + \sin(\phi).\cos(\theta).m.g\\ -T_{MR}.\cos(\beta_{1s}).\cos(\beta_{1c}) + \cos(\phi).\cos(\theta).m.g\end{bmatrix} \quad (5)$$

where $T_{MR}$ =main rotor thrust

$T_{TR}$ =tail rotor thrust

$\beta_{1s}$ =longitudinal flapping angle

$\beta_{1c}$ =lateral flapping angle

There are three parts in deriving torque equations. They are torques caused by main rotor, tail rotor and drag on main rotor. Torques caused by main rotor and tail rotor are derived from the equation ($\tau = F.d$) . Drag on main rotor causes counter torque. Torque caused by tail rotor, ${}^bM$ along ${}^by$ is zero [1].

So, final torque equation is as follow;

$${}^b\tau = \begin{bmatrix}{}^bL\\ {}^bM\\ {}^bN\end{bmatrix} = {}^b\tau_{MR} + {}^b\tau_{TR} + {}^b\tau_D = \begin{bmatrix}{}^bf_{y,MR}.h_m - {}^bf_{z,MR}.y_m + {}^bf_{y,TR}.h_t + Q_{MR}.\sin(\beta_{1c})\\ -{}^bf_{x,MR}.h_m - {}^bf_{z,MR}.l_m - Q_{MR}.\sin(\beta_{1s})\\ {}^bf_{x,MR}.y_m + {}^bf_{y,MR}.l_m - {}^bf_{y,TR}.l_t + Q_{MR}.\cos(\beta_{1c}).\cos(\beta_{1s})\end{bmatrix} \quad (6)$$

where $h_m$ is the distance from COG to the main rotor along ${}^bz$ axis. $h_t$ is the distance from COG to the tail rotor along ${}^bz$ axis. $l_m$ is the distance from COG to the main rotor along ${}^bx$ axis. $l_t$ is the distance from COG to the tail rotor along ${}^bx$ axis. $y_m$ is the distance from COG to the main rotor along ${}^by$ axis.

Using the above equations, force and torque equations are implemented in s function. After that, MATLAB SIMULINK model for rigid body dynamic is implemented using this s function.

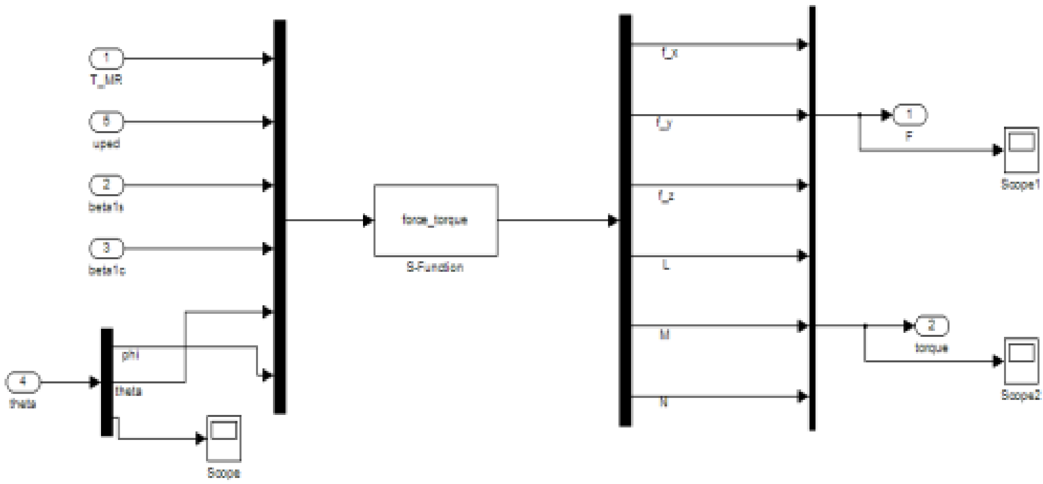


Figure 5. MATLAB SIMULINK Model for Force and Torque Equations

## IV. DEVELOPMENT OF THRUST AND FLAPPING EQUATIONS

The part of thrust and flapping equations consist of four control inputs ($u_{lat}$,$u_{long}$,$u_{long}$,$u_{ped}$). The outputs are thrusts caused by main rotor and tail rotor, and the flapping angles of the main rotor [1].

The main rotor thrust equation depends on induced velocity $v_i$ and is calculated with the numerical method. The main rotor thrust equation is as follow:

$$T_{MR} = (w_b - v_i).\frac{\rho.\Omega.R^2.a.B.c}{4} \tag{7}$$

where $\rho$ is the density of the air, Ω is the rotor angular rate, $R$ is rotor radius, a is the constant lift curve slope, $B$ is number of blades, $c$ is chord of blade, $w_b$ is the velocity of the main rotor blade and $v_i$ is the induced wind velocity, $w_r$ is the velocity of the main rotor disc.

From the torque equation, tail rotor thrust equation is derived by using yawing moment ${}^bN$.

$$T_{TR} = \frac{{}^bf_{y,MR}.l_m + {}^bf_{x,MR}.y_m + Q_{MR}.\cos(\beta_{1c}).\cos(\beta_{1s})}{l_t} + u_{ped} \tag{8}$$

There are two parts to get lateral and longitudinal flapping angles because the input is fed not only directly to the main rotor but also to the main rotor through the control rotor. The inputs from the swash plate, $A_{SP}$ and $B_{SP}$, become $\beta_{1s}$ and $\beta_{1c}$ as lateral and longitudinal flapping angles. The gain $K_{CR}$ is linked from the control rotor to the main rotor. The control rotor has flapping angles $\beta_{CR,1s}$ and $\beta_{CR,1c}$.The gain $K_{MR}$ is the mechanical linkage from the swash-plate to the main rotor. Therefore, the input from the swash-plate to the main rotor is obtained as follow:

$$A_{MR} = \mathrm{K_{MR}}.A_{SP} + K_{CR}.\beta_{CR,1s}$$
$$B_{MR} = K_{MR}.B_{SP} + K_{CR}.\beta_{CR,1c}$$

where $A_{MR}$ and $B_{MR}$ are the cyclic-input's contribution to the pitch of the blades.

Lateral and longitudinal flapping rates of the control rotor are as follow;

$$\dot\beta_{CR,1c}(t) = \frac{1}{4}\Omega.T_1.A_{SP}(t) - p(t) - \frac{1}{4}\Omega.T_2.q(t) - \frac{1}{2}\Omega.\beta_{CR,1s}(t) - \frac{1}{4}\Omega.\mathrm{T}_1.\beta_{CR,1c}(t) - \frac{\ddot\beta_{CR,1c}(t)}{2\Omega} \tag{9}$$

$$\dot\beta_{CR,1s}(t) = \frac{1}{4}\Omega.T_1.B_{SP}(t) - q(t) + \frac{1}{4}\Omega.T_2.p(t) + \frac{1}{2}\Omega.\beta_{CR,1s}(t) + \frac{1}{4}\Omega.\mathrm{T}_1.\beta_{CR,1c}(t) + \frac{\ddot\beta_{CR,1s}(t)}{2\Omega} \tag{10}$$

Using the above equations, the control rotor flapping equations are implemented in SIMULINK.

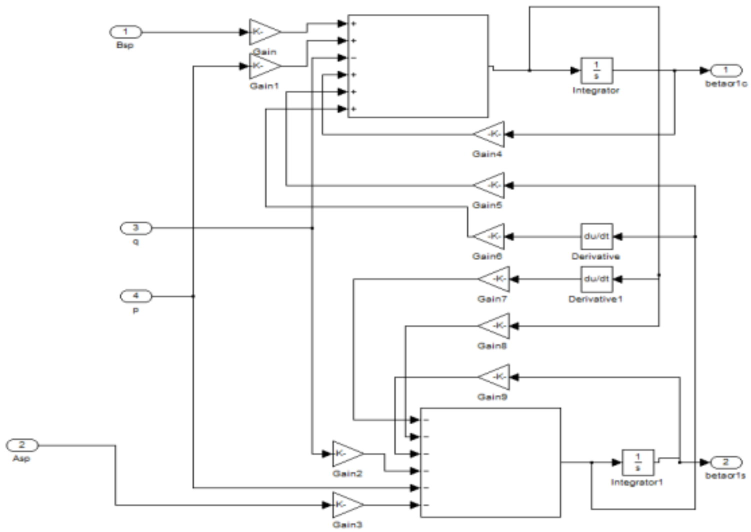


Figure 6 . MATLAB SIMULINK Model for control rotor flapping Equations

The main rotor flapping equations are

$$\beta_{1c(t)} = \left\{ -\frac{3.06.10^{-7}}{\pi^2} \left(3.26.10^6.B_{MR} + 816.97.\ {}^b v^2(t).B_{MR} + 3275.88.\ {}^b u(t).v_i + 2456.91.\ {}^b u^2(t).B_{MR} - 1637.94.A_{MR}.\ {}^b v(t).\ {}^b u(t) + 1.13.10^5 \pi p(t) + 4.67.10^5 \pi p(t) + 1.95.10^5 \pi u_{col}(t).\ {}^b u(t)\right) \right\} \quad (11)$$

$$\beta_{1s(t)} = \left\{ -\frac{3.06.10^{-7}}{\pi^2} \left(1.94.10^5 \pi.u_{col}(t).\ {}^b v(t) - 3.26.10^6.\pi^2.A_{MR} + 1637.94.B_{MR}.\ {}^b u(t).\ {}^b v(t).v_i + 4.67.10^5 \pi.p(t) - 3275.88.v_i + 2456.91.A_{MR}.\ {}^b v^2(t) - 818.96.\ {}^b u^2(t).A_{MR} + 1.12.10^5.q(t)\right) \right\} \quad (12)$$

The obtained thrust and flapping equations are implemented in s function. And then, MATLAB SIMULINK model for thrust and flapping equation part of top-down principle is implemented by using this s function.

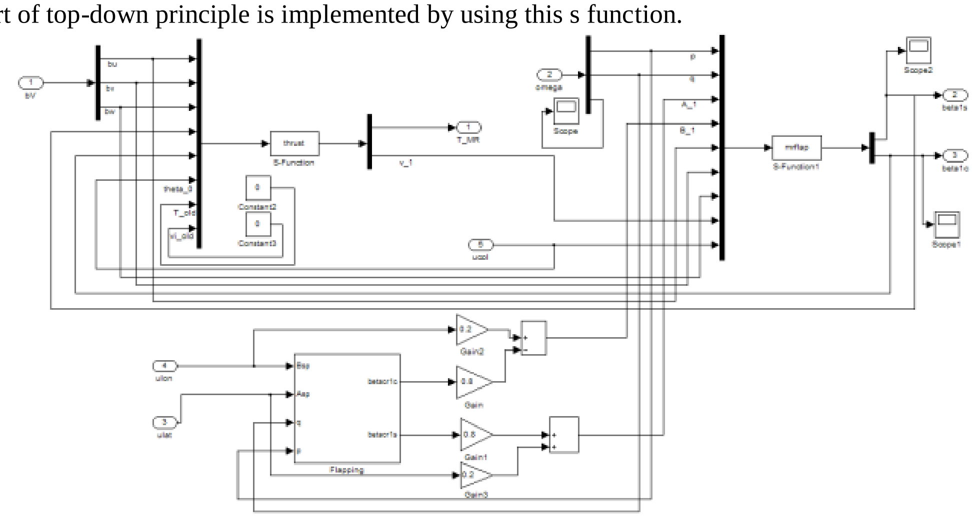


Figure 7. MATLAB SIMULINK Model for thrust and flapping Equations

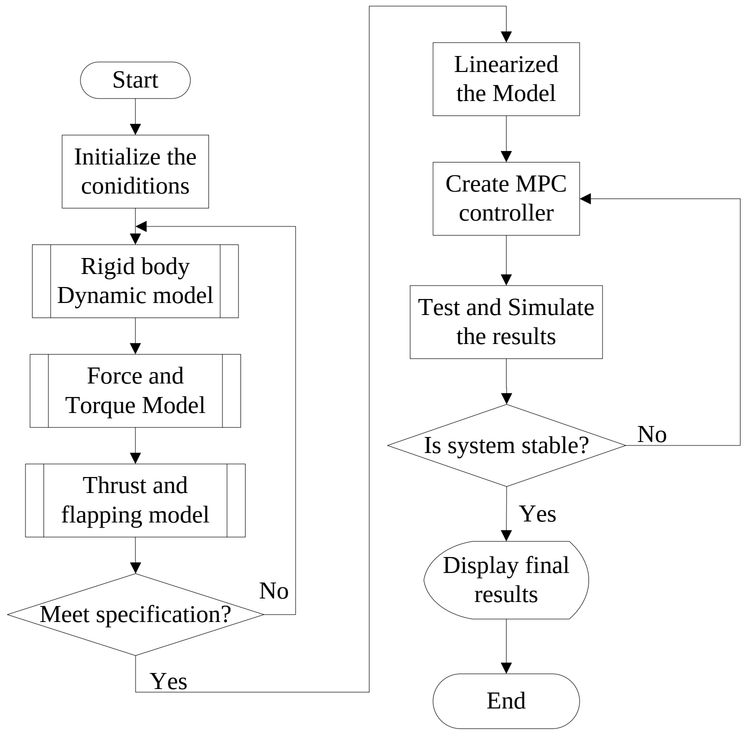


Figure 8. System Flowchart

## V. SIMULATION RESULTS

The Yahama R-50 helicopter has been implemented in SIMULINK MODEL. So the simulation results of the Euler angles, translational velocities, angular velocity and positions and flapping angles. Figure9,10,11,12, 13and 14 show the results of postions, Euler angles, angular velocities, translator velocities and flapping angles when the four inputs; $u_{lat}$=-0.000001, $u_{lon}$=-0.0000001,$u_{col}$=1 and $u_{ped}$=1.

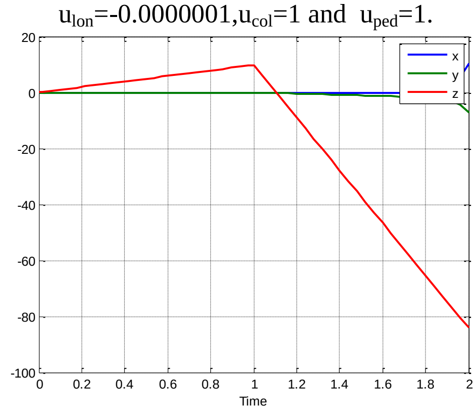


Figure 9.Position '$^e$V'
($u_{lat}$=-0.000001,$u_{lon}$=-0.0000001,$u_{col}$=1and $u_{ped}$=1)

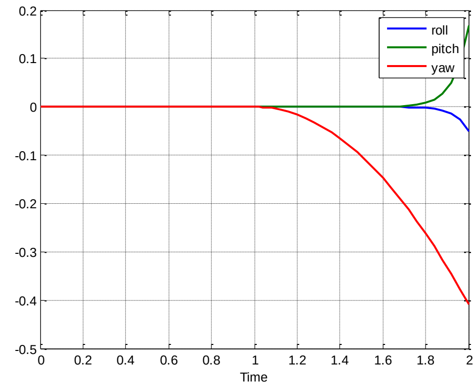


Figure 10. Euler angles
($u_{lat}$=-0.000001,$u_{lon}$=-0.0000001,$u_{col}$=1and $u_{ped}$=1)

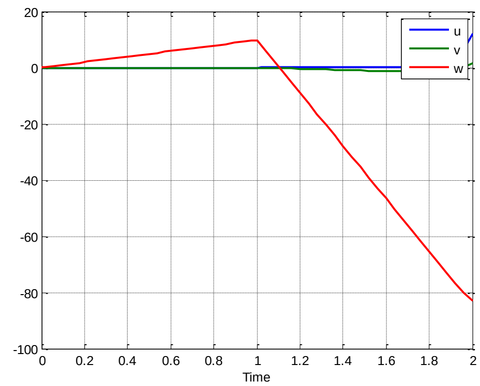


Figure 11. Translatory velocities '${}^{b}V$'
($u_{lat}$=-0.000001,$u_{lon}$=-0.0000001,$u_{col}$=1and $u_{ped}$=1)

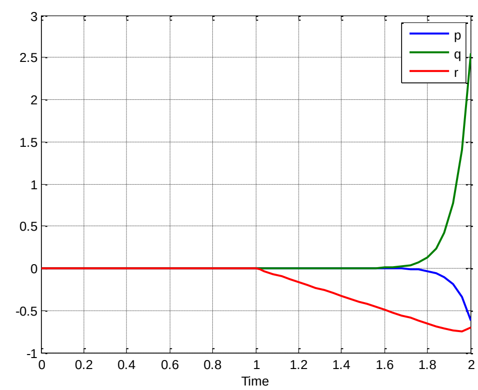


Figure 12. Angular velocities 'omega'
($u_{lat}$=-0.000001,$u_{lon}$=-0.0000001,$u_{col}$=1and $u_{ped}$=1)

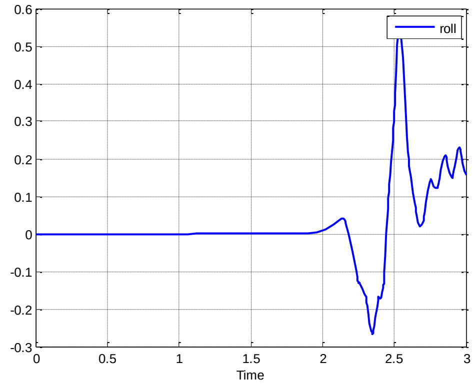


Figure 13. Lateral flapping angle
($u_{lat}$=-0.000001,$u_{lon}$=-0.0000001,$u_{col}$=1and $u_{ped}$=1)

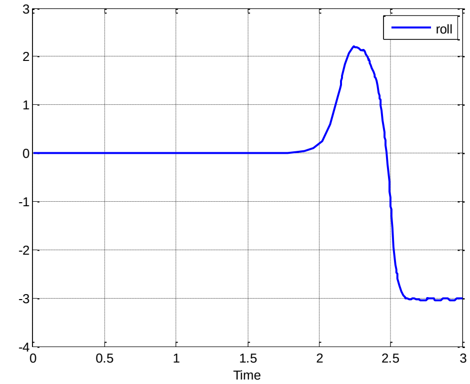


Figure 14.Longitudinal flapping angle
($u_{lat}$=-0.000001,$u_{lon}$=-0.0000001,$u_{col}$=1and $u_{ped}$=1)

## VI. STABILITY ANALYSIS

Using the linearized model obtained previously, the model predictive controller was tuned manually to obtain the responses shown in Figure. By using these matrices with the Model Predictive Control Toolbox, Prediction horizon is set as 10, Control horizon is set as 3. In this case, unity weights on both outputs and zero weights on both inputs are used. (Default values). Figure 15 to 19 show the stability analysis for hovering stage of UAH.

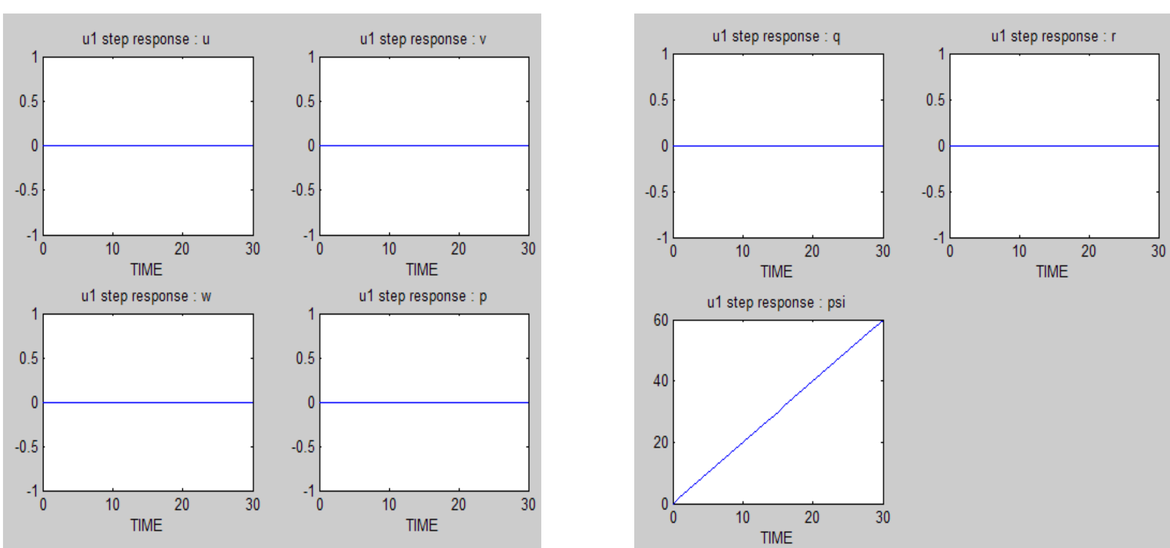


Figure 15. Linear and Angular Velocities Responses of Linearized R-50 Airframe when lateral control is used as input

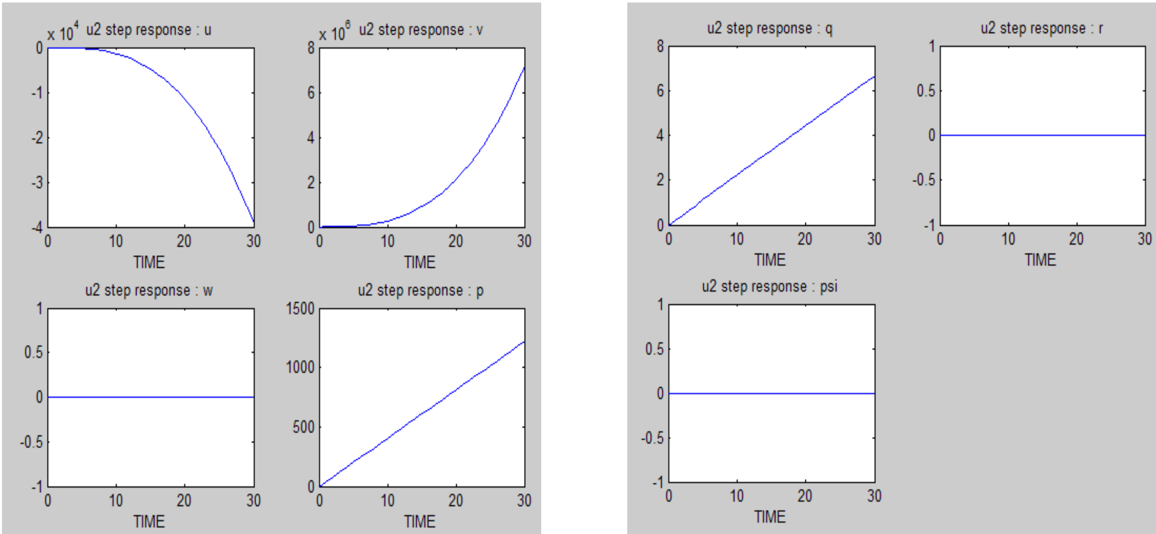


Figure 16. Linear and Angular Velocities Responses of Linearized R-50 Airframe when longitudinal control is used as input

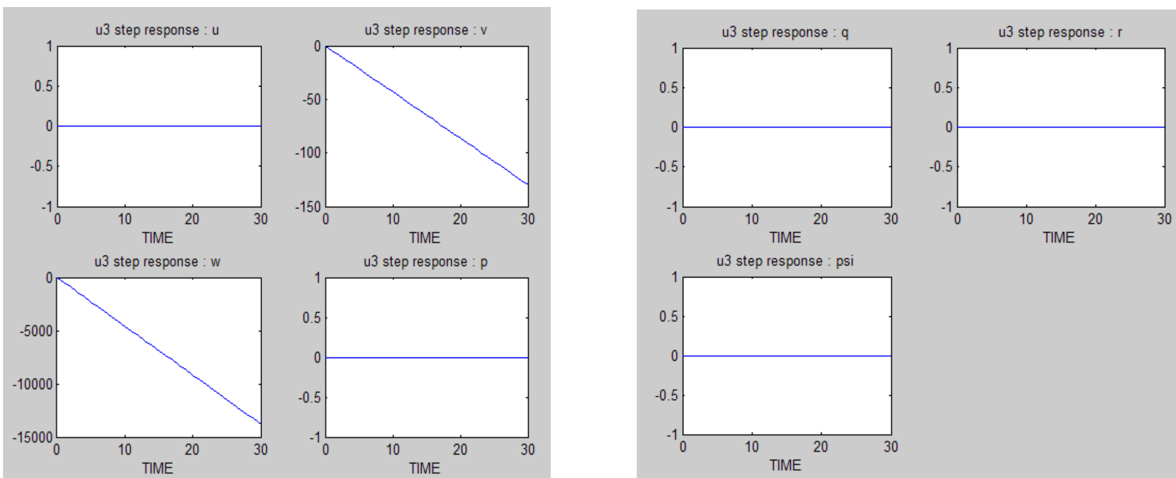


Figure 17. Linear and Angular Velocities Responses of Linearized R-50 Airframe when collective control is as input

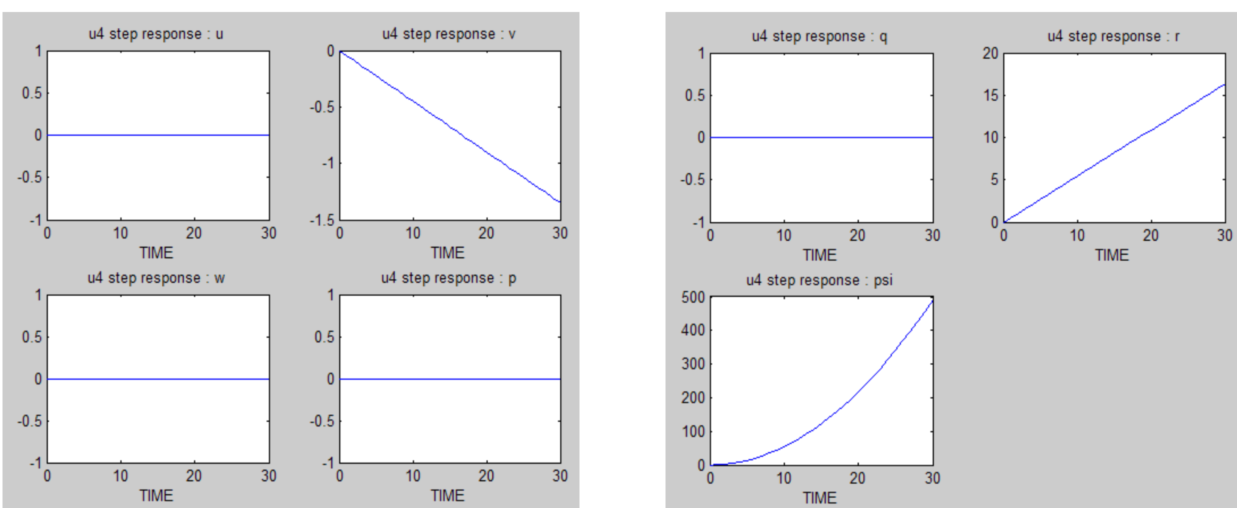


Figure 18. Linear and Angular Velocities Responses of Linearized R-50 Airframe when pedal control is used as input

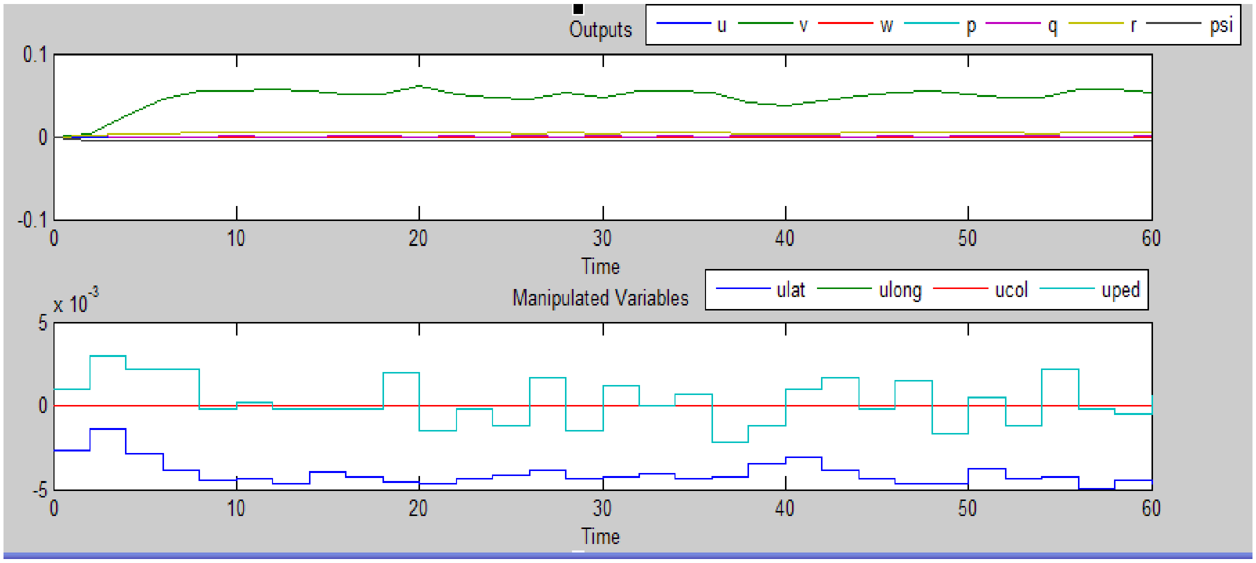


Figure 19. Comaprison Results for Extimated and Theoratical Response

## VII. CONCLUSION

The nonlinear mathematical R-50 helicopter model is implemented in SIMULINK by using minimum–complexity helicopter simulation math model (MCHSMM). When the collective control input ($u_{col}$) and pedal control input ($u_{ped)}$ is set positive, the lateral flapping angle ($\beta_{1s}$) and the longitudinal flapping angle ($\beta_{1c}$) become opposite due to the cross couplings of the lateral and longitudinal blade flapping. Since the helicopter model is still as Bare air-frame, it is not stable. So the controller is needed to stable. If this R-50 helicopter model is used with the controller, the results will possible to obtain better than without controller.

## ACKNOWLEDGMENT

I would wish to acknowledge the many colleagues at Mandalay Technological University who have contributed to the development of this paper.

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