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Non Linear Oscillator Systems and Solving Techniques

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Abstract- The paper involves thorough study of non-linear vibratory oscillators and numerical methodology to analyse and resolute the non-linear dynamical world. The study involves the analysis of non-linear oscillators like the *Van der Pol Oscillator* and *Duffing Oscillator*. Application of regular perturbation method in the oscillator is also demonstrated. The equilibrium and stability analysis of the oscillators with graphical representation is simulated through *XPP-AUT* and *MATLAB*. The graphical and mathematical depiction of damping with altering parameters in oscillators' equations is also shown. Saddle points, centers and equilibrium points of consequent curves are depicted in scale. Apart from the oscillators, implementation of "*Method of Multiple Scales*" and "*Method of Averaging*" in non-linear dynamical equations is also rendered numerically in the study with the conclusion that the "*Method of Multiple Scales*" produces better results than the "*Method of Averaging*".

Index Terms- Van der Pol Oscillator, Duffing Oscillator, Dynamics, Multiple-scales

I. INTRODUCTION

The prime interest lied in analyzing the dynamics of non-linear oscillator systems mainly Van der Pol Oscillator and Duffing Oscillator. The dynamical world was dealt with solving techniques and the results were compared. The methods under consideration are "Method of Multiple Scales" and "Method of Averaging". In Method of Multiple Scales we retard the actual time using the order of the parameter and the variable is written in terms of retarded time coefficients. On the contrary, Method of Averaging helps in optimizing the solution near the mean position only. Both these methods are depicted in sample oscillator equations and henceforth compared to conclude that which one of them produces better results.

II. THE VAN DER POL OSCILLATOR

In 1920, a Dutch scientist named Balthasar Van der Pol established experimental results on the dynamics of an oscillator in electrical circuits governed by a second order differential equation, which later came to be known as Van der Pol Oscillator. The consequent results also proved that Van der pol oscillator obeys Lienard's theorem which proves that it has stable limit cycle in the phase space. The Van der Pol Oscillator also describes how a pacemaker controls the irregular heartbeat of human heart where the whole cardiac system can be modelled as a working Van der Pol Oscillator.

Standard Equation: $\frac{d^2x}{dt^2} + \mu\frac{(-1+x^2)\,dx}{dt} + x = 0$
where 'x' is the dynamical variable and $\mu > 0$ a parameter.

When $\mu = 0$, the equation becomes Simple Harmonic Equation, $\frac{d^2x}{dt^2} + x = 0$

Regular Perturbation Method for Van Der Pol Oscillator:

$$\ddot{x} + \mu\frac{(-1+x^2)dx}{dt} + x = 0$$

For $\mu = 0$, $\ddot{x} + x = 0$

We have

$$x_o = A\sin t + B\cos t$$

where A and B are determined by initial conditions.

$$x = x_0 + \in x_1 + \in^2 x_2 + \in^3 x_3 \dots$$

$$x = x_0 + \in x_1 + O(\in^2) \text{ (Neglecting other higher orders)}$$

Hence, we get the equation,

$$\ddot{x}_0 + \in \ddot{x}_1 + \in (x_0^2 + \in^2 x_1^2 + 2 \in x_0 x_1 - 1)(\dot{x}_0 + \in \dot{x}_1) + x_0 + \in x_1 = 0$$

$$\ddot{x}_0 + x_0 + \in [(x_0^2 - 1)\dot{x}_0 + \ddot{x}_1 + x_1] + O(\in^2) = 0$$

$$O(\in^0) = \ddot{x}_0 + x_0 = 0$$

$$\Rightarrow x_0 = A \sin t + B \cos t$$

$$O(\in^1) = (x_0^2 - 1)\dot{x}_0 + \ddot{x}_1 + x_1 = 0$$

Substituting the value of x_0 we get,

$$O(\in^1) = \ddot{x}_1 + x_1 + (A^2 \sin^2 t + B^2 \cos^2 t - 2AB \sin t \cos t - 1)(A \cos t - B \sin t) = 0$$

$$\Rightarrow \ddot{x}_1 + x_1 = -[A^2 - 1 + (B^2 - A^2)\left(\frac{1 + \cos 2t}{2}\right) - AB \sin 2t](A \cos t - B \sin t)$$

$$\Rightarrow \ddot{\mathrm{x}}_1 + \mathrm{x}_1 = \mathrm{C} \sin \mathrm{t} + \mathrm{D} \cos \mathrm{t} + \mathrm{E} \sin 3\mathrm{t} + \mathrm{F} \cos 3\mathrm{t}$$

$$\therefore\ x_1 = \mathrm{G} \sin t + \mathrm{H} \cos t + \mathrm{I}t \sin t + \mathrm{J}t \cos t + \mathrm{K} \sin 3t + \mathrm{L} \cos 3t$$

So,

$$\mathrm{x} = \mathrm{A} \sin \mathrm{t} + \mathrm{B} \cos \mathrm{t} + \in (\mathrm{G} \sin \mathrm{t} + \mathrm{H} \cos \mathrm{t} + \mathrm{It} \sin \mathrm{t} + \mathrm{Jt} \cos \mathrm{t} + \mathrm{K} \sin 3\mathrm{t} + \mathrm{L} \cos 3\mathrm{t})$$

Equation becomes unstable at higher values of t.

For that,

$$\in t < 1$$

$$t < \frac{1}{\in}$$

$$\therefore t = O\left(\frac{1}{\in}\right)$$

Corrections:

We know that,

$$O(\in^0) = \ddot{x}_0 + x_0 = 0$$

$$x_0 = R \cos(t + \emptyset)$$

Also, $O(\in^1) = (x_0^2 - 1)\dot{x}_0 + \ddot{x}_1 + x_1 = 0$

$$\Rightarrow \ddot{x}_1 + x_1 - [R^2 cos^2(t + \emptyset) - 1]R \sin(t + \emptyset) = 0$$

$$\Rightarrow \ddot{x}_1 + x_1 = [R^2 cos^2(t + \emptyset) - 1]R \sin(t + \emptyset)$$

$$\Rightarrow \ddot{x}_1 + x_1 = \left[\frac{R^2}{2} - 1 + \frac{R^2}{2}\cos(2t + 2\emptyset)\right] R \sin(t + \emptyset)$$

$$\Rightarrow \ddot{x}_1 + x_1 = \left(\frac{R^2}{2} - 1\right) R \sin(t + \emptyset) + \frac{R^3}{2}\cos(2t + 2\emptyset) \sin(t + \emptyset)$$

$$\mathrm{x}_1 = \mathrm{A} \sin(\mathrm{t} + \emptyset) + \cos(\mathrm{t} + \emptyset) + \sin 3(\mathrm{t} + \emptyset) + \cos 3(\mathrm{t} + \emptyset)$$

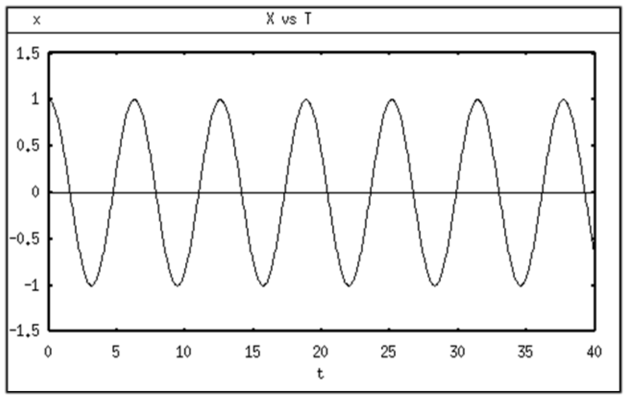
$$\therefore\ x = R \cos(t + \emptyset) + A \sin(t + \emptyset) + \cos(t + \emptyset) + \sin 3(t + \emptyset) + \cos 3(t + \emptyset)$$

The above solution gives a better idea of the Van Der Pol oscillator as the large values of “t” do not hamper the stability of the equation.

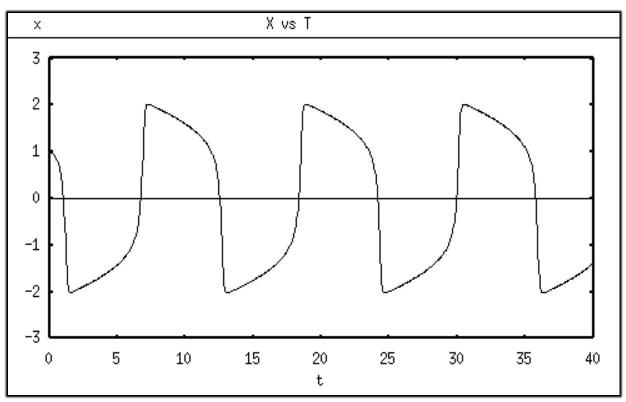
Graphical Analysis of Damping in Van der Pol Oscillator

Say $\mu \ll 1$, then $\mu \frac{(-1+x^2)\,dx}{dt} \to 0$
The equation reduces to $\dot{y} = x$, where $y = \frac{dx}{dt}$. (small damping)
Hence, the plot of ‘x’ vs. ‘t’ approximately becomes a sinusoidal curve because the parameter $\mu \ll 1,$ as the van der pol equation reduces to

$$\frac{d^2x}{dt^2} = x$$



When the value of ‘μ’ was increased, it was observed that the period ‘x’ also increased with loss of sinusoidal character in the plot of ‘x’ vs. ‘t’ and ‘y’ vs. ‘t’. The value of ‘μ’ was increased to 5 and the subsequent dynamics of ‘x’ vs. ‘t’ and ‘y’ vs. ‘t’ were observed respectively.



Equilibrium and Stability Analysis of the Van der Pol Oscillator

Considering the Van der Pol Oscillator, $\frac{d^2x}{dt^2} + \mu \frac{(-1+x^2)\,dx}{dt} + x = 0$

Say, $y = \frac{dx}{dt}$

We get $\frac{dy}{dt} = -x - \mu(-1 + x^2)\, y$; $\frac{dx}{dt} = y$

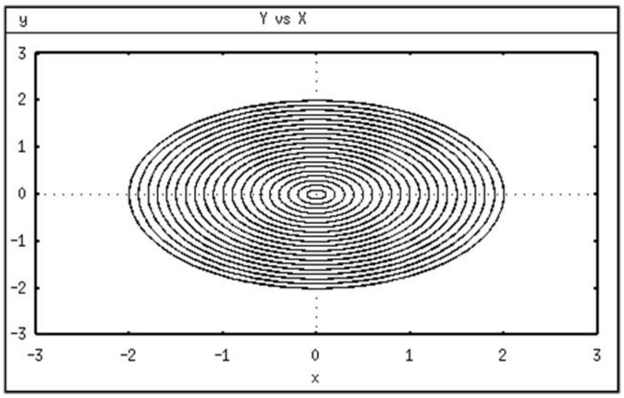
There is equilibrium at origin, and it is obvious from the slope vector that there are no other equilibria.

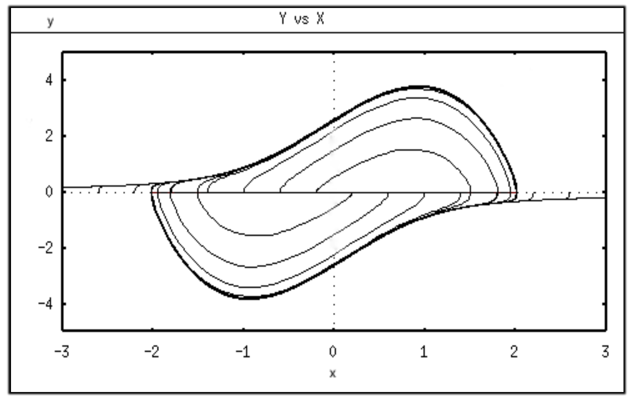
Eigenvalues:

$$\{\frac{1}{2}(\mu - \sqrt{\mu^2 - 4}),\ \frac{1}{2}(\mu + \sqrt{\mu^2 - 4})\}$$

We see that the equilibrium point ($x = 0, \dot{x} = 0$) is an unstable spiral equilibrium if $0 < \mu < 2$.

For $\mu > 2$, the equilibrium is an unstable node. Hence, the dynamics of the oscillator are bound to a restricted area around the origin.





DUFFING OSCILLATOR

The unforced duffing oscillator is given by:

$$\ddot{x} + \alpha x + \beta x^3 + \gamma x^5 = 0$$

General Application of the Oscillator:
Duffing's equation is used to model conservative double-well oscillators, which can occur, for example, in magneto-elastic mechanical systems. The system consists of a beam placed vertically between two magnets with the top end fixed and the bottom end free to swing. With velocity applied to the beam, the beam starts to oscillate between the two magnets. The beam finally comes to rest at a fixed point and remains in equilibrium.

Equilibrium Analysis:

The autonomous dynamical system can be written as,

$$\dot{x} = y,$$

$$\dot{y} = -(\alpha x + \beta x^3 + \gamma x^5)$$

For equilibirum, $\ddot{x} = 0 \Rightarrow y = 0$, and
When $\dot{y} = 0,\ x(\alpha + \beta x^2 + \gamma x^4) = 0$

$$\Rightarrow\ x = 0\ or\ (\alpha + \beta x^2 + \gamma x^4) = 0$$

$$\Rightarrow\ x = 0\ or\ x_{\pm} = \pm\sqrt{\frac{-\beta \pm \sqrt{\beta^2 - 4\gamma\alpha}}{2\gamma}}$$

Equilibrium exists under the conditions:

a) $-\beta \pm \sqrt{\beta^2 - 4\gamma\alpha} \geq 0$
b) $\beta^2 - 4\gamma\alpha \geq 0$

The eigen values satisfy the equation: $\lambda^2 = -\alpha - \beta {x_o}^2 - \gamma {x_o}^4$

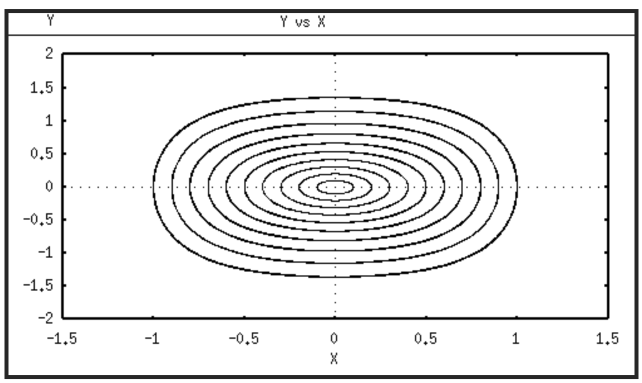
Where x_o is the X co-ordinate of the equilibrium point.

- The equilibrium points in the unforced Duffing oscillator are thus of the form (0, 0) and ($x_\pm$, 0).
- Varying the values of α, β and γ causes the behaviour of the solution x (t) to change, as depicted in some upcoming bifurcation diagrams. Equilibria can be either saddles or centers. The former are unstable points, whereas the latter are linearly stable.

The Phase Plane trajectories and analysis of Duffing's Oscillator is explained in the coming sections.

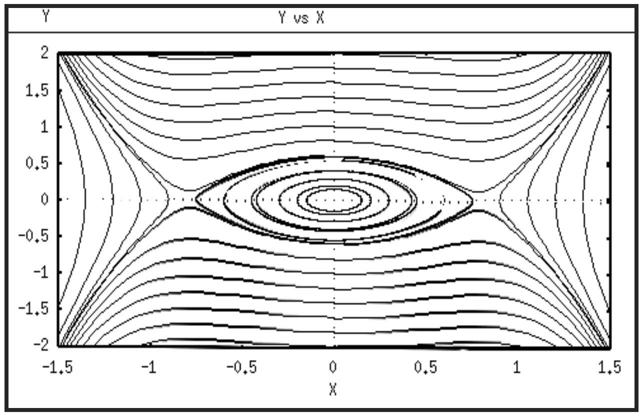
Dynamics of Duffing Oscillator with varying Parameters:

1. The phase plot of 'y' vs 'x' as shown in the adjacent figure shows the behaviour of the oscillator when $\alpha = 1, \beta = 1$ & $\gamma = 1$ The oscillator behaves periodically with a center (0,0).



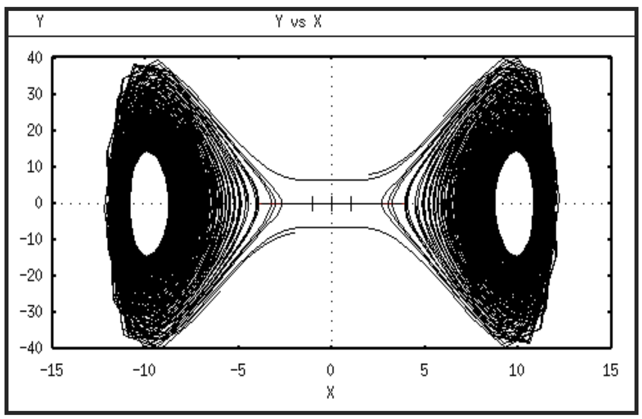
2. The phase plot of 'y' vs 'x' as shown in the adjacent figure shows the behaviour of the oscillator when $\alpha = 1, \beta = -1$ & $\gamma = -1$. We notice three equilibrium points from the phase plot. There is a center at (0,0) and there are two saddles present:

a) (+0.79, 0)
b) (-0.79, 0)



3. The phase plot of 'y' vs 'x' as shown in the adjacent figure shows the behaviour of the oscillator when $\alpha = 1, \beta = -1$ & $\gamma = 0.01$. Five equilibrium points are observed in the phase plot. There are three centers:

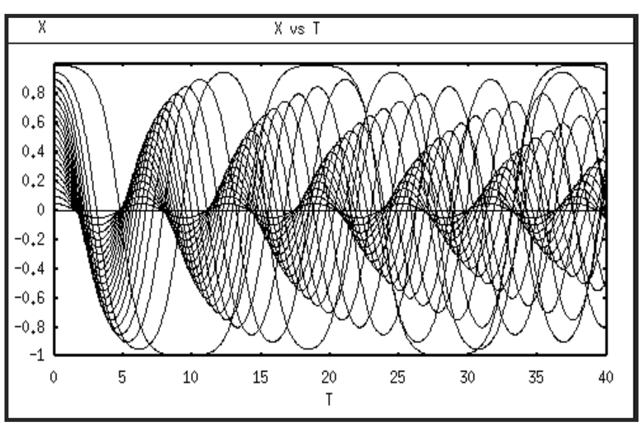
a) (0, 0)
b) (+1.005, 0)
c) (-1.005, 0)



We also observe two saddle points in the same phase plot:

a) (+9.95, 0)
b) (-9.95, 0)

The corresponding 'x' vs 't' curve is as shown.



III. METHOD OF MULTIPLE SCALES

Say an oscillator be defined as,

$$\ddot{x} + \in a_1(x^2-1)\dot{x} + \in a_2 x^3 + x = 0$$

$$T_0 = t \text{ (Actual time)}$$

$$T_1 = \in t \text{ (Slower time)}$$

$$T_2 = \in^2 t \text{ (Still slower time) and so on.}$$

$$\ddot{x} + \in a_1(x^2-1)\dot{x} + \in a_2 x^3 + x = 0$$

$$x(t) = x(T_0, T_{1,} T_2 \ldots)$$

$$\frac{dx}{dt} = \frac{\partial x}{\partial T_0}\frac{dT_0}{dt} + \frac{\partial x}{\partial T_1}\frac{dT_1}{dt} + \cdots$$

$$\frac{d}{dt} = \left(\frac{\partial}{\partial T_0} + \in \frac{\partial}{\partial T_1} + \in^2 \frac{\partial}{\partial T_2} + ..\right) = (D_0 + \in D_1 + \in^2 D_2 + \cdots)$$

$$\frac{d^2}{dt^2} = \frac{d}{dt}.\frac{d}{dt} = {D_0}^2 + \in (D_0 D_1 + D_1 D_0) + \in^2 \left(D_0 D_2 + {D_1}^2 + D_2 D_0\right) + O(\in^3) + \cdots$$

Substituting in the oscillator equation till $O(\in^1)$ terms, we get,

$${D_0}^2 x + 2 \in D_0 D_1 x + \in a_1(x^2-1) D_0 x + \in a_2 x^3 + x = 0$$

$$\text{Now, let } x = x_0 + \in x_1 + O(\in^2) + \cdots$$

$$D_0^2 x_0 + \in D_0^2 x_1 + 2 \in D_0 D_1 x_0 + \in a_1 D_0 x_0 (x_0^2 - 1) + x_0 + \in x_1 + \in a_2 x_0^3 + O(\in^2) = 0$$

$$O(\in_0) = D_0^2 x_0 + x_0 = 0$$

$$\Rightarrow \quad x_0 = R(T_1)\cos(T_0 + \emptyset(T_1))$$

$$O(\in_1) = D_0^2 x_1 + 2 D_0 D_1 x_0 + a_1 D_0 x_0 (x_0^2 - 1) + x_1 + a_2 x_0^3 = 0$$

Equating coefficients of $\sin(T_0 + \emptyset(T_1))$ and $\cos(T_0 + \emptyset(T_1))$ to Zero, we get,

$$\Rightarrow \; 2\frac{\partial R}{\partial T_1} - a_1 R + a_1 \frac{R^3}{4} = 0$$

$$\Rightarrow \quad 2R\cos(T_0 + \emptyset(T_1))\frac{\partial \emptyset}{\partial T_1} - 3a_2\frac{R^3}{4} = 0$$

$$\therefore \; \frac{\partial R}{\partial T_1} = D_1 R = \frac{a_1}{2}\left(R - \frac{R^3}{4}\right)$$

$$\therefore \; \frac{\partial \emptyset}{\partial T_1} = D_1 \emptyset = \frac{3a_2 R^2}{8}$$

METHOD OF AVERAGING

$f(x,t+T)=f(x,t)$, for some T

Average value of ‘x’ is defined as:

$$x_{av}=\frac{1}{T}\int_{t-\frac{T}{2}}^{t+\frac{T}{2}} xdt=\frac{1}{T}\int_{t-T}^{t} xdt=\frac{1}{T}\int_{0}^{T} xdt$$

$$x_{av}=\lim_{T\to\infty}\ \frac{1}{T}\int_{t-\frac{T}{2}}^{t+\frac{T}{2}} xdt$$

Say, $\dot{x}=\in f(x,t)$ where $0<\in\ll 1$

$$\Rightarrow\ \dot{x}_{av}=\frac{\in}{T}\int_{t-\frac{T}{2}}^{t+\frac{T}{2}} f(x_{av},t)dt$$

Application of Method of Averaging in an Oscillator

$$\dot{x}=\in \mathrm{sint}[1+\mathrm{x}(2-\mathrm{x})\mathrm{sint}]$$

$$T=2\pi$$

$$\dot{x}_{av}=\frac{\in}{T}\int_{t-\frac{T}{2}}^{t+\frac{T}{2}} f(x_{av},t)dt$$

$$\Rightarrow\ \dot{x}_{av}=\frac{\in}{2\pi}\int_{t-\pi}^{t+\pi}[\sin t+x_{av}(2-x_{av})\sin^2 t]dt$$

$$\Rightarrow\ \dot{x}_{av}=\frac{\in}{2\pi}x_{av}(2-x_{av})\int_{t-\pi}^{t+\pi}\frac{1}{2}(1-\cos 2t)dt$$

$$\Rightarrow\ \dot{x}_{av}=\frac{\in}{2\pi}x_{av}(2-x_{av})\left[\frac{t+\pi}{2}-\frac{t-\pi}{2}-\frac{\sin 2t}{4}+\frac{\sin 2t}{4}\right]$$

$$\Rightarrow\ \dot{x}_{av}=\frac{\in}{2\pi}x_{av}(2-x_{av})$$

where, $x_{av}=\lim_{T\to\infty}\ \frac{1}{T}\int_{t-\frac{T}{2}}^{t+\frac{T}{2}} xdt$

IV. CONCLUSION

Hence, the multiple scale method solution has the advantage that it provides a closed form solution with a good physical insight, whereas the averaging method does not provide a closed form solution and lacks this type of physical insight. The most important advantage of this method is that, by identification of a non-dimensional small parameter, which has a physical interpretation and by using several time scales, one can obtain a complete physical understanding about the behaviour of the system and the influence of different parameters and terms on the final response of the system. For complex equations and functions where the order of the differential equation reaches 2 or more, it becomes a very perplex situation to get the solution using the method of averaging. As a consequence, effects of nonlinearities are determined less accurately. Therefore, method of multiple scales proves out to be a better option to solve the oscillators instead of method of averaging.

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