On Direct Sum of Two Fuzzy Graphs

Dr. K. Radha*, Mr.S. Arumugam**

* P.G & Research Department of Mathematics, Periyar E.V.R. College, Tiruchirapalli-620023
** Govt. High School, Thinnanur, Tiruchirapalli-621006.

Abstract- In this paper, the direct sum $G_1 \oplus G_2$ of two fuzzy graphs G_1 and G_2 is defined. It is proved that when two fuzzy graphs are effective then their direct sum need not be effective. The degrees of the vertices in the direct sum $G_1 \oplus G_2$ of two fuzzy graphs G_1 and G_2 in terms of degrees of the vertices in the fuzzy graphs G_1 and G_2 are obtained. The lower and upper truncations of the direct sum of two fuzzy graphs are obtained. The regular property and connectedness of the direct sum of two fuzzy graphs are also studied.

***Index Terms*-** Fuzzy Graph, Direct Sum, Effective Fuzzy Graph, Regular Fuzzy Graph, Connectedness, Upper and Lower Truncations.

I. INTRODUCTION

Fuzzy graph theory was introduced by Azriel Rosenfeld in 1975. The properties of fuzzy graphs have been studied by Azriel Rosenfeld[7]. Later on, Bhattacharya[6] gave some remarks on fuzzy graphs, and some operations on fuzzy graphs were introduced by Mordeson.J.N. and Peng.C.S.[3]. The conjunction of two fuzzy graphs was defined by Nagoor Gani.A and Radha.K.[4].

In this paper, the direct sum of two fuzzy graphs is defined. The degree of a vertex in the direct sum $G_1 \oplus G_2$ of two fuzzy graphs G_1 and G_2 in terms of degrees of the vertices in the fuzzy graphs G_1 and G_2 is obtained. This has been illustrated through some examples. The regular properties of the direct sum of two fuzzy graphs have been studied. It is illustrated that the direct sum $G_1 \oplus G_2$ of two connected fuzzy graphs G_1 and G_2 need not be a connected fuzzy graph.

A fuzzy graph G is a pair of functions G:(σ, μ) where σ is a fuzzy subset of a non empty set V and μ is a symmetric fuzzy relation on σ. The underlying crisp graph of G:(σ, μ) is denoted by G*(V, E) where $E \subseteq V \times V$. Let G:(σ, μ) be a fuzzy graph.

The underlying crisp graph of G:(σ, μ) is denoted by G*:(V, E) where $E \subseteq V\times V$. A fuzzy graph G is an effective fuzzy graph if $\mu(uv) = \sigma(u) \wedge \sigma(v)$ for all $uv\in E$. G is complete if $\mu(uv) = \sigma(u)\wedge\sigma(v)$ for all $u,v\in V$. Therefore G is a complete fuzzy graph if and only if G is an effective fuzzy graph and G* is complete. (σ', μ') is a spanning fuzzy subgraph of (σ,μ) if $\sigma=\sigma'$ and $\mu' \subseteq \mu$, that is, if $\sigma(u) = \sigma'(u)$ for every $u\in V$ and $\mu'(e) \le \mu(e)$ for every $e\in E$.

The degree of a vertex u is defined as $d_G(u) = \sum_{u\neq v} \mu(uv)$. Since $\mu(uv) > 0$ for $uv \in E$ and $\mu(uv) = 0$ for $uv \notin E$, this can be expressed as $d_G(u) = \sum_{uv\in E} \mu(uv)$.

Let G:(σ, μ) be a fuzzy graph on G*:(V,E). If $d_G(v)=k$ for all $v\in V$, that is, if each vertex has same degree k, then G is said to be a regular fuzzy graph of degree k or a k-regular fuzzy graph.

Let G:(σ, μ) be a fuzzy graph on G*. The total degree of a vertex $u\in V$ is defined by $td_G(u) = \sum_{uv\in E} \mu(uv) + \sigma(u) = d_G(u) + \sigma(u)$. If each vertex of G has the same total degree k, then G is said to be a totally regular fuzzy graph of total degree k or a k-totally regular fuzzy graph.

The lower and upper truncations[2] of σ at level t, $0 < t \le 1$, are the fuzzy subsets $\sigma_{(t)}$ and $\sigma^{(t)}$ defined respectively by ,

$$\sigma_{(t)}(u) = \begin{cases} \sigma(u), & \text{if } u \in \sigma^t \\ 0, & \text{if } u \notin \sigma^t \end{cases} \text{ and } \sigma^{(t)}(u) = \begin{cases} t, & \text{if } u \in \sigma^t \\ \sigma(u), & \text{if } u \notin \sigma^t \end{cases}.$$

Let G:(σ,μ) be a fuzzy graph with underlying crisp graph G*:(V,E). Take $V_{(t)} = \sigma^t$, $E_{(t)} = \mu^t$. Then $G_{(t)}:(\sigma_{(t)},\mu_{(t)})$ is a fuzzy graph with underlying crisp graph $G_{(t)}^*:(V_{(t)}, E_{(t)})$. This is called the lower truncation[5] of the fuzzy graph G at level t. Here $V_{(t)}$ and $E_{(t)}$ may be proper subsets of V and E respectively. Take $V^{(t)} = V$, $E^{(t)} = E$. Then $G^{(t)}:(\sigma^{(t)}, \mu^{(t)})$ is a fuzzy graph with underlying crisp graph $G^{(t)}*:(V^{(t)},E^{(t)})$. This is called the upper truncation of the fuzzy graph G at level t.

II. DIRECT SUM

Let $G_1:(\sigma_1,\mu_1)$ and $G_2:(\sigma_2,\mu_2)$ denote two fuzzy graphs with underlying crisp graphs $G_1^*:(V_1,E_1)$ and $G_2^*:(V_2,E_2)$ respectively. Let $V = V_1 \cup V_2$ and let $E = \{uv / u,v\in V;\ uv \in E_1 \text{ or } uv \in E_2 \text{ but not both}\}$. Define G:(σ, μ) by

$$\sigma(u) = \begin{cases} \sigma_1(u) & , \text{if } u \in V_1 \\ \sigma_2(u) & , \text{if } u \in V_2 \\ \sigma_1(u) \vee \sigma_2(u) & , \text{if } u \in V_1 \cup V_2 \end{cases} \quad \text{and} \quad \mu(uv) = \begin{cases} \mu_1(uv), & \text{if } uv \in E_1 \\ \mu_2(uv), & \text{if } uv \in E_2 \end{cases}$$

Then if $uv \in E_1$, $\mu(uv) = \mu_1(uv) \le \sigma_1(u) \wedge \sigma_1(v) \le \sigma(u) \wedge \sigma(v)$, if $uv \in E_2$, $\mu(uv) = \mu_2(uv) \le \sigma_2(u) \wedge \sigma_2(v) \le \sigma(u) \wedge \sigma(v)$. Therefore (σ, μ) defines a fuzzy graph. This is called the direct sum of two fuzzy graphs.

ISSN 2250-3153

2.1Example

The following Fig.1 gives an example of the direct sum of two fuzzy graphs which have distinct edge sets.

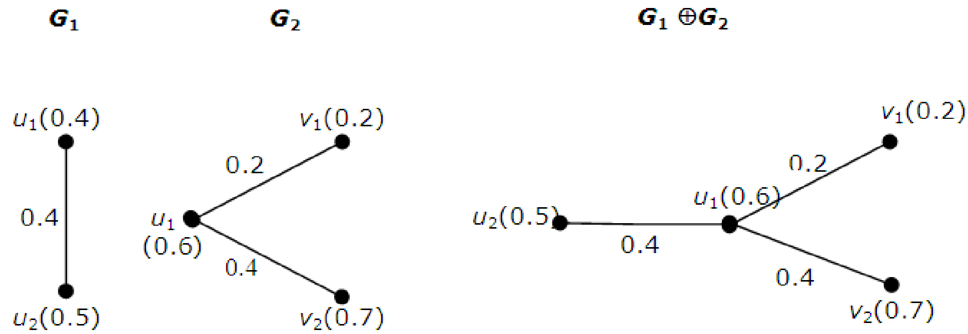


Figure 1: Direct sum of two fuzzy graphs with disjoint edge sets

2.2Example

The following Fig.2 gives an example of the direct sum of two fuzzy graphs in which the edge sets are not disjoint.

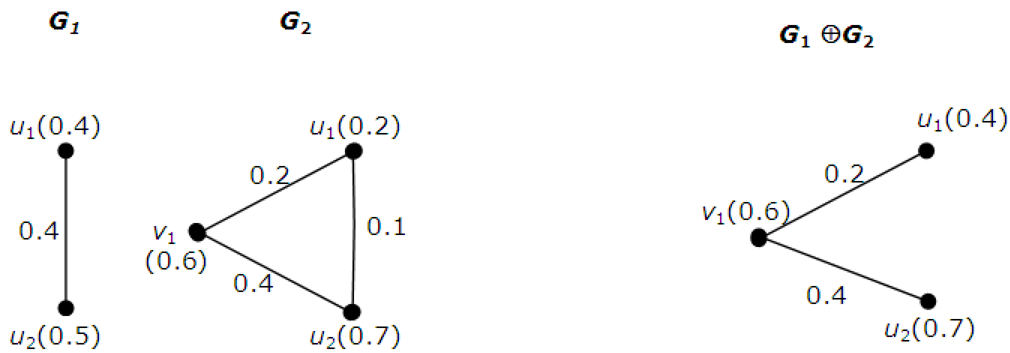


Figure 2: Direct sum of two fuzzy graphs with non-disjoint edge sets

2.3Remark

If G_1 and G_2 are two effective fuzzy graphs, their direct sum $G_1 \oplus G_2$ need not be an effective fuzzy graph which can be seen from the example in Figure 3.

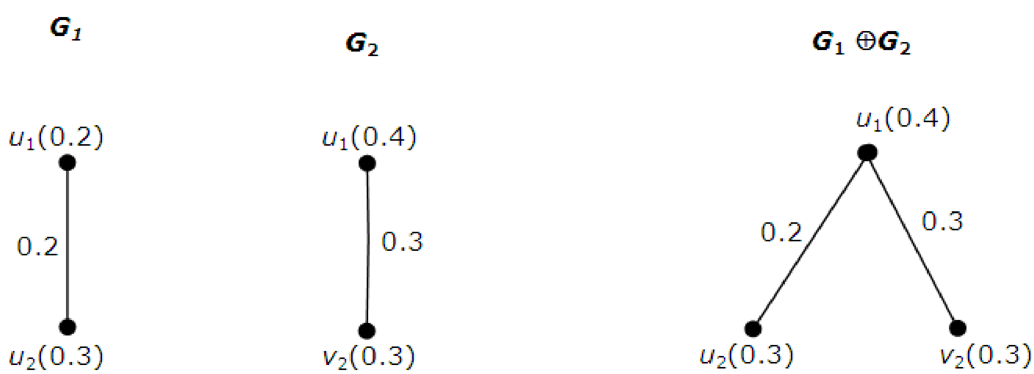


Figure 3: Direct sum of two effective fuzzy graphs

2.4Theorem

If G_1 and G_2 are two effective fuzzy graphs such that no edge of $G_1 \oplus G_2$ has both ends in $V_1 \cap V_2$ and every edge uv of $G_1 \oplus G_2$ with one end $u \in V_1 \cap V_2$ and $uv \in E_1$ (or E_2) is such that $\sigma_1(u) \geq \sigma_1(v)$ [or $\sigma_2(u) \geq \sigma_2(v)$], then $G_1 \oplus G_2$ is an effective fuzzy graph.

Proof: Let uv be an edge of $G_1 \oplus G_2$. We have two cases to consider.

Case 1: $u, v \notin V_1 \cap V_2$.

Then $u, v \in V_1$ or V_2 but not both.

Suppose that $u, v \in V_1$. Then $uv \in E_1$.

Therefore $\sigma(u) = \sigma_1(u)$, $\sigma(v) = \sigma_1(v)$ and $\mu(uv) = \mu_1(uv)$.

Also since G_1 is an effective fuzzy graph, $\mu(uv) = \mu_1(uv) = \sigma_1(u) \wedge \sigma_1(v) = \sigma(u) \wedge \sigma(v)$.

The proof is similar if $u, v \in V_2$.

Case 2: $u \in V_1 \cap V_2$, $v \notin V_1 \cap V_2$. (or vice versa).
Without loss of generality, assume that $v \in V_1$.
Then $\sigma(v) = \sigma_1(v)$. By hypothesis, $\sigma_1(u) \geq \sigma_1(v)$.
Now $\sigma(u) = \sigma_1(u) \vee \sigma_2(u) \geq \sigma_1(u) \geq \sigma_1(v) = \sigma(v)$. So $\sigma(u) \wedge \sigma(v) = \sigma(v)$.
Hence $\mu(uv) = \mu_1(uv) = \sigma_1(u) \wedge \sigma_1(v) = \sigma_1(v) = \sigma(v) = \sigma(u) \wedge \sigma(v)$.
Therefore $G_1 \oplus G_2$ is an effective fuzzy graph.

III. TRUNCATIONS OF THE DIRECT SUM OF TWO FUZZY GRAPHS

3.1Theorem

$(G_1 \oplus G_2)_{(t)}$ is a spanning fuzzy sub graph of $G_{1(t)} \oplus G_{2(t)}$.

Proof: First we prove that $(\sigma_1 \oplus \sigma_2)_{(t)} = \sigma_{1(t)} \oplus \sigma_{2(t)}$.
It follows from the definitions that if $u \in V_i$, $(\sigma_1 \oplus \sigma_2)_{(t)}(u) = \sigma_{i(t)}(u) = (\sigma_{1(t)} \oplus \sigma_{2(t)})(u)$, $i = 1,2$. Let $u \in V_1 \cap V_2$. Without loss of generality, assume that $\sigma_1(u) \leq \sigma_2(u)$.
Then $(\sigma_1 \oplus \sigma_2)(u) = \sigma_2(u)$ gives $(\sigma_1 \oplus \sigma_2)_{(t)}(u) = \sigma_{2(t)}(u)$ …..………………………… (1)
Now we claim that $\sigma_1(u) \leq \sigma_2(u)$ implies $\sigma_{1(t)}(u) \leq \sigma_{2(t)}(u)$.
If $t \leq \sigma_1(u) \leq \sigma_2(u)$, then $\sigma_{1(t)}(u) = \sigma_1(u) \leq \sigma_2(u) = \sigma_{2(t)}(u)$.
If $\sigma_1(u) < t \leq \sigma_2(u)$, then $\sigma_{1(t)}(u) = 0 < \sigma_2(u) = \sigma_{2(t)}(u)$.
If $\sigma_1(u) \leq \sigma_2(u) < t$, then $\sigma_{1(t)}(u) = 0 = \sigma_{2(t)}(u)$.
Hence $\sigma_{1(t)}(u) \leq \sigma_{2(t)}(u)$. Therefore $(\sigma_{1(t)} \oplus \sigma_{2(t)})(u) = \sigma_{2(t)}(u)$ …………………………….... (2)
From (1) & (2), $(\sigma_1 \oplus \sigma_2)_{(t)}(u) = (\sigma_{1(t)} \oplus \sigma_{2(t)})(u)$. Hence $(\sigma_1 \oplus \sigma_2)_{(t)} = \sigma_{1(t)} \oplus \sigma_{2(t)}$.
Next we prove that $(\mu_1 \oplus \mu_2)_{(t)} \leq \mu_{1(t)} \oplus \mu_{2(t)}$. For this, we consider the following three cases:
Case 1: $uv \in E_1 \cap E_2$ with either $\mu_1(uv) \geq t$ or $\mu_2(uv) \geq t$ but not both.
Suppose that $\mu_1(uv) \geq t$. Then $\mu_2(uv) < t$. So $\mu_{1(t)}(uv) = \mu(uv)$ and $\mu_{2(t)}(uv) = 0$.
Hence the edge uv will be in $G_{1(t)} \oplus G_{2(t)}$ with $(\mu_{1(t)} \oplus \mu_{2(t)})(uv) = \mu(uv)$.
Since $uv \in E_1 \cap E_2$, $(\mu_1 \oplus \mu_2)(uv) = 0 \Rightarrow (\mu_1 \oplus \mu_2)_{(t)}(uv) = 0$.
Therefore $(\mu_1 \oplus \mu_2)_{(t)}(uv) < (\mu_{1(t)} \oplus \mu_{2(t)})(uv)$. The proof is similar if $\mu_2(uv) \geq t$.
Case 2: $uv \in E_1 \cap E_2$ with either $\mu_1(uv) < t$, $\mu_2(uv) < t$ or $\mu_1(uv) \geq t$, $\mu_2(uv) \geq t$.
Since $uv \in E_1 \cap E_2$, $(\mu_1 \oplus \mu_2)_{(t)}(uv) = 0$.
If $\mu_i(uv) < t$, then $\mu_{i(t)}(uv) = 0$, $i = 1,2$. So $(\mu_{1(t)} \oplus \mu_{2(t)})(uv) = 0$.
If $\mu_i(uv) \geq t$, then $\mu_{i(t)}(uv) = \mu_i(uv) > 0$, $i = 1,2$. So $(\mu_{1(t)} \oplus \mu_{2(t)})(uv) = 0$.
Hence $(\mu_1 \oplus \mu_2)_{(t)} = \mu_{1(t)} \oplus \mu_{2(t)}$.
Case 3: $uv \in E_1$ or $uv \in E_2$ but not both.
If $uv \in E_i$, $(\mu_1 \oplus \mu_2)(uv) = \mu_i(uv)$, $i = 1,2$. Hence $(\mu_1 \oplus \mu_2)_{(t)}(uv) = \mu_{i(t)}(uv) = (\mu_{1(t)} \oplus \mu_{2(t)})(uv)$.
From the above three cases, we get $(\mu_1 \oplus \mu_2)_{(t)} \leq \mu_{1(t)} \oplus \mu_{2(t)}$.
Hence $(G_1 \oplus G_2)_{(t)}$ is a spanning fuzzy sub graph of $G_{1(t)} \oplus G_{2(t)}$.

3.2Remark:

From the proof of the above theorem, if for any $uv \in E_1 \cap E_2$, we have either $\mu_1(uv) < t$, $\mu_2(uv) < t$ or $\mu_1(uv) \geq t$, $\mu_2(uv) \geq t$, then $(G_1 \oplus G_2)_{(t)} = G_{1(t)} \oplus G_{2(t)}$.

3.3Theorem:

$(G_1 \oplus G_2)_{(t)} = G_{1(t)} \oplus G_{2(t)}$.

Proof: First we prove that $(\sigma_1 \oplus \sigma_2)_{(t)} = \sigma_{1(t)} \oplus \sigma_{2(t)}$.
It follows from the definitions that if $u \in Vi$, $(\sigma_1 \oplus \sigma_2)_{(t)}(u) = \sigma_{i(t)}(u) = (\sigma_{1(t)} \oplus \sigma_{2(t)})(u)$, $i = 1,2$.
Let $u \in V_1 \cap V_2$. Without loss of generality, assume that $\sigma_1(u) \leq \sigma_2(u)$.
Then proceeding as in the previous Theorem, that $(\sigma_1 \oplus \sigma_2)_{(t)}(u) = (\sigma_{1(t)} \oplus \sigma_{2(t)})(u)$.
Hence $(\sigma_1 \oplus \sigma_2)_{(t)}(u) = (\sigma_{1(t)} \oplus \sigma_{2(t)})(u)$.
Next we prove that $(\mu_1 \oplus \mu_2)_{(t)} = \mu_{1(t)} \oplus \mu_{2(t)}$.
For this, we consider the following two cases:
Case 1: $uv \in E_1 \cap E_2$.
Then $(\mu_1 \oplus \mu_2)(uv) = 0 \Rightarrow (\mu_1 \oplus \mu_2)_{(t)}(uv) = 0$. Since $\mu_i(uv) > 0$, $\mu_{i(t)}(uv) > 0$, $i = 1,2$.
Therefore the edge uv will not be in $G_{1(t)} \oplus G_{2(t)}$. So $(\mu_{1(t)} \oplus \mu_{2(t)})(uv) = 0$.
Hence $(\mu_1 \oplus \mu_2)_{(t)}(uv) = (\mu_{1(t)} \oplus \mu_{2(t)})(uv)$.
Case 2: $uv \in E_1$ or $uv \in E_2$ but not both.
If $uv \in E_i$, $(\mu_1 \oplus \mu_2)_{(t)}(uv) = \mu_i(uv)$, $i = 1,2$.

ISSN 2250-3153

Hence $(\mu_1 \oplus \mu_2)_{(t)}(uv) = \mu_i(t)(uv) = (\mu_{1(t)} \oplus \mu_{2(t)})(uv)$.
From the above two cases, we get $(\mu_1 \oplus \mu_2)_{(t)}(uv) = (\mu_{1(t)} \oplus \mu_{2(t)})(uv)$.
Hence **:** $(G_1 \oplus G_2)_{(t)} = G_{1(t)} \oplus G_{2(t)}$.

IV. DEGREE OF A VERTEX IN THE DIRECT SUM

In this section we find the degrees of the vertices in the direct sum $G_1 \oplus G_2$ of two fuzzy graphs G_1 and G_2 in terms of degrees of the vertices in the fuzzy graphs G_1 and G_2.

4.1Theorem:
The degree of a vertex in $G_1 \oplus G_2 : (\sigma, \mu)$ in terms of the degrees of the vertices in $G_1(\sigma_1, \mu_1)$ and $G_2(\sigma_2, \mu_2)$ is given by,

$$d_{G_1 \oplus G_2}(u) = \begin{cases} d_{G_1}(u), & \text{if } u \in V_1 - V_2 \\ d_{G_2}(u), & \text{if } u \in V_2 - V_1 \\ d_{G_1}(u) + d_{G_2}(u), & \text{if } u \in V_1 \cap V_2 \text{ and } E_1 \cap E_2 = \phi \\ [d_{G_1}(u) + d_{G_2}(u)] - \sum_{uv \in E_1 \cap E_2} [\mu_1(uv) + \mu_2(uv)], & \text{if } u \in V_1 \cap V_2 \text{ and } E_1 \cap E_2 \neq \phi \end{cases}$$

Proof:
For any vertex in the direct sum $G_1 \oplus G_2 : (\sigma, \mu)$ we have three cases to consider.
Case (1)
Either $u \in V_1$ or $u \in V_2$ but not both. Then no edge incident at u lies in $E_1 \cap E_2$.
So
$$(\mu_1 \oplus \mu_2)(uv) = \begin{cases} \mu_1(uv) & \text{if } u \in V_1 \ \text{(i.e) if } uv \in E_1 \\ \mu_2(uv) & \text{if } u \in V_2 \ \text{(i.e) if } uv \in E_2 \end{cases}$$

Hence if $u \in V_1$, then $d_{G_1 \oplus G_2}(u) = \sum_{uv \in E_1} \mu_1(uv) = d_{G_1}(u)$ and

if $u \in V_2$, then $d_{G_1 \oplus G_2}(u) = \sum_{uv \in E_2} \mu_2(uv) = d_{G_2}(u)$

Case (2)
$u \in V_1 \cap V_2$ but no edge incident at u lies in $E_1 \cap E_2$. Then any edge incident at u is either in E_1 or in E_2 but not in $E_1 \cap E_2$. Also all these edges are included in $G_1 \oplus G_2 : (\sigma, \mu)$.
Hence the degree of u in $G_1 \oplus G_2 : (\sigma, \mu)$ is given by,

$$d_{G_1 \oplus G_2}(u) = \sum_{uv \in E} (\mu_1 \oplus \mu_2)(uv)$$
$$= \sum_{uv \in E_1} \mu_1(uv) + \sum_{uv \in E_2} \mu_2(uv)$$
$$= [d_{G_1}(u) + d_{G_2}(u)]$$

Case (3)
$u \in V_1 \cap V_2$ and some edges incident at u are in $E_1 \cap E_2$. By the definition, any edge in $E_1 \cap E_2$ will not be included in $G_1 \oplus G_2 : (\sigma, \mu)$. Then the degree of u in the direct sum $G_1 \oplus G_2 : (\sigma, \mu)$ is

$$d_{G_1 \oplus G_2}(u) = \sum_{uv \in E} (\mu_1 \oplus \mu_2)(uv)$$
$$= \sum_{uv \in E_1 - E_2} \mu_1(uv) + \sum_{uv \in E_2 - E_1} \mu_2(uv)$$
$$= \sum_{uv \in E_1 - E_2} \mu_1(uv) + \sum_{uv \in E_2 - E_1} \mu_2(uv) + \sum_{uv \in E_1 \cap E_2} [\mu_1(uv) + \mu_2(uv)] - \sum_{uv \in E_1 \cap E_2} [\mu_1(uv) + \mu_2(uv)]$$
$$= \left\{ \sum_{uv \in E_1 - E_2} \mu_1(uv) + \sum_{uv \in E_1 \cap E_2} \mu_1(uv) \right\} + \left\{ \sum_{uv \in E_2 - E_1} \mu_2(uv) + \sum_{uv \in E_1 \cap E_2} \mu_2(uv) \right\} - \sum_{uv \in E_1 \cap E_2} [\mu_1(uv) + \mu_2(uv)]$$
$$= [d_{G_1}(u) + d_{G_2}(u)] - \sum_{uv \in E_1 \cap E_2} [\mu_1(uv) + \mu_2(uv)]$$

From the above two cases we conclude that the degree of the vertex in $G_1 \oplus G_2$ in terms of the degrees of the vertices in $G_1:(\sigma_1, \mu_1)$ and $G_2:(\sigma_2, \mu_2)$ is obtained as follows:

$$d_{G_1 \oplus G_2}(u) = \begin{cases} d_{G_1}(u), & \text{if } u \in V_1 - V_2 \\ d_{G_2}(u), & \text{if } u \in V_2 - V_1 \\ d_{G_1}(u) + d_{G_2}(u), & \text{if } u \in V_1 \cap V_2 \text{ and } E_1 \cap E_2 = \phi \\ [d_{G_1}(u) + d_{G_2}(u)] - \sum_{uv \in E_1 \cap E_2} [\mu_1(uv) + \mu_2(uv)], & \text{if } u \in V_1 \cap V_2 \text{ and } E_1 \cap E_2 \neq \phi \end{cases}$$

Hence the theorem is proved.

4.2Example:

Consider the two fuzzy graphs $G_1:(\sigma_1, \mu_1)$ and $G_2:(\sigma_2, \mu_2)$ in which the edge sets are disjoint and their sum $G_1 \oplus G_2 : (\sigma, \mu)$.

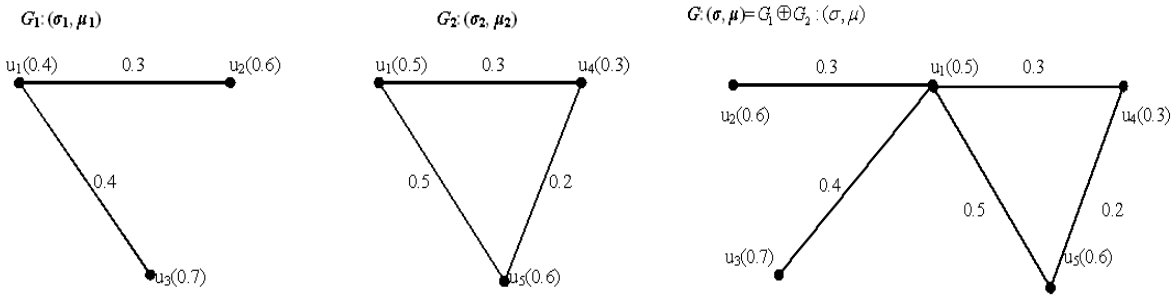


Figure 4: Degree of vertices in the Direct sum of two fuzzy graphs

The degrees of the vertices in the direct sum $G_1 \oplus G_2$ as follows:

$d_{G_1 \oplus G_2}(u_1) = 0.3 + 0.3 + 0.4 + 0.5 = 1.5$; $d_{G_1 \oplus G_2}(u_2) = 0.3$; $d_{G_1 \oplus G_2}(u_3) = 0.4$;

$d_{G_1 \oplus G_2}(u_4) = 0.3 + 0.2 = 0.5$; $d_{G_1 \oplus G_2}(u_5) = 0.5 + 0.2 = 0.7$

Now let us find the degrees of the vertices in the direct sum $G_1 \oplus G_2$ of two fuzzy graphs G_1 and G_2 in terms of degrees of the vertices in the fuzzy graphs G_1 and G_2.

Since there is no edge in $E_1 \cap E_2$ and $u_1 \in V_1 \cap V_2$ the degree of u_1 in $G_1 \oplus G_2$ is exactly the sum of the degrees of u_1 in G_1 and G_2.That is,

$$d_{G_1 \oplus G_2}(u_1) = d_{G_1}(u_1) + d_{G_2}(u_1) = (0.3 + 0.4) + (0.3 + 0.5) = 1.5$$

The vertices u_2 and u_3 are in V_1 only and not in V_2. That is, $u_2, u_3 \in V_1 - V_2$. Hence the degrees of u_2 and u_3 in $G_1 \oplus G_2$ are equal to the degrees of u_2 and u_3 in G_1. That is,

$$d_{G_1 \oplus G_2}(u_2) = d_{G_1}(u_2) = 0.3 \text{ and } d_{G_1 \oplus G_2}(u_3) = d_{G_1}(u_3) = 0.4$$

Similarly, since $u_4, u_5 \in V_2 - V_1$, we have,

$$d_{G_1 \oplus G_2}(u_4) = d_{G_2}(u_4) = 0.5 \text{ and } d_{G_1 \oplus G_2}(u_5) = d_{G_2}(u_5) = 0.7$$

4.3Example:

Here is another example in which the edge sets are not disjoint.

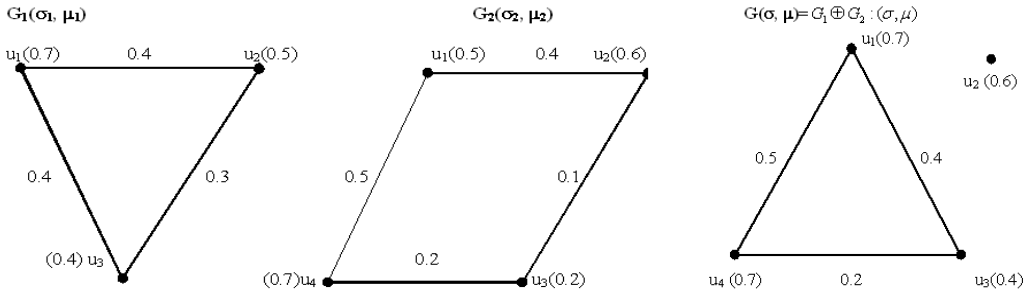


Figure 5: Degree of vertices in the Direct sum of two fuzzy graphs

Here, we have $E_1 \cap E_2 = \{u_1u_2, u_2u_3\}$.

From the graph of the direct sum $G_1 \oplus G_2 : (\sigma, \mu)$, we see that the degrees of the vertices of $G_1 \oplus G_2$ are: $\mu_{G_1 \oplus G_2}(u_1) = 0.4 + 0.5 = 0.9$; $\mu_{G_1 \oplus G_2}(u_3) = 0.2 + 0.4 = 0.6$; $\mu_{G_1 \oplus G_2}(u_4) = 0.2 + 0.5 = 0.7$.

Now we shall find the degrees of the vertices in $G_1 \oplus G_2$ in terms of the degrees of the vertices in $G_1:(\sigma_1, \mu_1)$ and $G_2:(\sigma_2, \mu_2)$. Since $E_1 \cap E_2 = \{u_1u_2, u_2u_3\}$ the edges u_1u_2, u_2u_3 are not in $G_1 \oplus G_2$. The vertex $u_4 \in V_2 - V_1$.

Hence by the previous case, the degree of u_4 in $G_1 \oplus G_2$ is that of u_4 in G_2.That is,

$$d_{G_1 \oplus G_2}(u_4) = 0.5 + 0.2 = 0.7 = d_{G_2}(u_4)$$

Since $E_1 \cap E_2 \neq \phi$ and $u_1 \in V_1 \cap V_2$ the degree of u_1 is given by,

$$\begin{aligned} d_{G_1 \oplus G_2}(u_1) &= [d_{G_1}(u_1) + d_{G_2}(u_1)] - \sum_{u_1u_i \in E_1E_2} [\mu_1(u_1u_i) + \mu_2(u_1u_i)] \\ &= [d_{G_1}(u_1) + d_{G_2}(u_1)] - [\mu_1(u_1u_2) + \mu_2(u_1u_2)] \\ &= [(0.4 + 0.4) + (0.5 + 0.4)] - [0.4 + 0.4] \\ &= 0.5 + 0.4 = 0.9 \end{aligned}$$

Similarly, since $E_1 \cap E_2 \neq \phi$ and $u_3 \in V_1 \cap V_2$ the degree of u_3 is given by,

$$\begin{aligned} d_{G_1 \oplus G_2}(u_3) &= [d_{G_1}(u_3) + d_{G_2}(u_3)] - \sum_{u_3u_i \in E_1E_2} [\mu_1(u_3u_i) + \mu_2(u_3u_i)] \\ &= [d_{G_1}(u_3) + d_{G_2}(u_3)] - [\mu_1(u_3u_2) + \mu_2(u_3u_2)] \\ &= [(0.4 + 0.3) + (0.2 + 0.1)] - [0.3 + 0.1] \\ &= 0.2 + 0.4 = 0.6 \end{aligned}$$

V. DIRECT SUM OF TWO REGULAR FUZZY GRAPHS

If $G_1:(\sigma_1, \mu_1)$ and $G_2:(\sigma_2, \mu_2)$ are two regular fuzzy graphs then their direct sum $G_1 \oplus G_2 : (\sigma, \mu)$ need not be a regular fuzzy graph. It is illustrated with the following examples.

ISSN 2250-3153

5.1Example:

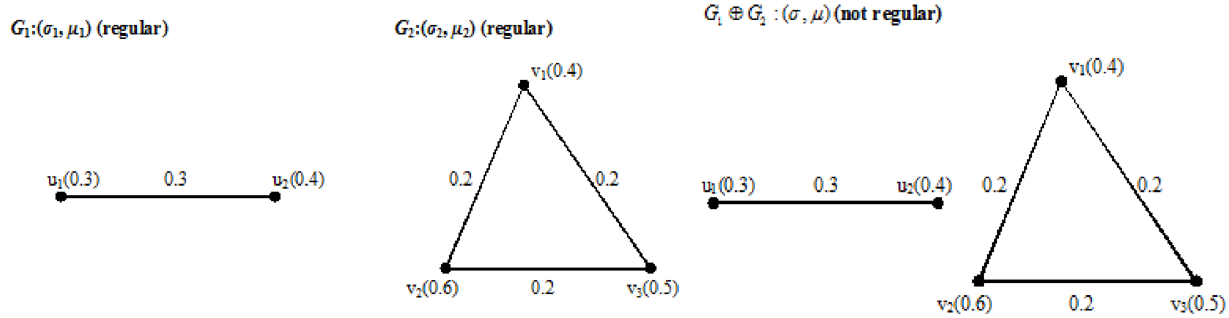


Figure 5: The Direct sum of two regular fuzzy graphs

5.2Example:

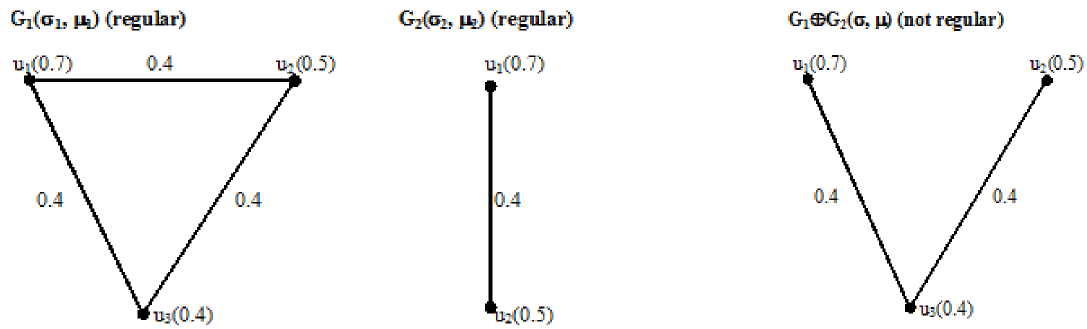


Figure 6: The Direct sum of two regular fuzzy graphs

5.3Remark:

For the direct sum $G_1 \oplus G_2 : (\sigma, \mu)$ to be regular, $G_1:(\sigma_1, \mu_1)$ and $G_2:(\sigma_2, \mu_2)$ need not be regular fuzzy graphs. It is illustrated with the following examples.

5.4Example:

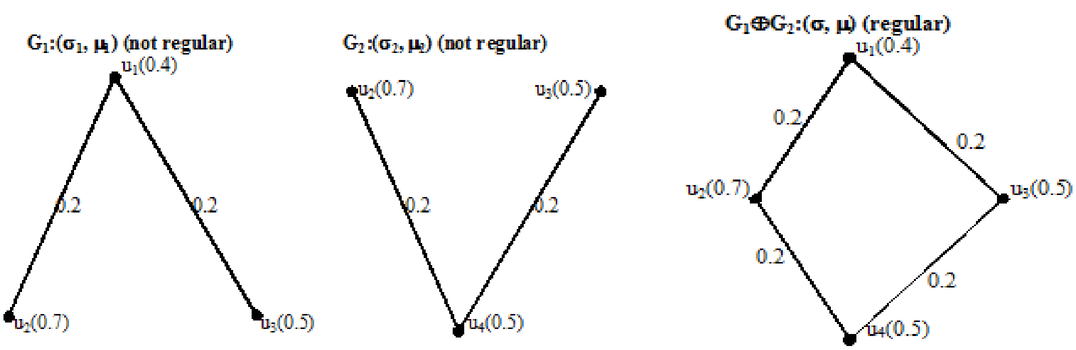


Figure 7: The Direct sum of two non-regular fuzzy graphs

5.5Example:

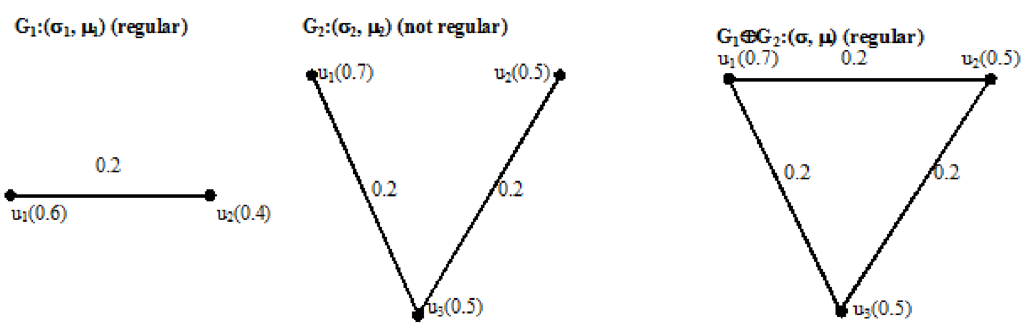


Figure 8: The Direct sum of a regular fuzzy graph and a non-regular fuzzy graph

5.6Example:

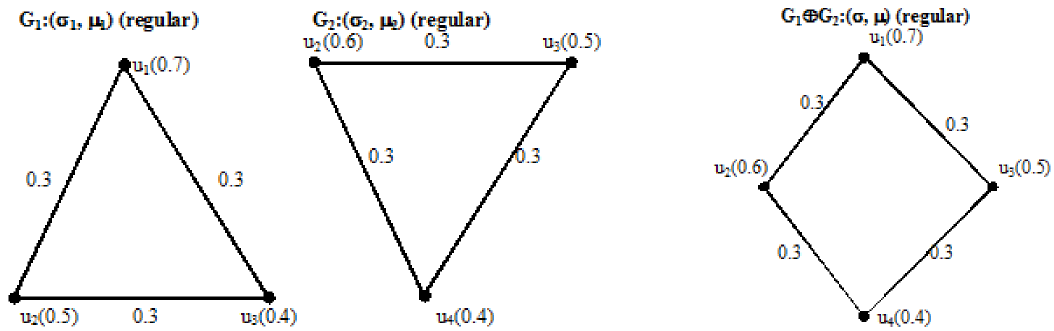


Figure 9: The Direct sum of two regular fuzzy graphs

From the above examples, we can see that there is no relationship between the regular property of the given fuzzy graphs and the direct sums of them. In the following result, we obtain the necessary and sufficient condition for the direct sum of two regular fuzzy graphs to be regular when $V_1 \cap V_2 = \phi$.

5.7Theorem:

If $G_1:(\sigma_1, \mu_1)$ and $G_2:(\sigma_2, \mu_2)$ are regular fuzzy graphs with degrees k_1 and k_2 respectively and $V_1 \cap V_2 = \phi$ then $G_1 \oplus G_2 : (\sigma, \mu)$ is regular if and only if $k_1 = k_2$.

Proof:

Let $G_1:(\sigma_1, \mu_1)$ be a k_1-regular fuzzy graph with underlying crisp graph $G_1^*:(V_1, E_1)$ and let $G_2:(\sigma_2, \mu_2)$ be a k_2-regular fuzzy graph with underlying crisp graph $G_2^*:(V_2, E_2)$ respectively such that $V_1 \cap V_2 = \phi$.

Assume that $G_1 \oplus G_2 : (\sigma, \mu)$ is regular.

We know that,

$$d_{G_1 \oplus G_2}(u) = \begin{cases} d_{G_1}(u), & \text{if } u \in V_1 - V_2 \\ d_{G_2}(u), & \text{if } u \in V_2 - V_1 \\ d_{G_1}(u) + d_{G_2}(u), & \text{if } u \in V_1 \cap V_2 \text{ and } E_1 \cap E_2 = \phi \\ [d_{G_1}(u) + d_{G_2}(u)] - \sum_{uv \in E_1 \cap E_2} [\mu_1(uv) + \mu_2(uv)], & \text{if } u \in V_1 \cap V_2 \text{ and } E_1 \cap E_2 \neq \phi \end{cases}$$

Since $V_1 \cap V_2 = \phi$,

$$d_{G_1 \oplus G_2}(u) = \begin{cases} d_{G_1}(u), & \text{if } u \in V_1 \\ d_{G_2}(u), & \text{if } u \in V_2 \end{cases}$$

$$d_{G_1 \oplus G_2}(u) = \begin{cases} k_1, & \text{if } u \in V_1 \\ k_2, & \text{if } u \in V_2 \end{cases}$$

Since $G_1 \oplus G_2 : (\sigma, \mu)$ is regular, $k_1 = k_2$.

Conversely assume that $G_1:(\sigma_1, \mu_1)$ and $G_2:(\sigma_2, \mu_2)$ are k-regular fuzzy graphs such that $V_1 \cap V_2 = \phi$.

Then the degree of any vertex in the direct sum is given by,

$$d_{G_1 \oplus G_2}(u) = \begin{cases} d_{G_1}(u), & \text{if } u \in V_1 \\ d_{G_2}(u), & \text{if } u \in V_2 \end{cases}$$

$$d_{G_1 \oplus G_2}(u) = \begin{cases} k, & \text{if } u \in V_1 \\ k, & \text{if } u \in V_2 \end{cases}$$

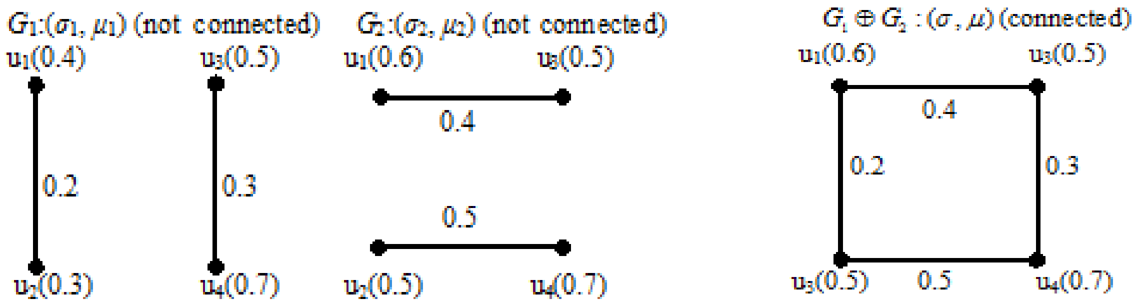
Therefore, $d_{G_1 \oplus G_2}(u) = k, \ \forall u \in V_1 \cup V_2$.

Hence $G_1 \oplus G_2 : (\sigma, \mu)$ is regular.

VI. DIRECT SUM OF CONNECTED FUZZY GRAPHS

If $G_1:(\sigma_1, \mu_1)$ and $G_2:(\sigma_2, \mu_2)$ are two connected fuzzy graphs then their direct sum $G_1 \oplus G_2 : (\sigma, \mu)$ need not be a connected fuzzy graph. It is illustrated with the following examples.

6.1Example:



6.2Theorem:

If $G_1:(\sigma_1, \mu_1)$ and $G_2:(\sigma_2, \mu_2)$ are two connected fuzzy graphs with underlying crisp graphs $G_1^*:(V_1, E_1)$ and $G_2^*:(V_2, E_2)$ respectively such that $E_1 \cap E_2 = \phi$ and $V_1 \cap V_2 \neq \phi$ then their direct sum $G_1 \oplus G_2 : (\sigma, \mu)$ is a connected fuzzy graph.

Proof:

Since $G_1:(\sigma_1, \mu_1)$ is a connected fuzzy graph, $\mu_1^{\infty}(u,v) > 0$ for all $(u,v) \in E_1$

and since $G_2:(\sigma_2, \mu_2)$ is a connected fuzzy graph, $\mu_2^{\infty}(u,v) > 0$ for all $(u,v) \in E_2$.

Also $V_1 \cap V_2 \neq \phi$.

Therefore there exists at least one vertex which is in $V_1 \cap V_2$. But there is no edge in $E_1 \cap E_2$.

Hence there exists a path between any two vertices in the direct sum $G_1 \oplus G_2 : (\sigma, \mu)$ of $G_1:(\sigma_1, \mu_1)$ and $G_2:(\sigma_2, \mu_2)$.

That is $\mu^{\infty}_{G_1 \oplus G_2}(u,v) > 0$ for all $(u,v) \in E$. This implies that $G_1 \oplus G_2 : (\sigma, \mu)$ is connected.

6.3Remark:

If $G_1:(\sigma_1, \mu_1)$ and $G_2:(\sigma_2, \mu_2)$ are two connected fuzzy graphs with underlying crisp graphs $G_1^*:(V_1, E_1)$ and $G_2^*:(V_2, E_2)$ respectively such that $n(V_1 \cap V_2) = 1$ then their direct sum $G_1 \oplus G_2 : (\sigma, \mu)$ is a connected fuzzy graph.

VII. CONCLUSION

In this paper, the direct sum $G_1 \oplus G_2$ of two fuzzy graphs G_1 and G_2 is defined. A formula to find the degree of a vertex in the direct sum $G_1 \oplus G_2 : (\sigma, \mu)$ of two fuzzy graphs $G_1:(\sigma_1, \mu_1)$ and $G_2:(\sigma_2, \mu_2)$ in terms of the degrees of the vertices in $G_1:(\sigma_1, \mu_1)$ and $G_2:(\sigma_2, \mu_2)$ is obtained. This has been illustrated with examples. Also some of the characteristics of the direct sum of effective, regular and connected fuzzy graphs have been illustrated. Operation on fuzzy graph is a great tool to consider large fuzzy graph as a combination of small fuzzy graphs and to derive its properties from those of the small ones. A step in that direction is made through this paper.

REFERENCES

[1] Frank Harary, Graph Thoery, Narosa / Addison Wesley, Indian Student Edition, 1988.

[2] John N. Modeson and Premchand S.Nair, Fuzzy Graphs and Fuzzy Hypergraphs, Physica-verlag Heidelberg, 2000.

[3] J.N.Mordeson and C.S. Peng, Operations on fuzzy graphs, Information Sciences 79 (1994), 159-170.

[4] Nagoorgani. A and Radha. K, "Conjunction of Two Fuzzy Graphs", International Review of Fuzzy Mathematics, 2008, Vol. 3, 95-105.

[5] Nagoorgani. A and Radha. K, "Some Properties of Truncations of Fuzzy Graphs", Advances in Fuzzy Sets and Systems, 2009, Vol.4, No.2, 215-227.

[6] P. Bhattacharya, Some remarks on fuzzy graphs, Pattern Recognition Letter 6 (1987), 297-302.

[7] Rosenfeld, A. (1975) "Fuzzy graphs". In: Zadeh, L.A., Fu, K.S., Tanaka, K., Shimura, M. (eds.), Fuzzy Sets and their Applications to Cognitive and Decision Processes, Academic Press, New York, ISBN 9780127752600, pp.77–95.

AUTHORS

First Author – Dr. K. Radha, M.Sc.,M.Phil.,Ph.D., P.G & Research Department of Mathematics, Periyar E.V.R. College, Tiruchirapalli-620023. E-mail: radhagac@yahoo.com

Second Author – Mr.S. Arumugam , M.Sc.,M.Phil.,B.Ed.,(Ph.D.), Govt. High School, Thinnanur, Tiruchirapalli-621006. E-mail: anbu.saam@gmail.com & arumugammathematics@gmail.com

Correspondence Author – Dr. K. Radha, M.Sc.,M.Phil.,Ph.D., P.G & Research Department of Mathematics, Periyar E.V.R. College, Tiruchirapalli-620023. E-mail: radhagac@yahoo.com & anbu.saam@gmail.com